Quadratic Equations

$\boxed{\text{QUESTION: How do we solve } ax^2 + bx + C = 0}$

We assume $a \neq 0$ and $a, b, c \in \mathbb{R}$. It turns out there are always two sol$^{\underline{n}}$'s, but sometimes they're duplicate or a double root and other times they're complex. There are five main methods to solve the quadratic eq$^{\underline{n}}$

(o.) Take roots of both sides (don't forget $\pm$)
(i.) factoring
(ii.) complete the square
(iii) use the quadratic eq$^{\underline{n}}$
(iv.) graphing

Each of these is important. I'll illustrate each with an example.

E60 (factoring) Solve $x^2 - 5x + 6 = 0$.

Notice $x^2 - 5x + 6 = \underbrace{(x-3)(x-2) = 0}$

a product is zero if either or both of it's factors is zero. We find sol$^{\underline{n}}$'s $\boxed{x = 3 \text{ and } x = 2}$

E61 ($\pm\sqrt{\ }$) Solve $x^2 = 7$

$$\Rightarrow x = \pm\sqrt{7}$$
$$\Rightarrow \boxed{x = \sqrt{7} \text{ or } x = -\sqrt{7}}$$

Technically, this is factoring of a slightly nonintuitive type; $x^2 = 7 \Leftrightarrow x^2 - 7 = 0$

$$\Leftrightarrow (x + \sqrt{7})(x - \sqrt{7}) = 0$$
$$\Leftrightarrow x = -\sqrt{7} \text{ or } x = \sqrt{7}$$

$\boxed{E62}$ (Completing Square)

$x^2 + 6x - 4 = 0$: [Given Problem]

$(x+3)^2 - 9 - 4 = 0$: [observe $(x+3)^2 = x^2 + 6x + 9$
so $x^2 + 6x = (x+3)^2 - 9$
I found the 3 by taking half of 6]

$(x+3)^2 = 13$: [isolate the squared term]

$x + 3 = \pm\sqrt{13}$: [take $\sqrt{\ }$ of both sides, remember we must allow for $\pm$ when taking an even-root]

$x = -3 \pm \sqrt{13}$: [Solved for x]

$\boxed{x = -3 + \sqrt{13} \quad \text{or} \quad x = -3 - \sqrt{13}}$

Remark: Could just factor if you have the sight,

$$x^2 + 6x - 4 = (x + 3 - \sqrt{13})(x + 3 + \sqrt{13})$$

But, I can't recommend this for this problem, I mean would you see the step above?

$\boxed{E63}$ (factoring) $x^2 = 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$\Rightarrow \boxed{x = 0 \quad \text{or} \quad x = 3}$

Quadratic Equation

We can repeat the steps in $\boxed{E62}$ for an arbitrary quadratic eq$^{\underline{n}}$. Let's see how that goes

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$: divided by $a \neq 0$.

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$: you can check that $\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$: isolated the squared term

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$: made common denominator.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$: to $\sqrt{\ }$ of both sides, allowed for $\pm$ possibility.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$\underbrace{\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}}_{\text{QUADRATIC EQUATION}}$$

: cleaned up the expression notice $\sqrt{4a^2} = 2\sqrt{a^2} = \pm 2a$ but this $\pm$ is logically disjoint with the $\pm$ from before, thus there is no cancellation in general and overall we just get the $\pm$ in the box

Remark: $b^2 - 4ac$ is called the discriminant it discriminates which type of sol$^{\underline{n}}$ we'll find,

I.) $b^2 - 4ac > 0 \Rightarrow$ distinct real sol$^{\underline{n}}$'s.
II.) $b^2 - 4ac = 0 \Rightarrow$ repeated real sol$^{\underline{n}}$
III.) $b^2 - 4ac < 0 \Rightarrow$ conjugate pair of complex sol$^{\underline{n}}$'s

$\boxed{E64}$ (Quadratic Equation, case I)

$$x^2 - 5x + 6 = 0 \qquad \left[\begin{array}{l} ax^2+bx+c \quad \text{compare coefficients} \\ a=1,\ b=-5,\ c=6 \end{array}\right]$$

Use quadratic eq^n:

$$x = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5\pm\sqrt{1}}{2} = \frac{5+1}{2} \text{ or } \frac{5-1}{2} = \boxed{3 \text{ or } 2}$$

Good, this agrees with $\boxed{E60}$. I would argue factoring is easier (and safer) when it works. Many students stumble in calculating the $\sqrt{b^2-4ac}$ term. I do allow use of a scientific calculator.

$\boxed{E65}$ (Quadratic Eq^n, case II)

$$x^2 + 2\sqrt{3}\,x + 3 = 0, \qquad a=1, \quad b=2\sqrt{3}, \quad c=3$$

$$x = \frac{-2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 12}}{2}$$

$$= -\sqrt{3} \pm \frac{1}{2}\sqrt{12-12}$$

$$= -\sqrt{3} \pm 0 \implies \boxed{x = -\sqrt{3} \text{ twice}}$$

$\boxed{E66}$ (Quadratic Eq^n, case III) [<u>there are no real number sol^n's here</u>]

$$x^2 + 4x + 5 = 0, \qquad a=1, \quad b=4, \quad c=5$$

$$x = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= -2 \pm \frac{1}{2}\sqrt{-4}$$

$$= -2 \pm \frac{1}{2}\sqrt{4}\sqrt{-1}$$

$$= -2 \pm i \implies \underbrace{\boxed{x = -2+i \quad \text{or} \quad x = -2-i}}_{}$$

conjugate pair: $(-2+i)^* = -2-i$

<u>Remark</u>: $x^2+4x+5 = (x+2-i)(x+2+i)$ it factors over $\mathbb{C}$.