Quadratic Functions and Models (§3.1)

We have discussed quadratic and polynomial eq^{n}'s. Now we discuss quadratic and polynomial functions.

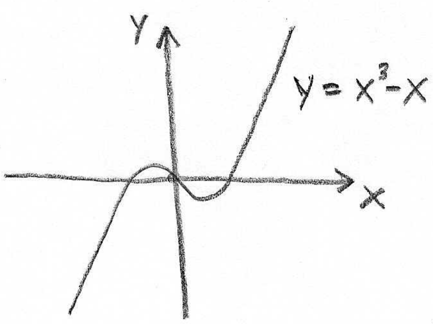
$$\boxed{\begin{array}{ll} P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0 : & \text{Polynomial Function} \\ \hline f(x) = ax^2 + bx + c,\ a \neq 0 : & \text{Quadratic Function} \\ g(x) = ax^3 + bx^2 + cx + d,\ a \neq 0 : & \text{Cubic Function} \\ h(x) = mx + b,\ m \neq 0 : & \text{Linear or Affine Function} \end{array}}$$

Now f, g, h are all polynomial functions of a special type. The leading coefficient $a_n \neq 0$ indicates what the graph looks like for large $\pm$ values of x.

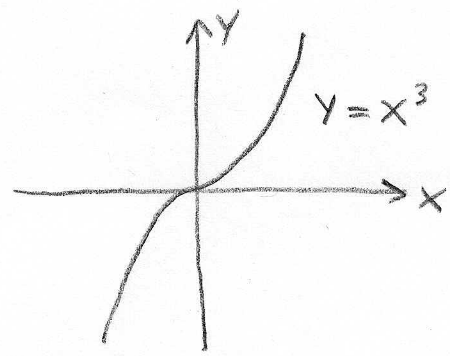
Big Picture

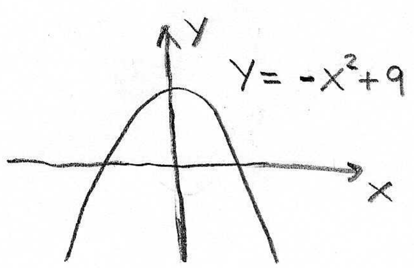
$Y = P(x) = a_n x^n + \cdots + a_1 x + a_0$ same longterm shape as $Y = a_n x^n$

$\boxed{\text{E141}}$ The notation $|x| \gg 1$ means $|x|$ is very large.

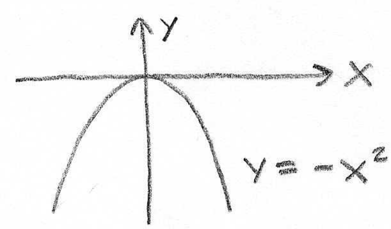


is roughly same as $Y = x^3$ for $|x| \gg 1$





is roughly same as $Y = -x^2$ for $|x| \gg 1$

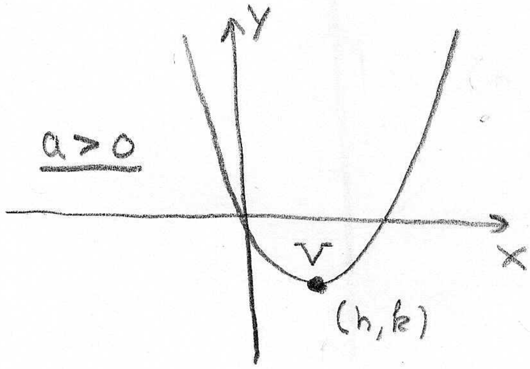


Remark: The leading coefficient gives us the BIG PICTURE but, details matter. We'll need to factor to figure out the details.

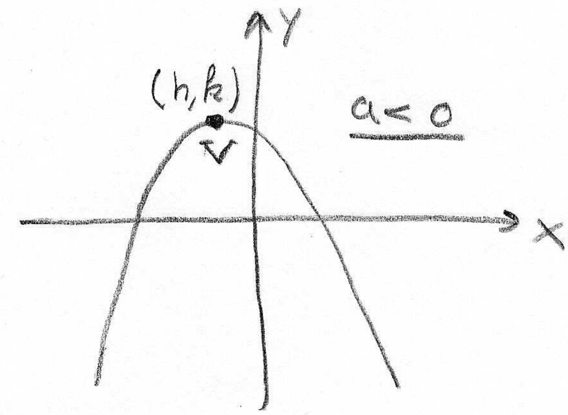
EVERYTHING QUADRATIC

Let's collect all the trivia concerning quadratic polynomials.

