Determinants

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc.$$: Definition of 2×2 Determinant

E241

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - 2(3) = -2$$

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 - 0 = 1$$

$$\det\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 2 - 2 = 0$$

$$\det\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = 3(2) - 4 = 2$$

Remark: if two rows are the same then the determinant works out to zero. Also, if we swap two rows then the sign of the determinant changes.

Observation: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A^{-1} exists $\Leftrightarrow \det(A) \neq 0$. Notice $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We can't compute A^{-1} if $\det(A) = 0$.

Def^{n}/ $$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \equiv a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

E242

$$\det\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = (0)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - (1)\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$

$$= -(24 - 30) + 2(21 - 24)$$

$$= 30 - 24 + 42 - 48$$

$$= \boxed{0}$$

$\boxed{E243}$ Calculate the determinant of A from $\boxed{E231}$,

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = (1)\det\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} - 2\det\begin{bmatrix} 0 & 5 \\ 0 & 6 \end{bmatrix} + 3\det\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$$

$$= 1(24-0) - 2(0-0) + 3(0-0)$$

$$= \boxed{24}$$

Remark: for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ we saw A^{-1} exists in $\boxed{E231}$. We now find $\det(A) \neq 0$.

$\boxed{E244}$ Calculate determinant.

$$\underbrace{\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 2 & 0 & -1 \end{bmatrix}}_{A} = \det\begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix} - \det\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \det\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= 1 - 0 - (-1-6) + (0+2)$$

$$= 1 + 7 + 2$$

$$= \boxed{10}.$$

This matrix is A from $\boxed{E234}$. We found A^{-1} existed in that example. See a pattern? Again $\det(A) \neq 0 \Leftrightarrow A^{-1}$ exists.

$\boxed{E245}$ The matrix below is taken from $\boxed{E237}$, we found many sol$^{\underline{n}}$'s in that example. In contrast, whenever A^{-1} existed we found a unique sol$^{\underline{n}}$.

$$\det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \det\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} - \det\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} + \det\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$= (6-6) - (6-6) + (6-6)$$

$$= \boxed{0}.$$

Remark: We see determinants tell us if $A\vec{x} = \vec{c}$ has a unique sol$^{\underline{n}}$. The criteria is simply that $\det(A) \neq 0$.

Kramer's Rule

We can solve $A\vec{x} = \vec{c}$ if $\det(A) \neq 0$ by a simple procedure known as Kramer's Rule. Let's see how it works by example.

E246 Solve $\underbrace{\begin{matrix} x + 2y = 3 \\ 4x + 5y = 9 \end{matrix}}$ via Kramer's Rule.

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \vec{c}$$

Kramer's Rule says,

$$x = \frac{\det\begin{bmatrix} 3 & 2 \\ 9 & 5 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}} = \frac{15 - 18}{5 - 8} = \frac{-3}{-3} = 1.$$

$$y = \frac{\det\begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}} = \frac{9 - 12}{5 - 8} = \frac{-3}{-3} = 1$$

Thus we find that $\boxed{x = 1, \ y = 1}$

E247 Solve $\left.\begin{matrix} 13x - y = -3 \\ 7x + 4y = 12 \end{matrix}\right\} \longrightarrow \underbrace{\begin{bmatrix} 13 & -1 \\ 7 & 4 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 12 \end{bmatrix}}_{\vec{c}}$

$$x = \frac{\det\begin{bmatrix} -3 & -1 \\ 12 & 4 \end{bmatrix}}{\det\begin{bmatrix} 13 & -1 \\ 7 & 4 \end{bmatrix}} = \frac{-12 + 12}{52 + 7} = 0.$$

$$y = \frac{\det\begin{bmatrix} 13 & -3 \\ 7 & 12 \end{bmatrix}}{\det\begin{bmatrix} 13 & -1 \\ 7 & 4 \end{bmatrix}} = \frac{156 + 21}{52 + 7} = \frac{177}{59} = 3.$$

Thus we find $\boxed{x = 0 \text{ and } y = 3}$ by Kramer's Rule

Summary of 2x2 Kramer's Rule

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{with } ad-bc \neq 0 \text{ has sol}^{n}$$

$$x = \frac{\det\begin{bmatrix} c_1 & b \\ c_2 & d \end{bmatrix}}{\det(A)} \qquad y = \frac{\det\begin{bmatrix} a & c_1 \\ c & c_2 \end{bmatrix}}{\det(A)}$$

Let's do system with 3 linear eq^{n}'s and 3 unknowns via Kramer's Rule.

E248 Solve $\begin{cases} x + z = 2 \\ y + z = 2 \\ 13x - 3z = 10 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 13 & 0 & -3 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}}_{\vec{c}}$

$$\det(A) = \det\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} - 0\det\begin{bmatrix} 0 & 1 \\ 13 & -3 \end{bmatrix} + \det\begin{bmatrix} 0 & 1 \\ 13 & 0 \end{bmatrix} = -3-13 = -16.$$

Kramer's Rule says to divide by $\det(A)$ in the formulas below,

$$x = \frac{\det\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \\ 10 & 0 & -3 \end{bmatrix}}{-16} = \frac{-1}{16}\left(2\det\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} + \det\begin{bmatrix} 2 & 1 \\ 10 & 0 \end{bmatrix}\right) = \frac{-16}{-16} = 1.$$

$$\begin{aligned} y = \frac{-1}{16}\det\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 13 & 10 & -3 \end{bmatrix} &= \frac{-1}{16}\left(\det\begin{bmatrix} 2 & 1 \\ 10 & -3 \end{bmatrix} - 2\det\begin{bmatrix} 0 & 1 \\ 13 & -3 \end{bmatrix} + \det\begin{bmatrix} 0 & 2 \\ 13 & 10 \end{bmatrix}\right) \\ &= \frac{-1}{16}\big((-6-10) - 2(0-13) + (0-26)\big) \\ &= 1. \end{aligned}$$

$$\begin{aligned} z = \frac{-1}{16}\det\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 13 & 0 & 10 \end{bmatrix} &= \frac{-1}{16}\left(\det\begin{bmatrix} 1 & 2 \\ 0 & 10 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 1 \\ 13 & 0 \end{bmatrix}\right) \\ &= \frac{-1}{16}(10 - 0 + 2(0-13) \\ &= 1. \end{aligned}$$

We find the sol^{n} is $\boxed{x=1,\ y=1,\ z=1}$