Linear and Nonlinear Systems of Equations (§6.1)

There are essentially two methods we employ here

1.) use graphical method: the two equations correspond to two graphs. If they intersect then those points give sol^{n}s. This is an <u>approximate</u> method due to the limits of graphing precisely.

2.) Substitution. We have two equations in variables X and Y. Choose X or Y and solve for it, then stick that into the remaining equation. This is sometimes harder than graphing BUT we are rewarded with <u>exact</u> solutions.

$\boxed{\text{E205}}$ Solve the following <u>system</u> of <u>linear equations</u>

$2x + Y = 6$: Eq^{n} I

$-x + Y = 0$: Eq^{n} II

Solve II for Y and get <u>$Y = X$</u> III

<u>Substitute</u> III into I,

$$2x + x = 6 \Rightarrow 3x = 6 \Rightarrow \boxed{x = 2}$$

III says $Y = X = 2$

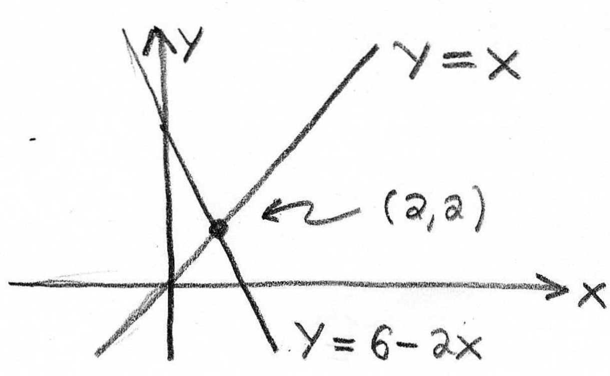
thus $\boxed{Y = 2}$

We just used the "<u>method of substitution</u>". Let's see how the graphical method compares.

I: $Y = 6 - 2x$

II: $Y = X$

Remark: we'd need some graph paper to do this accurately



E206 Solve $\begin{cases} x + y = 0 & \text{Eq}^{\underline{n}} \text{ I} \\ x^3 - 5x - y = 0 & \text{Eq}^{\underline{n}} \text{ II} \end{cases}$

Clearly I is easy to substitute into II. I $\Rightarrow$ $\underline{y = -x}$ III

Substitute III into II,

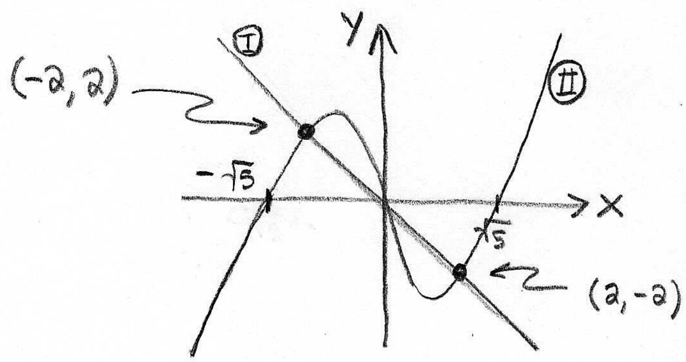
$$x^3 - 5x + x = 0$$

$$x(x^2 - 4) = 0$$

$$\underline{x = 0} \quad \text{or} \quad x^2 = 4 \Rightarrow \underline{x = \pm 2}$$

We find solutions $\boxed{(0,0),\ (2,-2),\ (-2,2)}$

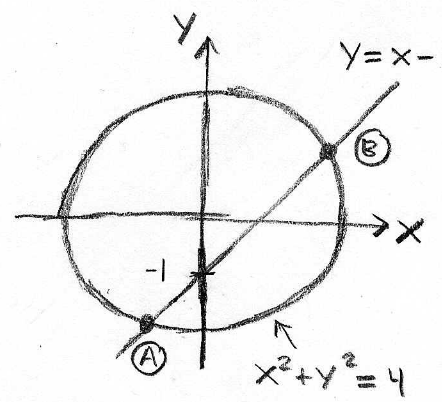
Let's critique our sol$^{\underline{n}}$ graphically (this is §6.1 # 10)



I : $y = -x$

II : $\underbrace{y = x^3 - 5x = x(x^2 - 5)}_{\text{cubic with zeros at } x = 0 \text{ \& } x = \pm\sqrt{5}}$

