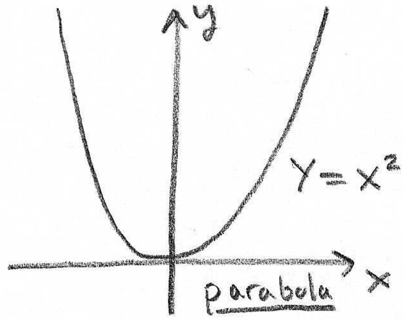
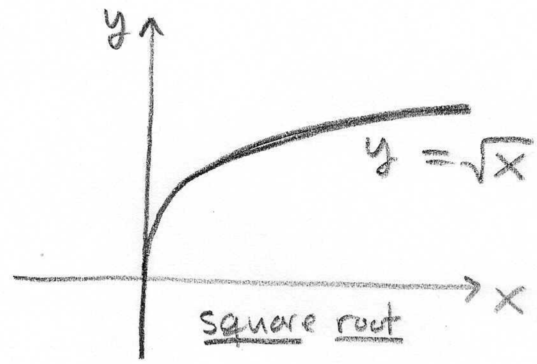
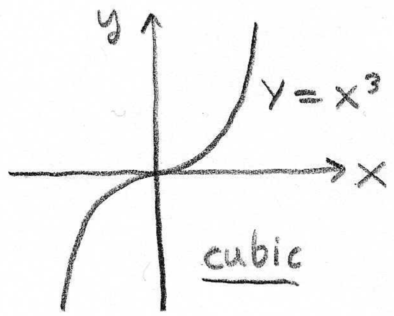
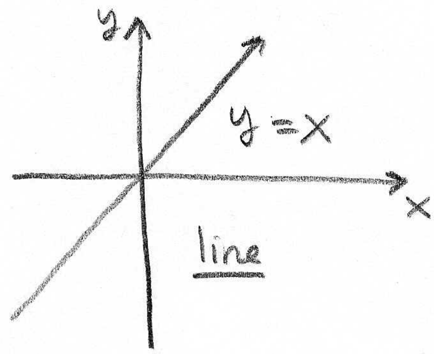
Standard "Parent" Functions

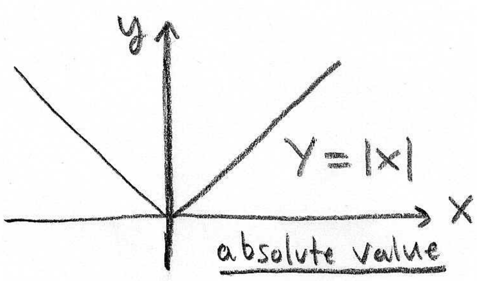
The examples below should be memorized. We use these to build other graphs via transformations.

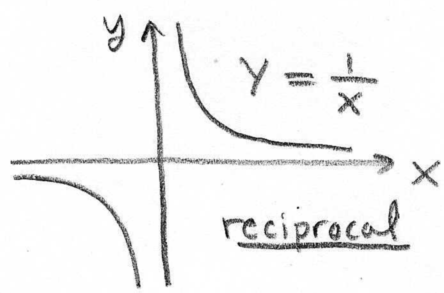












I hope you know these already, or at least sometime soon.

Strategy: take one of the Parent functions and transform it to make new graphs via

1.) Vertical Shift : $f(x) + c$
2.) Horizontal Shift : $f(x - c)$
3.) Vertical Stretch : $c f(x)$
4.) Horizontal Stretch : $f(x/c)$

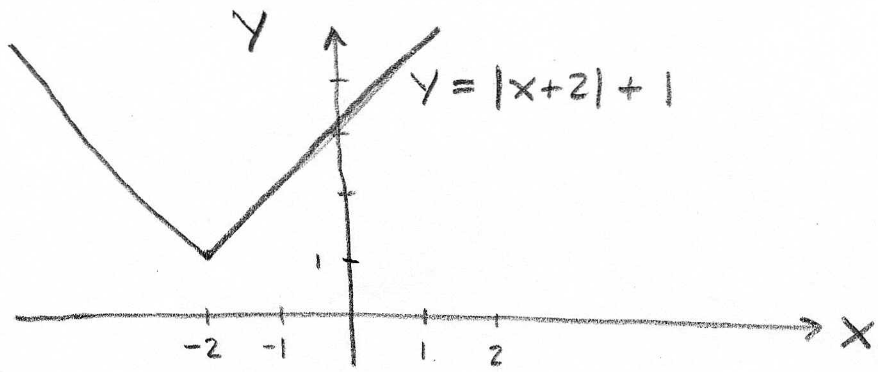
$\left.\begin{array}{l} \text{3.), 4.)} \end{array}\right\}$ $\underline{c > 1}$

