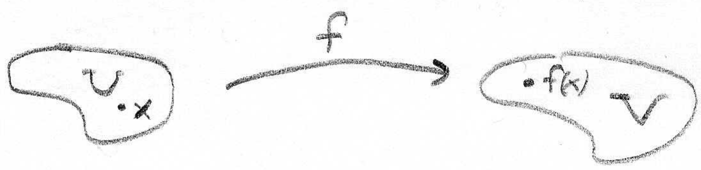
Real - Valued Functions of a Real Variable

Def^{n}/ Let U, V be subsets of $\mathbb{R}$. A function f from U to V is denoted $f : U \longrightarrow V$. We call $U = \text{dom}(f)$ and $V = \text{codomain}(f)$. We require that for each $x \in U$ the function f assigns a unique value $f(x) \in V$. In other notation $x \longmapsto f(x)$. like a machine

$$\xrightarrow{\ x\ } \boxed{\ f\ } \xrightarrow{\ f(x)\ }$$

Mapping Picture



I also like this sort of abstract picture. It helps me picture what the function is doing.

Remark: there are many ways to define the action of the function: verbally, graphically, set-notation, formula, ...

Custom: When we say $f(x) =$ stuff in x then the $\text{dom}(f)$ is taken to be the largest subset of $\mathbb{R}$ for which the formula makes sense. Also the codomain is by default taken to be the range (f).

$$\text{range}(f) = \{ f(x) \mid x \in \text{dom}(f) \} = \text{set of outputs}$$

$$\text{dom}(f) = \text{all possible inputs for } f(x).$$

It is always possible to shrink the $\text{dom}(f)$ to be smaller, we just need to state that in addition to the formula.

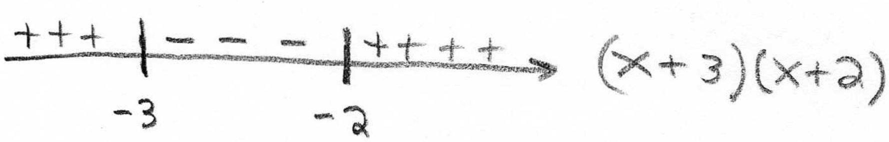
E100 $f(x) = x^2$ has $\text{dom}(f) = \mathbb{R}$

$\boxed{E101}$ Let $f(x) = \dfrac{1}{x-1}$ what is the dom (f) by custom?
Well, we cannot divide by zero so $\text{dom}(f) = \{x \mid x \neq 1\}$ or in other words $\text{dom}(f) = (-\infty, 1) \cup (1, \infty)$.

$\boxed{E102}$ Let $f(x) = \dfrac{1}{x^2}$ for $x \geq 1$. Here we would have trouble at $x = 0$ but our formula is only given for $x \geq 1$ thus $\text{dom}(f) = [1, \infty)$

$\boxed{E103}$ Let $f(x) = \sqrt{x^2 + 5x + 6}$, find domain.
We need the radicand $x^2 + 5x + 6 \geq 0$

$$\Rightarrow \quad (x+3)(x+2) \geq 0$$



critical #'s
$x = -3, -2$

$$\text{dom}(f) = (-\infty, -3] \cup [-2, \infty)$$

$\boxed{E104}$ Let $g(x) = \sqrt{x-2}$, find dom (g).
We need $x - 2 \geq 0 \Rightarrow x \geq 2$. $\boxed{\text{dom}(g) = [2, \infty)}$

$\boxed{E105}$ Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ for $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$ find dom (f). This one is easy, there is no division by zero or even root to worry over. The domain is all $\mathbb{R}$.

$$\boxed{\text{dom}(f) = (-\infty, \infty) = \mathbb{R}}$$

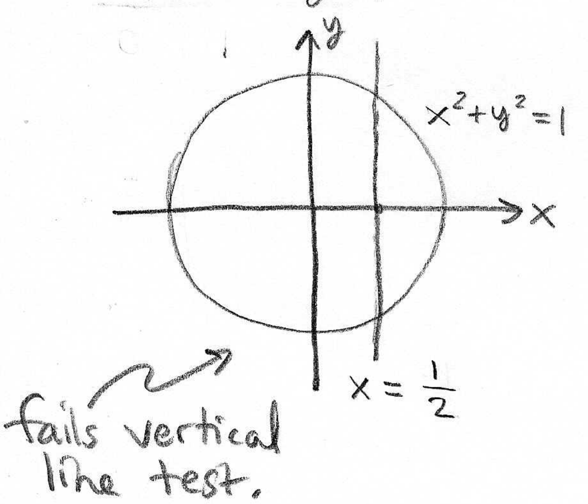
$\boxed{E106}$ Suppose $f(1) = 1$ and $f(1) = 2$ can f be a function? Answer: NO! we must have one output for the input 1.