E207 Solve $\begin{cases} x = y + 1 & \text{Eq}^{\underline{n}} \text{ I} \\ x^2 + y^2 = 4 & \text{Eq}^{\underline{n}} \text{ II} \end{cases}$



Notice I substitutes nicely into II,

$$(y+1)^2 + y^2 = 4$$

$$y^2 + 2y + 1 + y^2 - 4 = 0$$

$$2y^2 + 2y - 3 = 0$$

$$y = \frac{-2 \pm \sqrt{4 - 4(2)(-3)}}{4} = \frac{-2 \pm \sqrt{28}}{4}$$

Thus $y = \underline{\dfrac{-1 \pm \sqrt{7}}{2}} = \underline{0.8229 \text{ or } -1.823}$ then the x-values $\underline{\text{follow from I : } x = 1.8229 \text{ or } x = -0.823}$

The sol$^{\underline{n}}$'s are $\boxed{\underbrace{(-0.823, -1.823)}_{A} \text{ and } \underbrace{(1.8229, 0.8229)}_{B}}$

You can see the graph checks our algebra.

$\boxed{E208}$ Solve $\begin{cases} x^2 + y = 4 & \text{eq}^{\underline{n}}\ \text{I} \\ e^x - y = 0 & \text{eq}^{\underline{n}}\ \text{II} \end{cases}$

Solve II for $\underline{y = e^x}$ (III) then substitute into I,

$$\underbrace{x^2 + e^x = 4}$$

this is a transcendental eq$^{\underline{n}}$.
We cannot solve it algebraically.

<u>THIS IS WHY YOU MUST DO §6.1 # 55 & 56 GRAPHICALLY</u>

Method of Elimination, Two Variable Linear Systems (§6.2)

I commonly refer to this as adding or subtracting equations. Basically given

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

Multiply one or both rows by nonzero #'s such that either a_{11} & a_{21} match or a_{12} & a_{22} match then add or subtract to eliminate either x or y. Solve for remaing variable then back substitute to find the other variable. (see pg. 460 for nice neatly written algorithm). <u>CHECK ANSWER</u>

$\boxed{E209}$

$$\begin{array}{r} (x + y = 7) \\ +\ (x - y = 5) \\ \hline 2x = 12 \Rightarrow \boxed{x = 6} \end{array}$$

Then $6 - y = 5 \Rightarrow y = 6 - 5 = 1 \Rightarrow \boxed{y = 1}$

$\boxed{\text{The solution is } (6,1)}$ (we should write sol$^{\underline{n}}$ as a pair for systems although no danger of confusion exists here in contrast with $\boxed{E207}$)

Check Answer:

$6 + 1 = 7$ ✓

$6 - 1 = 5$ ✓

$\boxed{E210}$ Solve $\begin{cases} 2x - 7y = 1 \\ 3x + y = 0 \end{cases} \to \begin{pmatrix} 6x - 21y = 3 \\ 6x + 2y = 0 \end{pmatrix}$

$$-23y = 3$$
$$y = -3/23$$

Then $3x - 3/23 = 0 \Rightarrow 3x = \frac{3}{23} \Rightarrow x = \frac{1}{23}$

$\boxed{\text{The solution is } (1/23, -3/23)}$

Check Answer: $2\left(\frac{1}{23}\right) - 7\left(\frac{-3}{23}\right) = \frac{2+21}{23} = \frac{23}{23} = 1$ ✓
$3\left(\frac{1}{23}\right) - \frac{3}{23} = 0$ ✓

$\boxed{E211}$ Solve $\begin{cases} 0.1x - 0.7y = 2 : \underline{R1} \\ 0.13x - 0.35y = 1 : \underline{R2} \end{cases}$