5.) Reflection over x-axis : $-f(x)$
6.) Reflection over y-axis : $f(-x)$

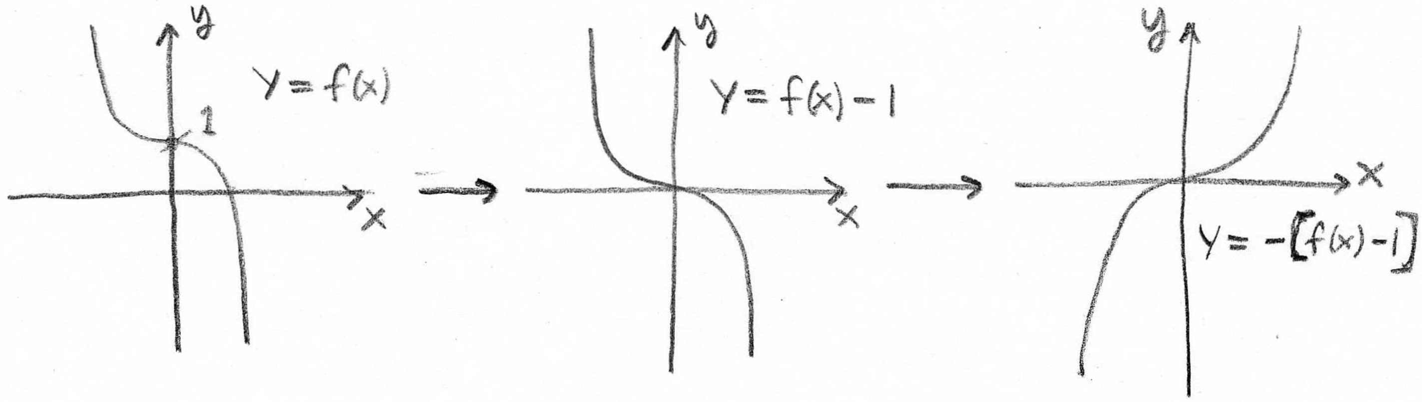
E121 Shift $y = |x|$ up one unit and two to the left.

$$f(x) = |x| \longmapsto f(x) = |x| + 1 \longmapsto \underline{f(x) = |x+2| + 1}$$

The graph of $Y = |x+2| + 1$ is as follows:



E122 Find eq^{n} of following graph using transformations



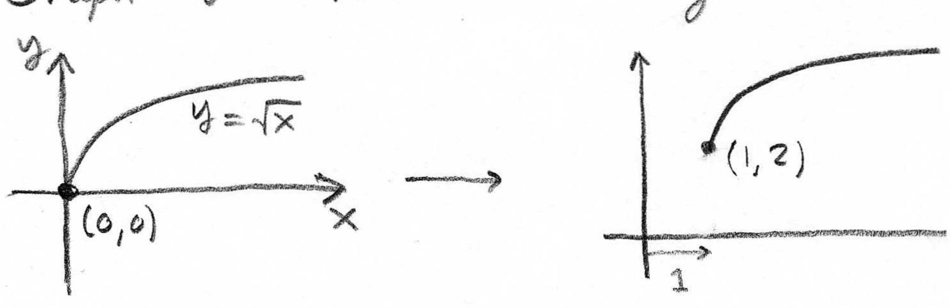
$\underbrace{\hphantom{xxxxxxxx}}_{\text{this is the cubic. } Y = x^3}$

