Augmented Matrix Method

Goal: Given a system of linear eq$^{\underline{n}}$'s find the sol$^{\underline{n}}$ by matrix calculations. We are allowed to use calculator and website on coursepage to perform a nontrivial matrix operation called "ref" or "rref" or "row reduction". We could perform that step ourselves via "Gauss-Jordan" elimination. But, I'm not teaching that algorithm, we let the calculator or computer do those steps for us. Come back and read this again after a few examples.

$\boxed{\text{E235}}$ Solve $\begin{cases} x + z = 2 \\ y + z = 2 \\ 13x - 3z = 10 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 13 & 0 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}}_{\vec{c}}$

Form the "augmented matrix"

$$[A \,\vdots\, \vec{c}\,] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 13 & 0 & -3 & 10 \end{array}\right]$$

I can't make this show up in the calculator. It is to remind us where the coefficient matrix A stops and the constant vector $\vec{c}$ begins.

Define $B = [A \,\vdots\, \vec{c}\,]$ and enter it into TI-89 or other sufficiently sophisticated calculator (or use website)

$$\underbrace{\text{rref}(B) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]}_{\text{this corresponds to the eq}^{\underline{n}}\text{'s below}}$$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \end{aligned}}$$

$\boxed{E236}$ Suppose $\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 0 \\ 3x + 3y + 3z = 0 \end{cases}$ find sol^{n}'s if there are any. This is $A\vec{x} = \vec{c}$

The augmented matrix is found by reading off the coefficients,

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]}_{\text{call this } B}$$

Use technology,

$$\text{rref}(B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$ this row translates to $0x + 0y + 0z = 1$

that is $0 = 1$

$\Rightarrow$ $\boxed{\text{no sol}^{n}\text{'s here}}$

$\boxed{E237}$ $\begin{cases} x + y + z = 0 \\ 2x + 2y + 2z = 0 \\ 3x + 3y + 3z = 0 \end{cases}$ Solve if possible.

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

this time the rows of zeros cause no trouble. We find there is just one equation here.

$$x + y + z = 0$$

There is no unique sol^{n}, instead there are infinitely many. We could say $\boxed{x = x,\ y = y,\ z = -x - y}$ (two free variables x & y)

Remark: Graphically $ax + by + cz = d$ is a plane. three planes can either intersect at a point, line, plane or nowhere at all. The unique sol^{n} case is when the planes intersect at a single point. (See pg. 474 for nice pictures of this)

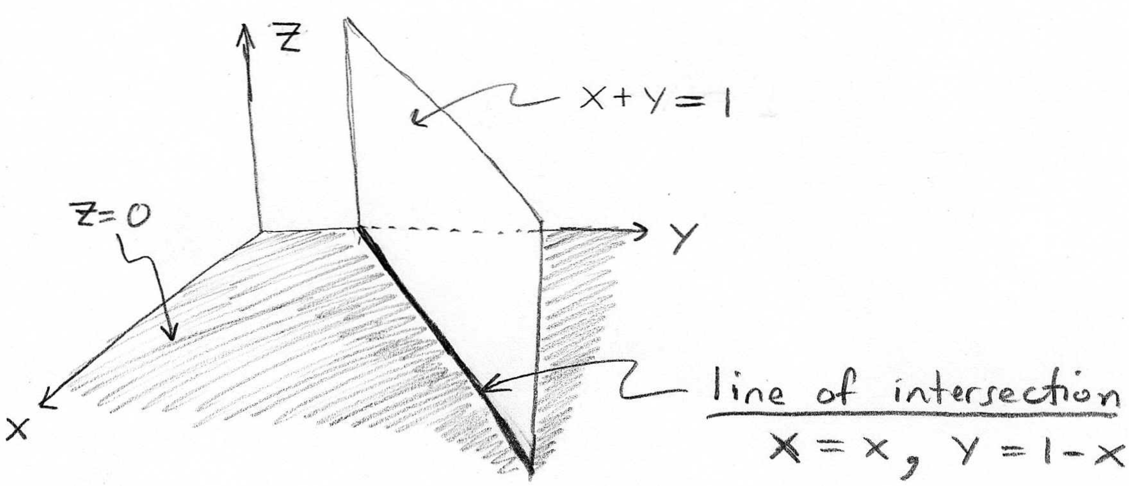
E238 Solve $\begin{cases} x + y = 1 \\ 2x + 2y = 2 \\ z = 0 \end{cases}$ if possible.