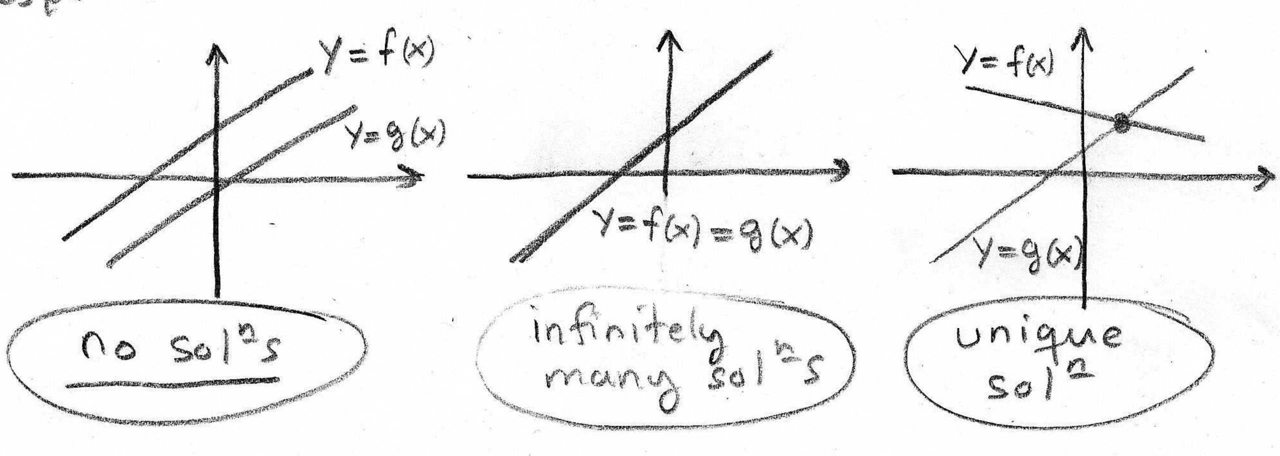
$$\begin{matrix} 10R_1 : \\ 20R_2 : \end{matrix} \begin{pmatrix} x - 7y = 20 \\ 26x - 7y = 20 \end{pmatrix}$$

$$-25x = 0 \Rightarrow x = 0$$

Then $0 - 7y = 20 \Rightarrow y = -20/7$

$\therefore$ $\boxed{(0, -20/7) \text{ is the sol}^{\underline{n}}}$

Remark: if you think about what we're doing graphically it is strange we always find just one sol$^{\underline{n}}$ in our examples. Two linear equations correspond to two lines: $y = f(x)$ & $y = g(x)$ thus



$\boxed{E212}$ Solve $\begin{cases} x + y = 1 & : R1 \\ 2x + 2y = 2 & : R2 \end{cases}$

$$\begin{aligned} 2(R1) &: \ \ (2x + 2y = 2) \\ R2 &: -(2x + 2y = 2) \\ \hline & \quad 0 = 0 \end{aligned} \Rightarrow y = 1 - x \text{ is the sol}^{\underline{n}}$$

$$\boxed{\text{Sol}^{\underline{n}} \text{ Set} = \{ (x, y) \mid y = 1 - x, \ x \in \mathbb{R} \}}$$

Graphically: R1 & R2 are the same line.

$\boxed{E213}$ Solve $\begin{cases} 2x - 2y = 1 \\ x - y = 3 \end{cases}$

$$\begin{aligned} R1 &: \ \ (2x - 2y = 1) \\ 2(R2) &: -(2x - 2y = 6) \\ \hline & \quad 0 = -5 \end{aligned}$$

a contradiction, this is impossible $\Rightarrow$ $\boxed{\text{no sol}^{\underline{n}}\text{'s}}$

Graphically: R1 & R2 are paralell lines.

Multivariable Linear Systems (§6.3)

All the possibilities for two eq$^{\underline{n}}$s and 2 unknowns exist as well for 3 unknowns. For 3 linear equations our sol$^{\underline{n}}$ will be a triple

$\boxed{E214}$ Show $(-1, 0, 4)$ is a sol$^{\underline{n}}$ of $\begin{cases} 3x - y + z = 1 & : R1 \\ 2x - 3z = -14 & : R2 \\ 5y + 2z = 8 & : R3 \end{cases}$

A sol$^{\underline{n}}$ will solve all three equations simultaneously

$$x = -1, \quad y = 0, \quad z = 4$$

$3(-1) - 0 + 4 = 1$ Solves R1 ✓

$2(-1) - 3(4) = -14$ solves R2 ✓

$5(0) + 2(4) = 8$ Solves R3 ✓

We solve systems of 3 linear eq^{n}'s and 3 unknowns by essentially the same methods as in § 6.1 or § 6.3. We can use substitution or elimination.