$\boxed{E107}$ Suppose $f(1) = 3$ and $f(2) = 3$ can f be a function? Answer: YES! Certainly we do not require that the outputs all be different.

$\boxed{\text{Def}^{n}/ \text{ The graph of a function is a subset of } \mathbb{R}^2, \text{ in particular } \text{graph}(f) = \{(x,y) \mid y = f(x) \text{ and } x \in \text{dom}(f)\}}$

Remark: we can test an equation to see if it is a function. If the graph of an equation satisfies the so-called vertical line test then the equation can often be solved to give $y = f(x)$.

$\boxed{E108}$ Consider $x^2 + y^2 = 1$. Is this the graph of some function f, and if so what is the formula for $f(x)$? This is a circle of radius one centered at the origin,

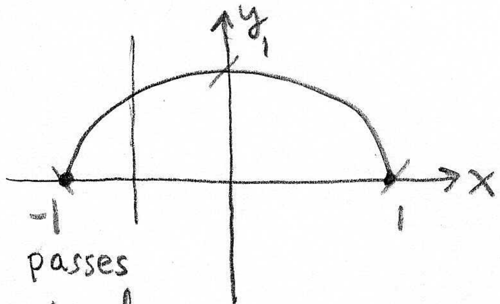


when $x = \frac{1}{2}$ we find

$$\frac{1}{4} + y^2 = 1$$
$$y^2 = 1 - \frac{1}{4} = \frac{3}{4}$$
$$\underbrace{y = \pm\frac{\sqrt{3}}{2}}$$

two outputs for single input
not the graph (f).

109 $x^2+y^2=1$ with $y \geq 0$. Is the graph of this the graph (f) and if so what is the formula for $f(x)$?



passes vertical line test.
(Vertical line only intersects the graph of the eq$^{\underline{n}}$ at most once each x.)

$$x^2+y^2=1$$
$$y^2=1-x^2$$
$$y=\pm\sqrt{1-x^2}$$

Choose $(+)$ since $y \geq 0$

$\therefore\ y=\sqrt{1-x^2}$

This is graph of f with

$$\boxed{f(x)=\sqrt{1-x^2}}$$

You can also deduce $\mathrm{dom}(f)=[-1,1]$ and $\mathrm{range}(f)=[0,1]$.

<u>Terminology & Apology</u>: The function $f(x)$ and the graph $y=f(x)$ are often <u>identified</u> as the same thing. It makes sense to say $y=x^2$ is a function. Just keep in mind this is short for saying $f:\mathbb{R}\to[0,\infty)$ with $f(x)=x^2$ has graph $y=x^2$. Moreover, we keep straight which variable is the output and which is the input by saying

<u>Usually</u> $\begin{cases} x = \text{independent variable (input)} \\ y = \text{dependent variable (output)} \end{cases}$

E110 Can mix things up. Don't be a slave to notation,

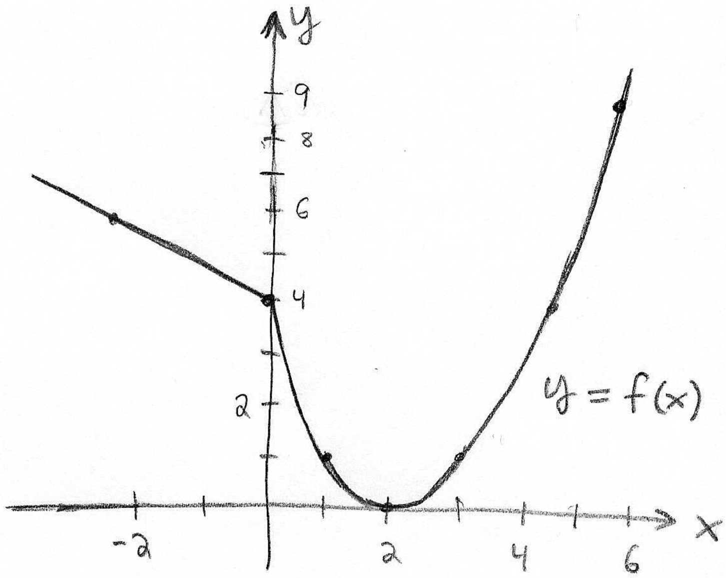
$g(y)=y^2-y$ then $x=y^2-y$ has indep. var. y, dep. var x.

$h(t)=t^2=☺$ has independent variable t, dependent variable ☺

$\boxed{E111}$ Let $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \le 0 \\ (x-2)^2, & x > 0 \end{cases}$ } piecewise-defined function.

