Matrices and Systems of Equations (Chapter 7)

My goal in the remainder of this course is to teach you the basics of how matrix math allows us to find orderly, neat solutions to linear systems. Then for fun we'll see how matrix inverses & multiplication give a way of doing simple encryption.

Road Map

1.) What is the Matrix? Add, subtract and Mutiply Matrices. We then discover that certain matrices can be "divided" but here division means multiplication by the inverse. I give the formula for the 2x2 inverse and we leave the 3x3 inverse for the calculator.

2.) I show how systems of linear equations can be rewritten as a single matrix eq: $A\vec{x} = \vec{c}$. Then we see how multiplication by A^{-1} gives the <u>unique</u> solution $\vec{x} = A^{-1}\vec{c}$.

3.) We discuss the general & efficient method of row-reducing $[\vec{A} \mid \vec{c}]$ to solve $A\vec{x} = \vec{c}$. We leave the heavy lifting to the calculator in $\boxed{E235}$ – $\boxed{E240}$.

4.) Determinants of 2x2 and 3x3 matrices are examined. Then we see how Kramer's Rule is yet another method to solve $A\vec{x} = \vec{c}$.

5.) Encryption by Matrix Multiplication.

<u>Remark</u>: we will avoid some of the stickier calculations. Gaussian Elimination is terribly useful but we'll not cover it in the style of Chapter 7. Our philosophy for this course is to focus on the 2x2 case and easy aspects of 3x3 case.

What is a MATRIX?

Here are a few examples,

E222

i.) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$: this is a (2×2) matrix, its square

ii.) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: this is a (3×3) matrix, also a square matrix

iii) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: this is a (4×4) matrix. Again we say this is a square matrix.

iv.) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$: this is a (2×4) matrix. It is not square. We say it is rectangular but that's no so interesting since all matrices are rectangular

v.) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$: this is a (3×2) matrix

number of rows number of columns.

vi.) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: this is a (2×1) matrix. This very special matrix has another name, it's a 2-dimensional column vector

vii.) $[1 \ -2 \ \ 3] = [\underbrace{1, -2, 3}]$: this is a (1×3) matrix. It is called a row vector of dimension 3.

commas optional add them if it helps clarify the entries.

General Notation for Matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & & & \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix} = [A_{ij}] = \underline{p \times q \text{ matrix}}$$

We say that A_{ij} is the (ij)-th <u>component</u> of A. Also,

$$\begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{p1} \end{bmatrix} = [A_{i1}] = \text{the } 1^{st} \text{ column of } A.$$

$$[A_{11}\ A_{12} \cdots A_{1q}] = \text{the } 1^{st} \text{ row of } A.$$

You can easily see a $p \times q$ matrix is made of p-rows of dimension q pasted together. Or we can look at it as q-columns of dimension p pasted together. If $A_{ij} \in \mathbb{R}$ for $i = 1, 2, .., p$ and $j = 1, .., q$ then we say A is a matrix with real entries. Moreover, we may state $A \in \mathbb{R}^{p \times q}$ = set of all $p \times q$ real-entried matrices.

Def^n/ Two matrices are equal when <u>all</u> their components are equal. Matrices of different size cannot be equal.

E223

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

not equal

(zero matrix)

$$\underbrace{\begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

a matrix equation. This is same as saying

$$x + y = 2$$
$$x - y = 0$$

Adding, Subtracting, Mutiplying by a Number

E224

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1+a & 2+b \\ 3+c & 4+d \end{bmatrix}.$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5-1 & 6-2 \\ 7-3 & 8-4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 0+8 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}.$$

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2(3) \\ 3(3) & 4(3) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}.$$

E225 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ calculate $A + 2B$ and $A - A$.

$$\begin{aligned} A + 2B &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix} = A + 2B} \end{aligned}$$

in this context this means the zero matrix

$$A - A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

General Formulas for Matrix Operations

Let $A = [A_{ij}]$ and $B = [B_{ij}]$ and c a number then,

$$A + B = [(A+B)_{ij}] \text{ where } (A+B)_{ij} \equiv A_{ij} + B_{ij} \quad \forall i,j.$$

$$cA = [(cA)_{ij}] \text{ where } (cA)_{ij} \equiv cA_{ij} \quad \forall i,j.$$

Matrix Multiplication

We can multiply certain products of matrices. We need the number of columns for the left factor to match the number of rows for the right factor. In other words, $(P \times q) \cdot (q \times R)$ makes sense. Moreover the product of a $(P \times q) : A$ with $(q \times R) : B$ will yield AB a $(P \times R)$ sized matrix. However, BA may not even make sense. (would need $P = R$ for BA to be a reasonable matrix multiplication)

$\boxed{\text{E226}}$ Let's see how it works for a square matrix multiplied by a column vector,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1(5) + 2(6) \\ 3(5) + 4(6) \end{bmatrix} = \begin{bmatrix} 5 + 12 \\ 15 + 24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 0 + 7y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 7y \end{bmatrix}.$$

Notice $(2 \times 2)(2 \times 1) = (2 \times 1)$ in both cases above.