We found $Y = x^3 = -[f(x) - 1]$

Thus the given graph is

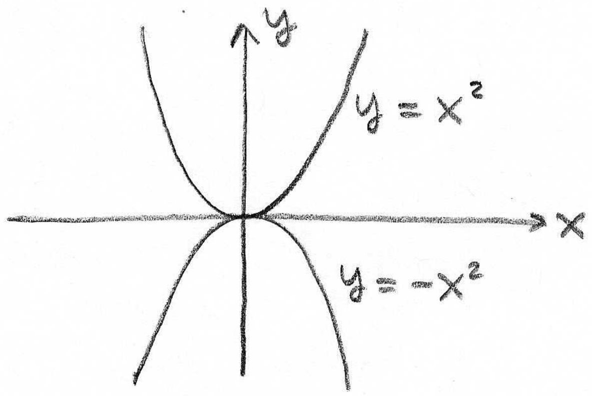
$$\boxed{f(x) = 1 - x^3 = Y}$$

E123 Graph $y = \sqrt{x-1} + 2$ using transformations on $y = \sqrt{x}$

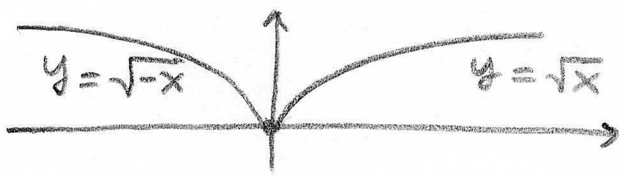


Remark: for complicated examples it helps to pick points to follow.

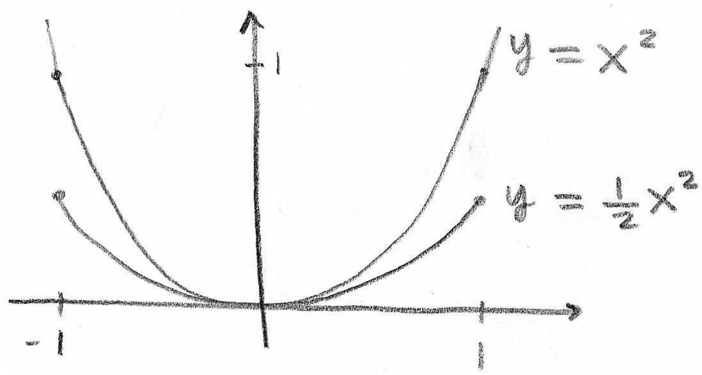
$\boxed{E124}$ Graph $y = x^2$ and $y = -x^2$



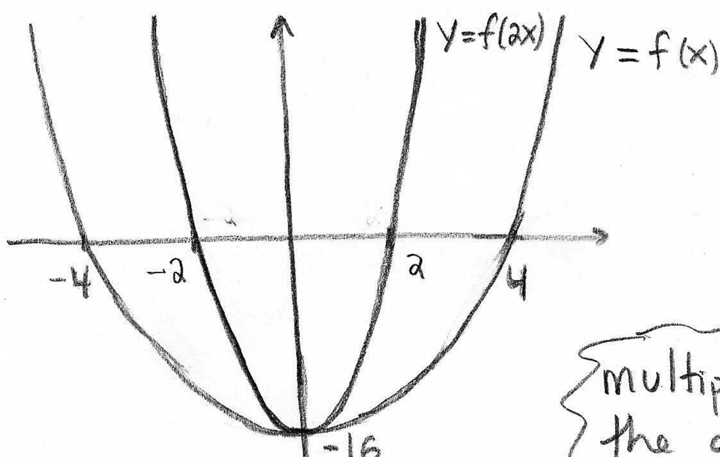
$\boxed{E125}$ Graph $y = \sqrt{x}$ and $y = \sqrt{-x}$



$\boxed{E126}$ Graph $y = x^2$ and $y = \frac{1}{2}x^2$



$\boxed{E127}$ Graph $y = (x+4)(x-4) = f(x)$ and $y = f(2x)$



$$\begin{aligned} f(2x) &= (2x+4)(2x-4) \\ &= 4(x+2)(x-2) \\ &= 4(x^2-4) \\ &= 4x^2-16 \end{aligned}$$

multiplying x by 2 compresses the graph by 50% horizontally.