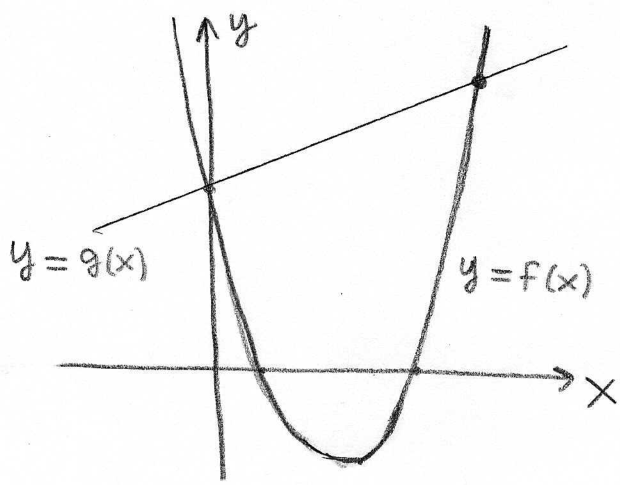
Let's construct a table of values

x	$f(x)$
-2	$-\frac{1}{2}(-2)+4 = 6$
0	4
0.001	$(0.001-2)^2 \cong 4.$
1	$(1-2)^2 = 1$
2	0
3	1
4	4
5	9



We can use the table to construct a sketch of the graph

$\boxed{E112}$ Let $f(x) = x^2 - 6x + 5$ and let $g(x) = 5 + x$. Graph both functions and find their intersection points algebraically. Notice $f(x) = x^2 - 6x + 5 = (x-1)(x-5)$



(not to scale, just a sketch)

The points of intersection have $f(x) = g(x)$

$$x^2 - 6x + 5 = 5 + x$$

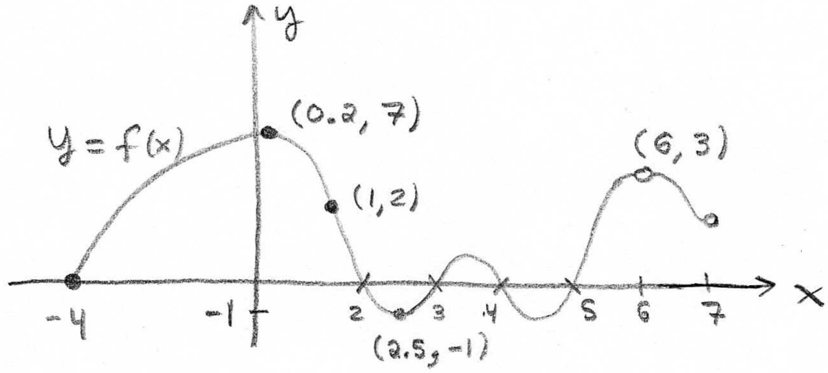
$$\Rightarrow x^2 - 7x = 0$$

$$x(x-7) = 0$$

$$\underline{x = 0 \text{ and } x = 7}$$

The points of intersection are $\underline{(0, 5) \text{ and } (7, 12)}$.

$\boxed{E113}$ Given a graph we can read much information just from the graph.



$f(0.2) = 7$
$f(1) = 2$
$f(6)$ undefined
$f(7)$ undefined

$\mathrm{dom}(f) = [-4, 6) \cup (6, 7)$

$\mathrm{range}(f) = [-1, 7]$

By the way, $f(0.2) = 7$ is a local maximum, whereas $f(2.5) = -1$ is a local minimum.

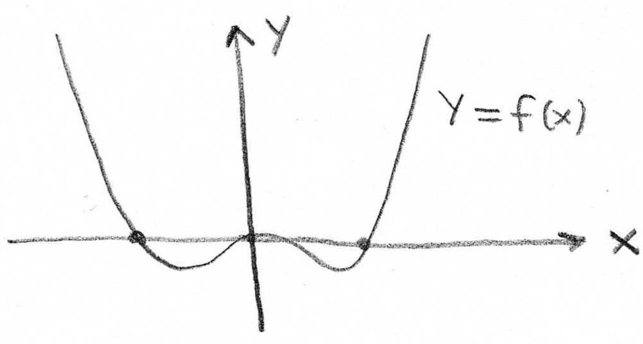
> Def^{n}/ A zero for $y = f(x)$ is $x \in \mathrm{dom}(f)$ such that $f(x) = 0$.

$\boxed{E114}$ The zeros for $f(x)$ from $\boxed{E113}$ are $\underline{x = -4, 2, 3, 4, 5}$.

> <u>Remark</u>: We can find the zeros for a <u>polynomial function</u> $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ by <u>factoring</u>. The zeros for $y = f(x)$ are the sol^{n}'s to the polynomial eq^{n} $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0 = 0$.