Form the augmented coefficient matrix and use technology to do the Gauss-Jordan reduction,

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow x + y = 1 \\ \rightarrow z = 0 \\ \rightarrow \text{no direct info.} \end{array}$$

I say no "direct info" because the row of zeros show us that there is no unique sol$^{\underline{n}}$. Instead for this problem we find $\boxed{x = x,\ y = 1 - x,\ z = 0}$ (one free variable x)

Remark: I can attempt a picture here



We treat these ideas with some care in calculus III, don't worry I don't expect you to be able to reproduce such pictures. I'm merely trying to explain why there are different cases when solving three eq$^{\underline{n}}$'s and three unknowns.

$\boxed{E239}$ Let's solve the system from $\boxed{E234}$ via the augmented coefficient matrix method.

$$\begin{cases} x + y + z = 1 \\ x - y + 3z = 3 \\ 2x - z = -1 \end{cases}$$

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & 3 \\ 2 & 0 & -1 & -1 \end{array}\right] = \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]}_{} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \boxed{\begin{array}{c} x = 0 \\ y = 0 \\ z = 1 \end{array}}$$

when A^{-1} exists
the augmented
Coefficient matrix $[\vec{A} \mid \vec{c}]$
reduce to $[I \mid A^{-1}\vec{c}]$

$\vec{x} = A^{-1}\vec{c}$ is the sol^n.

<u>Remark</u>: Since I do not allow graphing calculators on tests this means I cannot ask you to find rref (B). However, I could give you a matrix which has been Gauss-Jordan eliminated and then ask questions about the matrix. For example, I could ask you to match each system with its reduced reduced aug-coeff. matrix

a.) $\begin{cases} x + y + z = 3 \\ y + z = 2 \\ x - y + z = 1 \end{cases}$

1.) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b.) $\begin{cases} x - 13y + z = 0 \\ x + y + 2z = 0 \\ x - y + z = 0 \end{cases}$

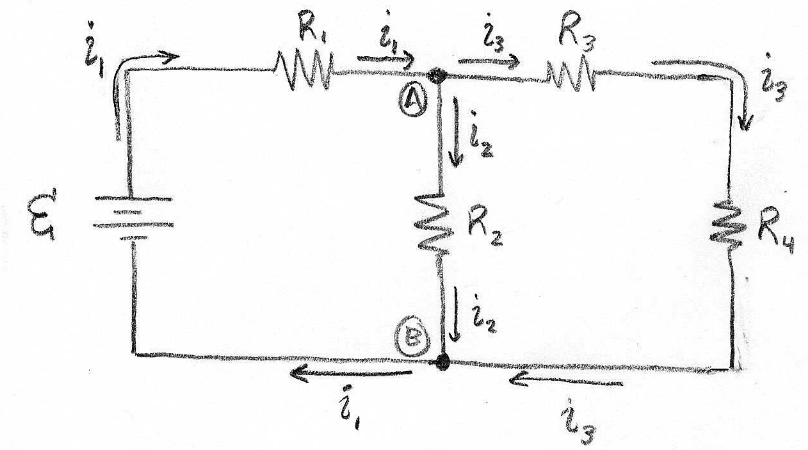
2.) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

c.) $\begin{cases} x + y + 2z = 0 \\ x + y + 2z = 0 \\ x + y + 2z = 0 \end{cases}$

3.) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Can you determine which reduced augmented matrix goes with which system?

E240 Consider the circuit below. Find the total current flowing through the circuit.



Kirchoff's Laws and the conservation of charge say that the current going into a node and the current leaving a node must be equal.

NODE (A) : $i_1 = i_2 + i_3$

NODE (B) : $i_2 + i_3 = i_1$

Left Loop : $\mathcal{E} - i_1 R_1 - i_2 R_2 = 0$

Right Loop : $i_2 R_2 - i_3 R_3 - i_3 R_4 = 0$

Let $\mathcal{E} = 10$ and $R_1 = 1,\ R_2 = 2,\ R_3 = 3,\ R_4 = 4$ then

$$\left.\begin{array}{l} i_1 = i_2 + i_3 \\ 10 - i_1 - 2i_2 = 0 \\ 2i_2 - 3i_3 - 4i_3 = 0 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} i_1 - i_2 - i_3 = 0 \\ i_1 + 2i_2 = 10 \\ -2i_2 + 7i_3 = 0 \end{array}\right.$$

We find the problem is equivalent to

$$\text{rref}\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & -10 \\ 0 & -2 & 7 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 90/23 \\ 0 & 1 & 0 & 70/23 \\ 0 & 0 & 1 & 20/23 \end{array}\right] \begin{array}{l} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \boxed{\begin{array}{l} i_1 = 90/23 \\ i_2 = 70/23 \\ i_3 = 20/23 \end{array}}$$

<u>Remark</u>: to be careful and correct physically we ought to include units for voltage and resistance and current. I just wanted to show how the algebra we're doing solves real world problems.