Combinations of Functions : Composite Functions

Def^{n}/ Suppose $\text{dom}(f) \cap \text{dom}(g)$ is nonempty. Then we define new functions from $\text{dom}(f) \cap \text{dom}(g) \longrightarrow \mathbb{R}$ as follows :

1.) $f+g$ where $(f+g)(x) \equiv f(x)+g(x)$ (sum)
2.) $f-g$ where $(f-g)(x) \equiv f(x)-g(x)$ (difference)
3.) fg where $(fg)(x) \equiv f(x)\,g(x)$ (product)
4.) $\frac{f}{g}$ where $\left(\frac{f}{g}\right)(x) \equiv \frac{f(x)}{g(x)}$ (quotient)

In case 4.) $\text{dom}(f/g)$ excludes x such that $g(x)=0$.

$\boxed{E128}$ Let $f(x)=x^2$ and $g(x)=\sqrt{x-2}$ then,

$$(f+g)(x) = x^2+\sqrt{x-2}\ , \quad \text{dom}(f+g)=[2,\infty)$$
$$(f-g)(x) = x^2-\sqrt{x-2}\ , \quad \text{dom}(f-g)=[2,\infty)$$
$$(fg)(x) = x^2\sqrt{x-2}\ , \quad \text{dom}(fg)=[2,\infty)$$
$$\left(\frac{f}{g}\right)(x) = \frac{x^2}{\sqrt{x-2}}\ , \quad \text{dom}\left(\frac{f}{g}\right)=(2,\infty)$$

(have to exclude $x=2$) where $g(2)=0$.

Remark: most students have little trouble with the pointwise definitions given above. I bet you would have known how to do these w/o extra guidance. The composite of f with g is denoted $f\circ g$, this requires more care.

$$fg \neq f\circ g$$

NOTATIONAL ALERT!

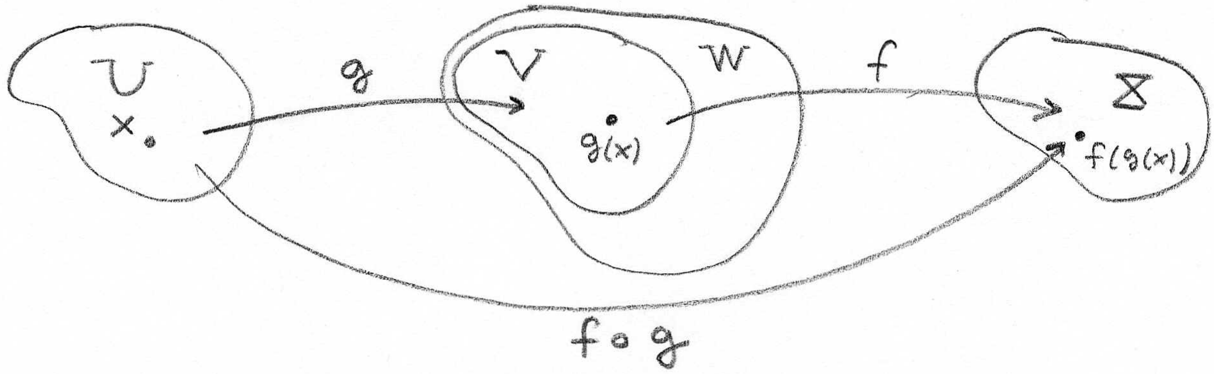
DANGER.

Composite Functions

Def$^{\underline{n}}$/ Let $g : U \longrightarrow V$ and $f : W \longrightarrow X$ such that $V \subseteq W$ then for each $x \in U$ we define

$$(f \circ g)(x) \equiv f(g(x))$$

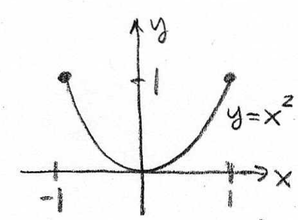
where $f \circ g : U \longrightarrow X$ is the composite of f with g.



Remark: we need range$(g) \subset$ dom(f). If the given function violates this we can just shrink the domain of g so that the new domain has a range which fits in dom(f).

$\boxed{E129}$ Let $f(x) = \sqrt{1-x}$ $\quad g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{1-x^2}$$



We cannot use $U = \text{dom}(g) = \mathbb{R}$ since range$(g) = [0, \infty)$ in that case. But, dom$(f) = (-\infty, 1]$. A moment's reflection reveals we should restrict dom(g) down to $[-1, 1]$ since $g([-1,1]) = \text{range}(g_{restricted}) = [0, 1]$

Hence, $\boxed{(f \circ g)(x) = \sqrt{1-x^2} \text{ with } \text{dom}(f \circ g) = [-1, 1]}$

In this case dom$(f \circ g) \subset$ dom(f) but this is not always the case, see $\boxed{E130}$

Remark: my definition is greedy we can't assume $V \subseteq W$ w/o work. this is the tricky part.