FACTS: A quadratic function $f(x) = ax^2 + bx + c$ has $a \neq 0$. The vertex of the parabola $y = f(x)$ is at $(h, k) = V$ where $f(x) = a(x-h)^2 + k$ and graphically



OR



As you can see the vertex V is either gives

$$\boxed{\begin{array}{ll} \text{1.)} & \text{Minimum for } y = f(x) \text{ in the case } a > 0 \\ \text{2.)} & \text{Maximum for } y = f(x) \text{ in the case } a < 0 \end{array}}$$

The min/max value will be $y = k$ the y-coordinate of the vertex V.

QUESTION: Given $f(x) = ax^2 + bx + c$ how can we find the vertex (h, k) where $f(x) = a(x-h)^2 + k$?

ANSWER: I'll do it the hard way then you'll get formulas.

$$f(x) = a(x-h)^2 + k = a(x^2 - 2xh + h^2) + k$$
$$= ax^2 - 2ahx + ah^2 + k$$

But this has to match $f(x) = ax^2 + bx + c$ thus comparing coefficients of x^2, x and x^0 yields

x^2) $a = a$

x) $-2ah = b \longrightarrow h = -b/2a$

x^0) $ah^2 + k = c \longrightarrow k = c - ah^2$

$$= c - a\left(\frac{b^2}{4a^2}\right)$$

$$\boxed{\begin{array}{c} h = -b/2a \\ \hline k = c - \dfrac{b^2}{4a} \end{array}}$$

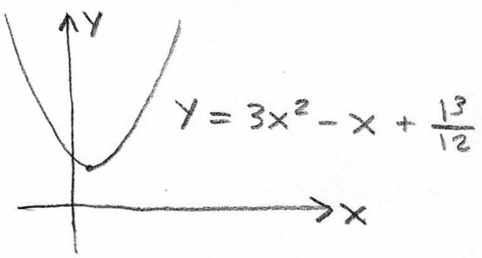
E142 Find vertex of $Y = 3x^2 - x + \frac{13}{12}$. What is the minimum value of Y on the graph?
Using my derivation on 61 we found

$$h = \frac{-b}{2a} = \frac{-(-1)}{2(3)} = \frac{1}{6}$$

$$k = c - \frac{b^2}{4a} = \frac{13}{12} - \frac{1}{12} = \frac{12}{12} = 1$$

The vertex is $\boxed{(1/6, 1) \text{ and } y_{min} = 1}$

We can sketch the parabola



You can see there are no x-intercepts. This means that the quadratic eqⁿ $3x^2 - x + \frac{13}{12} = 0$ has no real solⁿ's. (otherwise our graph would cross the x-axis)

CONNECTION BETWEEN QUADRATIC EQUATIONS AND FUNCTIONS

The x-intercepts of $Y = f(x) = ax^2 + bx + c$ are the Solⁿ's to the quadratic equation $ax^2 + bx + c = 0$.

<u>Remark</u>: the objects $ax^2 + bx + c = 0$ and $f(x) = ax^2 + bx + c$ are very different types. We have an equation in one variable verses a function of one variable. We only set $f(x) = ax^2 + bx + C = 0$ when we wish to find <u>particular</u> x-values, namely those where $Y = f(x) = 0$, the "x-intercepts".

(there can be two, one or no x-intercepts for a parabola)

Roots, Zeroes, x-intercepts, for $y = f(x) = ax^2+bx+c$ intersects x-axis

The x-intercepts have $y=0$ thus $ax^2+bx+c=0$. We find sol^n's via factoring or the quadratic formula,

$$\boxed{r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}}$$

These give $f(r_1) = f(r_2) = 0$. (We consider the case $b^2-4ac \geq 0$ in this discussion). We call r_1 and r_2 the <u>roots</u>. It can be shown,

$$\boxed{f(x) = a(x-r_1)(x-r_2)}$$

Notice that we can find a few extra formulas to check our work. Remember $f(x) = ax^2+bx+c$ <u>and</u> $f(x) = a(x-r_1)(x-r_2) = ax^2 - a(r_1+r_2)x + ar_1r_2$. We can compare coefficients of x and x^0 to find,

$$b = -a(r_1+r_2) \Rightarrow \boxed{r_1+r_2 = -b/a}$$
$$c = ar_1r_2 \Rightarrow \boxed{r_1r_2 = c/a}$$