$\boxed{\text{E227}}$ What about a row-vector with a square matrix,

$$[5, 6] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [5(1) + 6(3), 5(2) + 6(4)] = [23, 34].$$

$$[x, y, z] \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = [x + z, 2y + z, z].$$

Notice that $(1 \times 2)(2 \times 2) = (1 \times 2)$ whereas $(1 \times 3)(3 \times 3) = (1 \times 3)$.

E228 *Multiplying Square Matrices.* Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 0(4)+1(6) & 0(5)+1(7) \\ \hline 2(4)+3(6) & 2(5)+3(7) \end{array}\right]$$

$$= \boxed{\begin{bmatrix} 6 & 7 \\ 26 & 31 \end{bmatrix} = AB}$$

$$BA = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 0\cdot 4 + 2\cdot 5 & 1\cdot 4 + 3\cdot 5 \\ \hline 0\cdot 6 + 2\cdot 7 & 1\cdot 6 + 3\cdot 7 \end{array}\right]$$

$$= \boxed{\begin{bmatrix} 10 & 19 \\ 14 & 27 \end{bmatrix} = BA}$$

Notice that $(2\times 2)(2\times 2) = (2\times 2)$ in both cases above. Also we note <u>matrix</u> <u>multiplication</u> is <u>not</u> <u>commutative</u>; $AB \neq BA$ in general.

E229 There are certain matrices which commute with each other. For example, the <u>inverse</u> <u>matrix</u> of A is found via the following formula

$$\underbrace{A^{-1} = \frac{1}{0-2}\begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}}_{\text{I give formula in general on next page.}} = \frac{-1}{2}\begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1 & 0 \end{bmatrix}.$$

Identity Matrix

$$AA^{-1} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3/2 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3+3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^{-1}A = \begin{bmatrix} -3/2 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix I_2 is like 1 for 2×2 matrices.

Formula for 2x2 Matrix Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad-bc \neq 0$. When $ad-bc \neq 0$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ we can say A is <u>invertible</u>.

E230 Find inverse of $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{3-0}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 0 & 1 \end{bmatrix}.$$

E231 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and $B = \frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

$$AB = \left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right)\left(\frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}\right)$$

Can pull "scalar multiples" out front.

$$= \frac{1}{24}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{24}\left[\begin{array}{c|c|c} 1(24)+2(0)+3(0) & 1(-12)+2(6)+3(0) & 1(-2)+2(-5)+3(4) \\ 0(24)+4(0)+5(0) & 0(-12)+4\cdot(6)+5\cdot(0) & 0(-2)+4(-5)+5(4) \\ 0(24)+0(0)+6(0) & 0(-12)+0(6)+6(0) & 0(-2)+0(-5)+6(4) \end{array}\right]$$

$$= \frac{1}{24}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{thus} \quad \boxed{B = A^{-1}}$$

<u>Remark</u>: I found B with my TI-89 calculator. I don't allow such calculators on our tests. However, I also would not ask you to find A^{-1} for a 3x3 on a test. In homework it will come up

Remark on not calculating 3x3 inverse on tests continued

I may ask you to find A^{-1} for a 3×3 in homework. I don't expect you do it by hand (E3 on pg. 557 is not something I expect you to learn in this course) Instead, you can use a calculator or the website I linked to on our course webpage. I find that googling "matrix calculator" brings up a few options. There are website calculators for many things these days.