$\boxed{E130}$ Let $f(x) = \sqrt{x}$ and $g(x) = x^2$.
By default $\text{dom}(f) = [0,\infty)$ and $\text{dom}(g) = (-\infty,\infty)$.
It's easy to see $\text{range}(g) = [0,\infty)$ thus

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \\ &= \sqrt{x^2} \quad \text{for } x \in (-\infty,\infty).\end{aligned}$$

The $\text{dom}(f \circ g) = \mathbb{R}$ even though $\text{dom}(f) = [0,\infty)$.

Ok, enough about domains! My main desire is for you to learn how to calculate the formula for $(f \circ g)(x)$.

$\boxed{E131}$ Let $f(x) = x^2 + 3$, $g(x) = \frac{1}{x}$, $h(x) = \sqrt{1+x^3}$

$$(f \circ g)(x) = f(g(x)) = f\left(\tfrac{1}{x}\right) = \left(\tfrac{1}{x}\right)^2 + 3 = \frac{1}{x^2} + 3.$$

$$(g \circ f)(x) = g(f(x)) = g(x^2+3) = \frac{1}{x^2+3}.$$

$$(f \circ g \circ h)(x) = (f \circ g)(h(x)) = (f \circ g)(\sqrt{1+x^3}) = \frac{1}{(\sqrt{1+x^3})^2} + 3$$

$$\begin{aligned}(h \circ f)(x) &= h(f(x)) = h(x^2+3) \\ &= \sqrt{1 + (x^2+3)^3}.\end{aligned}$$

$$(f \circ f)(x) = f(f(x)) = f(x^2+3) = (x^2+3)^2 + 3 = x^4 + 6x^2 + 12.$$

Perhaps the following step helps.

$$(g \circ g)(x) = g(g(x)) = g(u) = \frac{1}{u} = \frac{1}{\frac{1}{x}} = x$$

let $u = g(x) = \frac{1}{x}$

The function $\mathcal{I}(x) = x$ is the identity function; $g \circ g = \mathcal{I}$.

E132 Let $f(x) = x^2$ and $g(x) = \sqrt{x} - 2$.

$$\begin{aligned}[f\circ(f+g)](x) &= f((f+g)(x)) \\ &= f(f(x)+g(x)) \\ &= f(x^2+\sqrt{x}-2) \\ &= u^2 \\ &= \underline{(x^2+\sqrt{x}-2)^2}.\end{aligned}$$

let $u = x^2 + \sqrt{x} - 2$

if this helps you then use this in-between step.

Inverse Functions (§ 2.7)

Defn/ Let f and g be two functions such that

1.) $f(g(x)) = x$ for each $x \in \mathrm{dom}(g)$

2.) $g(f(x)) = x$ for each $x \in \mathrm{dom}(f)$

then we write $g = f^{-1}$ and write the conditions above compactly as $f\circ f^{-1} = \mathrm{id}_{\mathrm{dom}(f^{-1})}$ and $f^{-1}\circ f = \mathrm{id}_{\mathrm{dom}(f)}$.

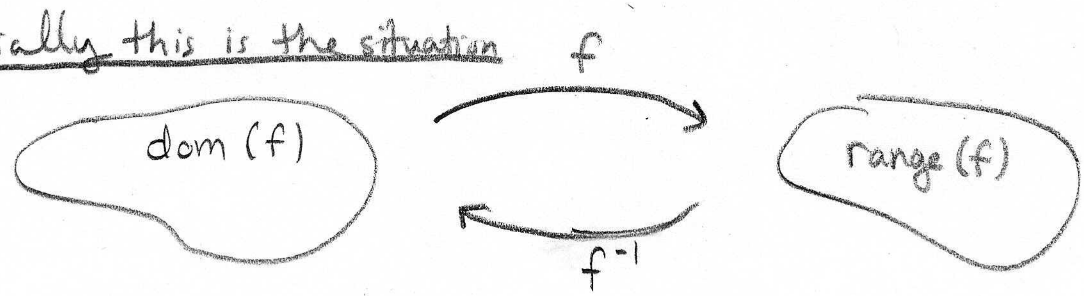
E133 Let $f(x) = x+3$ is $g(x) = x-3$ the inverse fnct?