$\boxed{E115}$ Let $f(x) = x^4 - x^2$. Find zeros and graph this function.

$f(x) = x^2(x^2 - 1) = x^2(x+1)(x-1) = 0$ for $\underline{x = 0, \pm 1}$



Def^{n}/ Average Rate of change on $a \le x \le b$.

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

E116 Let $f(x) = 3x - 2$ find average over $[0, 2]$

$$\frac{\Delta f}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{3(2) - 2 - (-2)}{2} = \frac{6}{2} = \boxed{3 = \text{avg. rate of change}}$$

E117 Let $s(t) = 64t - 16t^2$. Thinking of s as the position at time t we can calculate the average rate of change during $0 \le t \le 2$, this is the so-called average velocity

$$V_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(2) - s(0)}{2 - 0} = \frac{128 - 64 - 0 - 0}{2} = 32 = V_{avg}$$

Def^{n}/ An even function has $f(x) = f(-x)$ for all $x \in dom(f)$
An odd function has $f(x) = -f(-x)$ for all $x \in dom(f)$

We cannot define even and odd unless both x and $-x$ are in $dom(f)$ for each $x \in dom(f)$. We need $dom(f) = \mathbb{R}$ or $[-a, a]$ or $(-a, a)$ etc...

E118 Power functions are simple polynomials, they have the form $f(x) = x^n$ for $n = 1, 2, 3, 4, 5, \dots$

$f(x) = x^2$ is even since $f(-x) = (-x)^2 = x^2 = f(x)$
$f(x) = x^3$ is odd since $f(-x) = (-x)^3 = -x^3 = -f(x)$

Similar arguments will show

$f(x) = x^n$ for $n = 2, 4, 6, 8, 10, \dots$ are even functions.
$f(x) = x^n$ for $n = 1, 3, 5, 7, 9, \dots$ are odd functions.

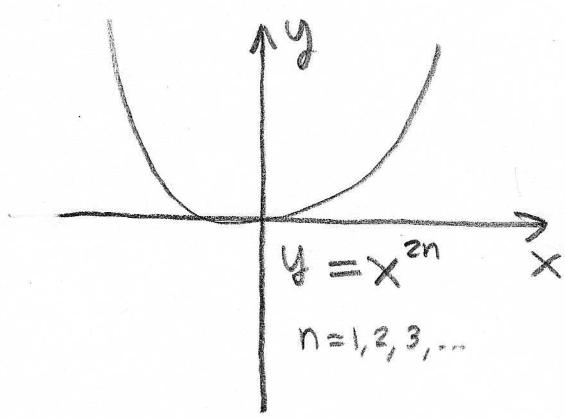
$\boxed{E119}$ Let $f(x) = x + x^2$. This function is neither even nor odd since $f(-x) = -x + (-x)^2 = -x + x^2 \neq f(x)$. However, $f_o(x) = x$ is odd and $f_e(x) = x^2$ is even so we see $f(x) = f_o(x) + f_e(x)$. It turns out we can write any $f : \mathbb{R} \to \mathbb{R}$ as a sum of an even and an odd function.

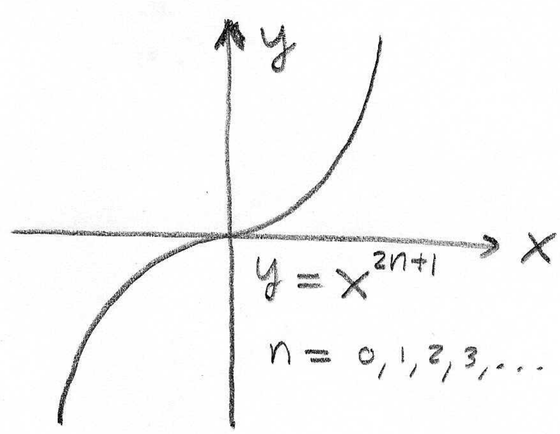
$$f(x) = \underbrace{\tfrac{1}{2}\big(f(x) - f(-x)\big)}_{f_{odd}(x)} + \underbrace{\tfrac{1}{2}\big(f(x) + f(-x)\big)}_{f_{even}(x)}$$

I simply added zero to get the formula above. You can earn a bonus point by showing f_{odd} is odd and f_{even} is even.

$\boxed{E120}$ <u>Graphical Utility of even and odd-ness</u>

<u>Roughly</u> the sketch of $y = x^n$ goes as follows





graphs of $y = x^2, x^4, x^6, x^8, \ldots$ look like a bowl (symmetric with respect to y-axis)

graphs of $y = x^3, x^5, x^7$ look like the cubic graph. (Symmetric with respect to origin)