$\boxed{E232}$ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ as in $\boxed{E231}$. Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and suppose $\vec{c} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$. Solve $A\vec{x} = \vec{c}$. We found the inverse matrix in $\boxed{E231}$ and we saw $AA^{-1} = I_3$. It turns out that $A^{-1}A = I_3$ as well. I said I is like 1 for matrix multiplication. Notice

$$I_3\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+0+0 \\ 0+y+0 \\ 0+0+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{x}.$$

Thus we can solve $A\vec{x} = \vec{c}$ as follows:

1.) Find A^{-1}, make sure it exists, set the details aside until 4.).
2.) Multiply by A^{-1} on the left (don't write out the components yet)
$$A^{-1}A\vec{x} = A^{-1}\vec{c}$$
3.) Simplify using $A^{-1}A\vec{x} = I\vec{x} = \vec{x}$ to find $\boxed{\vec{x} = A^{-1}\vec{c}}$
4.) Work out component sol^n's if need be,

$$\vec{x} = A^{-1}\vec{c} = \frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} -24-12 \\ 12-30 \\ 24 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} -36 \\ -18 \\ 24 \end{bmatrix}$$

E232 Continued:

We found $A\vec{x} = \vec{c} \Rightarrow \vec{x} = \frac{1}{24}\begin{bmatrix} -36 \\ -18 \\ 24 \end{bmatrix} = \begin{bmatrix} -36/24 \\ -18/24 \\ 24/24 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -3/4 \\ 1 \end{bmatrix}$

Since $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we find $\boxed{x = -3/2, \ y = -3/4, \ z = 1}$

Let's think about what we've done here.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \vec{c} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

Writing $A\vec{x} = \vec{c} \iff \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$

$$\iff \begin{aligned} x + 2y + 3z &= 0 \\ 4y + 5z &= 2 \\ 6z &= 6 \end{aligned}$$

Thus solving $A\vec{x} = \vec{c}$ was equivalent to solving $\begin{cases} x + 2y + 3z = 0 \\ 4y + 5z = 2 \\ 6z = 6 \end{cases}$

Remark: We can trade the trouble of solving several eq^{n}'s simultaneously for the trouble of solving a single matrix eq^{n} $A\vec{x} = \vec{c}$. We saw in $\boxed{E232}$ multiplication by A^{-1} helped us deduce $\vec{x} = A^{-1}\vec{c}$. This often works but finding A^{-1} is not easy for systems of 3 or more eq^{n}'s.

$\boxed{E233}$ Solve $\begin{cases} x + y = 3 \\ x - 2y = -3 \end{cases}$ by converting to a matrix eq$^{\underline{n}}$ $A\vec{x} = \vec{c}$ then multiplying by A^{-1}.

$$\begin{aligned} x + y &= 3 \\ x - 2y &= -3 \end{aligned} \implies \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Thus the given system of eq$^{\underline{n}}$'s is $A\vec{x} = \vec{c}$ with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

We can find A^{-1} using the nice 2×2 inverse formula,

$$A^{-1} = \frac{1}{-2-1}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}.$$

Notice, $A\vec{x} = \vec{c} \implies A^{-1}A\vec{x} = A^{-1}\vec{c}$

$\implies I_2\vec{x} = A^{-1}\vec{c}$

$\implies \underline{\vec{x} = A^{-1}\vec{c}}$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \boxed{\begin{matrix} x = 1 \\ y = 2 \end{matrix}}$$

You can check this sol$^{\underline{n}}$ is correct. This is the third method we've found to solve 2×2 systems. Obviously elimination is a more efficient method but this requires less insight.

<u>We just follow the recipe:</u>

1.) Identify A and $\vec{c}$ so $A\vec{x} = \vec{c}$ is equivalent to the given system of equations.

2.) Find A^{-1} via $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (for 2×2)

or via calculator for 3×3

3.) Multiply $A^{-1}\vec{c}$ to read off the sol$^{\underline{n}}$'s from $\vec{x} = A^{-1}\vec{c}$.

E234 Find solution of $\begin{cases} x + y + z = 1 \\ x - y + 3z = 3 \\ 2x - z = -1 \end{cases}$

Follow the recipe from 121:

1.) $$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 2 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}_{\vec{c}}$$

2.) $$A^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 1 & 4 \\ 7 & -3 & -2 \\ 2 & 2 & -2 \end{bmatrix}$$

(I like to factor out fraction so I have less fraction math to deal with in step 3.)

3.) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 1 & 4 \\ 7 & -3 & -2 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1+3-4 \\ 7-9+2 \\ 2+6+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can read from the 1st, 2nd and 3rd rows

$$\boxed{x = 0, \quad y = 0, \quad z = 1}$$

Remark: multiplication by inverse is a conceptually satisfying method in my opinion. However, I should tell you this method is not general, there are cases it misses. Whenever we can't find A^{-1} the method fails. Next, I show a robust method which solves any linear system. We will use the calculator or website to do the heavy-lifting. Your primary duty is translating and interpreting the matrix eq^{n}'s.