$$(f\circ g)(x) = f(g(x)) = f(x-3) = x-3+3 = x$$

$$(g\circ f)(x) = g(f(x)) = g(x+3) = x+3-3 = x$$

Thus $g(x) = \boxed{f^{-1}(x) = x-3}$

Pictorially this is the situation



We need $\mathrm{dom}(f^{-1}) = \mathrm{range}(f)$

$\mathrm{range}(f^{-1}) = \mathrm{dom}(f)$

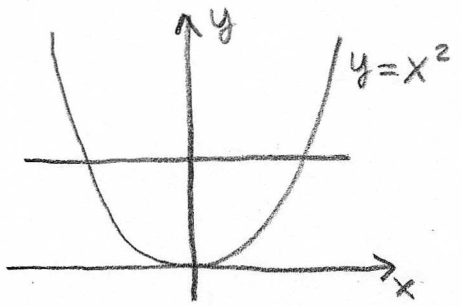
One - One Functions

Def^{n}/ f is one-one on U if $f(a) = f(b) \Rightarrow a = b$ for all pairs $a, b \in U \subset \mathrm{dom}(f)$. If f is one-one on $\mathrm{dom}(f)$ then f is one-one.

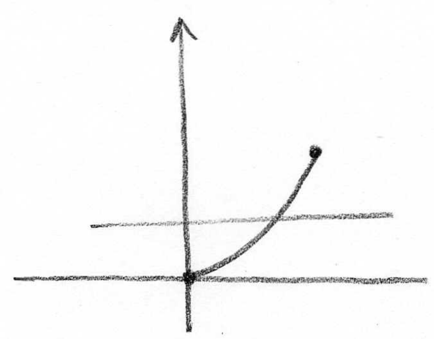
E134 Let $f(x) = x^2$.

Suppose $f(a) = f(b) \Rightarrow a^2 = b^2$
$\Rightarrow \underline{a = \pm b}$.

We can deduce $f(x)$ is <u>not</u> one to one on $\mathbb{R}$ since $f(-1) = f(1) = 1$. However $f(x)$ is one-one on $[0, 1]$ since $a, b \in [0,1] \Rightarrow a = b$ is only sol^{n}



fails horizontal line test $f(a) = f(b)$ yet $a \neq b$.



passes horizontal line test. Each x-value outputs <u>unique</u> y-value.

<u>Observation</u>: If a function is one-one then we can find an inverse function.

$$f(x) = y \Rightarrow \underbrace{f^{-1}(y) = x}$$

one-one gaurantees there is a unique x-value. This makes f^{-1} a <u>function</u>.

Observation Continued

I claimed if f is one-one we can <u>define</u> f^{-1} by

$$f(x) = y \implies f^{-1}(y) = x \quad \text{for each } y \in \text{range}(f).$$

Suppose I'm wrong. Suppose

$$f^{-1}(y) = x \quad \text{and} \quad f^{-1}(y) = x_2$$
$$f(x) = y \qquad\qquad f(x_2) = y$$

But, f is one-one thus $f(x) = f(x_2) \Rightarrow x = x_2$.
So my rule $f(x) = y \Rightarrow f^{-1}(y) = x$ must work for 1-1 fncts.

E135 Suppose f is one-one and $f(1) = 3$ what is $f^{-1}(3) = ?$ Use observation to see $\underline{f^{-1}(3) = 1}$.

E136 Let f be one-one. Let's sketch how to find the formula for f^{-1}.

$f(x) = \dfrac{3+x}{7}$: write given function

$y = \dfrac{3+x}{7}$: replace $f(x)$ with y

$x = \dfrac{3+y}{7}$: switch x and y

this reflects across $y = x$

$7x = 3 + y$

$y = 7x - 3$: solved for y

$\boxed{f^{-1}(x) = 7x - 3}$: change y to $f^{-1}(x)$.

<u>Remark</u>: this algorithm essentially says take the graph of f, flip it over $y = x$ then find the eq$^{\underline{n}}$ of the reflected graph, its the graph of f^{-1}.