$\boxed{E215}$ Solve
$$\begin{cases} x + y + z = 4 & : R1 \\ x - y + z = 2 & : R2 \\ 2x - z = 0 & : R3 \end{cases}$$

$$\begin{array}{r|l} R1 + R2 & 2x + 2z = 6 \\ R3 & -(2x - z = 0) \\ \hline & 3z = 6 \Rightarrow \boxed{z = 2} \end{array}$$

$R3: \quad 2x - 2 = 0 \Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$

$R2: \quad 1 - y + 2 = 2 \Rightarrow \boxed{y = 1}$ $\boxed{\text{the sol}^{n} \text{ is } (1,1,2)}$

$\boxed{E216}$ Elimination as in $\boxed{E215}$ requires insight. Substitution on the other hand just requires careful calculation. Solve

$$x + 2y + 3z = 5 \quad : R1$$
$$2x - y + 2z = 11 \quad : R2$$
$$3x + y - z = 10 \quad : R3$$

$x = 2, \; y = -1, \; z = 1$

Solve R3 for $\underline{y = 10 + z - 3x}$ (I). Substitute (I) into R2 and then solve for z,

$$2x - (10 + z - 3x) + 2z = 11$$
$$2x + 3x - 10 - z + 2z = 11 \Rightarrow \underline{z = 21 - 5x} \text{ (II)}$$

We have made use of $\boxed{R2}$ & $\boxed{R3}$ we now use $\boxed{R1}$ to finish, substitute (I) and (II) into $\boxed{R1}$

$$x + 2(10 + z - 3x) + 3(21 - 5x) = 5$$
$$x + 20 + 2z - 6x + 63 - 15x = 5$$

over

E216 Continued : Collect terms from the last step,

$$-20x + 2z = 5 - 20 - 63$$

Now substitute (II) to eliminate z,

$$-20x + 2(21 - 5x) = -78$$

$$-20x + 42 - 10x = -78$$

$$-30x = -78 - 42$$

$$x = \frac{-120}{-30} = 4 \quad \therefore \boxed{x = 4}$$

Thus (II) says $z = 21 - 5(4) = 21 - 20 \Rightarrow \boxed{z = 1}$

Returning to (I) $y = 10 + z - 3x = 10 + 1 - 12 = -1 \Rightarrow \boxed{y = -1}$

We find the sol^{n} $\boxed{(4, -1, 1)}$

Remark: Elimination is much quicker when you see it.

$\boxed{\text{E217}}$ Solve

$$\begin{aligned} 2x + y + 3z &= 1 \quad : R1 \\ 2x + 6y + 8z &= 3 \quad : R2 \\ 6x + 8y + 18z &= 5 \quad : R3 \end{aligned}$$

R2 − R1 $\quad 5y + 5z = 1 \Rightarrow z = \frac{1}{5}(1 - 5y)$ (I)

$$\begin{aligned} 3(R2) :&\quad 6x + 18y + 24z = 9 \\ R3 : -&\quad 6x + 8y + 18z = 5 \\ \hline &\quad 10y + 6z = 4 \Rightarrow z = \tfrac{1}{6}(4 - 10y) \text{ (II)} \end{aligned}$$

(I) & (II) $\Rightarrow \frac{1}{5}(1 - 5y) = \frac{1}{6}(4 - 10y)$

$$\Rightarrow 6 - 30y = 20 - 50y$$

$$\Rightarrow 20y = 14 \Rightarrow \boxed{y = \frac{14}{20} = \frac{7}{10}}$$

Then (II): $z = \frac{1}{6}\left(4 - 10\left(\frac{7}{10}\right)\right) = \frac{1}{6}(4 - 7) = \frac{-3}{6} \Rightarrow \boxed{z = \frac{-1}{2}}$

R1 : $2x + \frac{7}{10} - \frac{3}{2} = 1 \Rightarrow 2x = 1 + \frac{3}{2} - \frac{7}{10} = \frac{10 + 15 - 7}{10} = \frac{18}{10}$

Thus, $2x = \frac{18}{10}$ and $\boxed{x = 9/10}$ Hence $\boxed{(9/10,\ 7/10,\ -1/2) \text{ is the sol}^{n}.}$