E137 Let $f(x) = \sqrt[3]{x}$

$$y = \sqrt[3]{x}$$

$$x = \sqrt[3]{y} \quad \text{switch } x \,\&\, y \text{ towards finding } f^{-1}(x)$$

$$x^3 = y$$

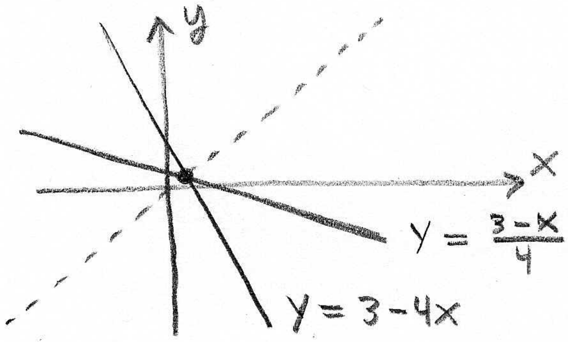
$$\boxed{f^{-1}(x) = x^3}$$

E.138 Graph $y = f(x)$ and $y = g(x)$, check if these are inverses. Let $f(x) = 3 - 4x$ and $g(x) = \dfrac{3-x}{4}$

Notice $f(g(x)) = f\left(\frac{3-x}{4}\right) = 3 - 4\left(\frac{3-x}{4}\right) = 3 - 3 + x = x.$

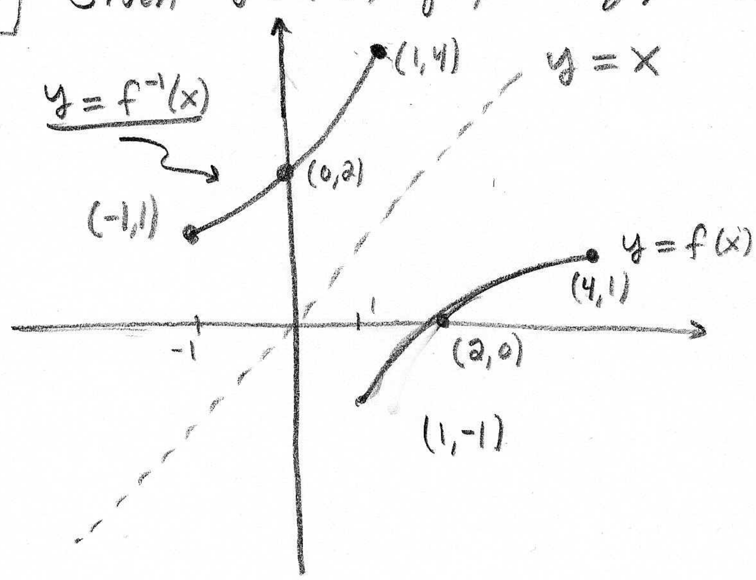
$g(f(x)) = g(3-4x) = \dfrac{3-(3-4x)}{4} = \dfrac{4x}{4} = x.$

Thus $g = f^{-1}$. What does this mean graphically?



$y = f(x)$ and $y = g(x)$ are reflections of each other across the line $y = x$.

E139 Given $y = f(x)$ graphically, construct graph of $\underbrace{y = f^{-1}(x)}$.



such points have

$f(y) = x$

We have

$f(1) = -1$

$f(2) = 0$

$f(4) = 1$

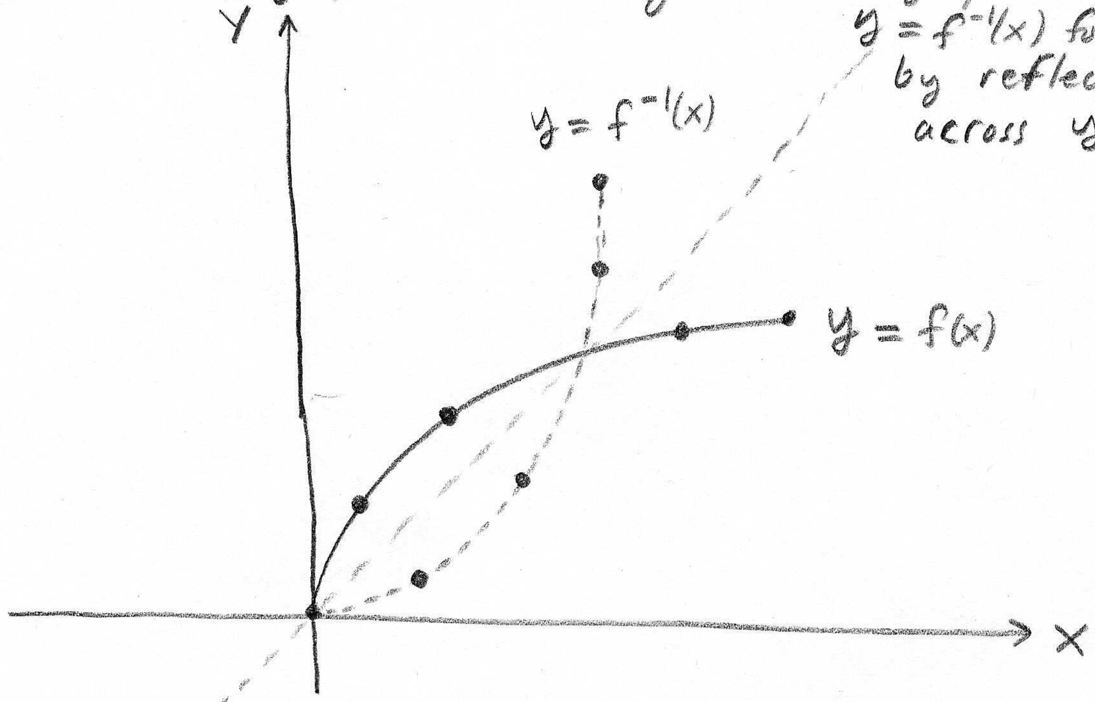
These tell us

$f^{-1}(-1) = 1$

$f^{-1}(0) = 2$

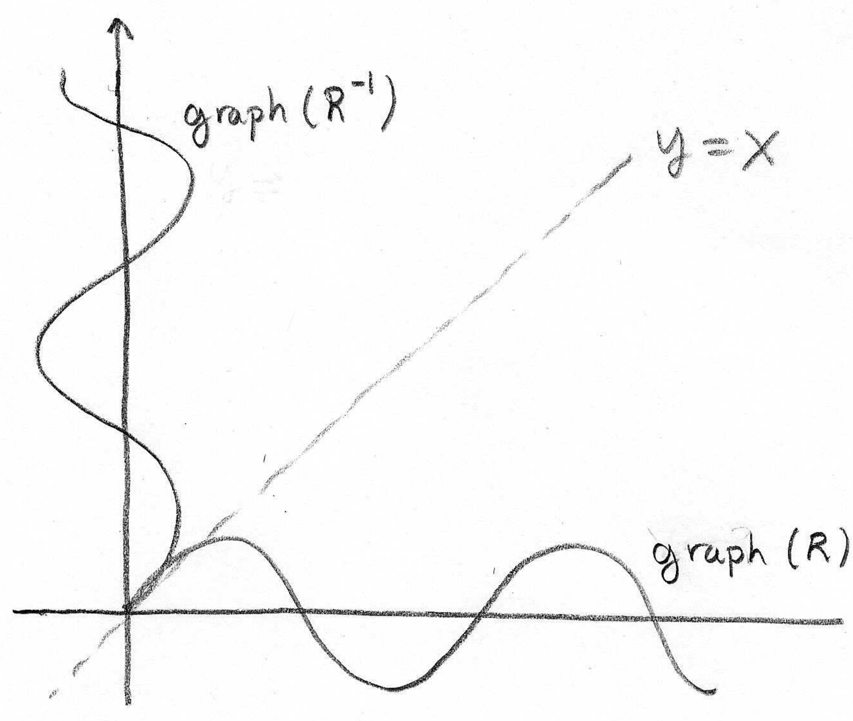
$f^{-1}(1) = 4$

$\boxed{E140}$ The solid graph $y = f(x)$ is given. The graph $y = f^{-1}(x)$ follows by reflection across $y = x$.



Remark: the inverse function has the same graph as $Y = f(x)$ it's just run upwards instead of to the right.

How a Relation R and it's Inverse R^{-1} are related. $Y = f(x)$ and $Y = f^{-1}(x)$ are a very special case. (math 200)



(Relations are not troubled by the vertical line test)