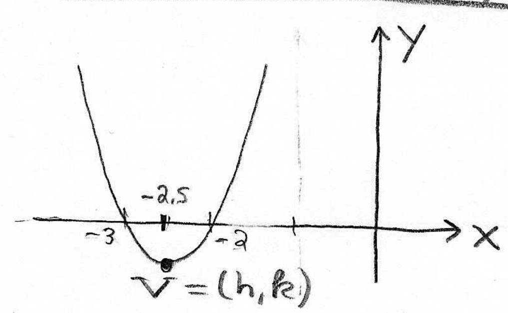
I don't use these much, but they provide good check on other work

E143 Find x-intercepts and vertex for $y = x^2+5x+6$

We see $y = (x+3)(x+2)$ thus $r_1 = -3$ and $r_2 = -2$.

Recall $h = \frac{-b}{2a} = \frac{-5}{2}$ and $k = c - \frac{b^2}{4a} = 6 - \frac{25}{4} = \frac{-1}{4}$

The vertex is at $(-5/2, -1/4) = V$



Check work:

$r_1 + r_2 = -3-2 = -5 = \frac{-b}{2a}$ ✓

$r_1r_2 = (-3)(-2) = 6 = \frac{c}{a}$ ✓

<u>Remark</u>: The formulas $r_1 + r_2 = -b/a$ and $r_1 r_2 = c/a$ still work when $b^2 - 4ac < 0$. The fact that $(r_1)^* = r_2$ causes the sum and product of the roots to be real numbers.

Def^n/ Let $y = ax^2 + bx + c = a(x-h)^2 + k$ where $a \neq 0$ and $a, b, c, h, k \in \mathbb{R}$. Then the <u>axis of symmetry</u> is $x = h$.

$\boxed{E144}$ Let $y = -x^2 + 12x - 11$. Find the vertex, x-intercepts and axis of symmetry. Also find the y-intercept and sketch its graph.

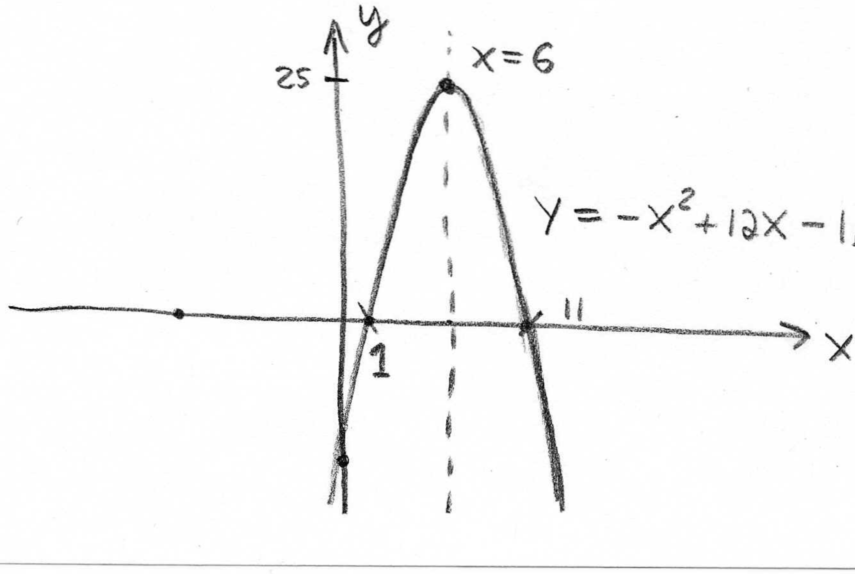
$$y = -(x^2 - 12x + 11) = -(x-1)(x-11) = 0 \Rightarrow \underbrace{x = 1 \text{ or } x = 11}_{x\text{-intercepts}}$$

The vertex (h, k) has

$$h = \frac{-b}{2a} = \frac{-12}{-2} = 6$$

$$k = c - \frac{b^2}{4a} = -11 + \frac{144}{4} = -11 + 36 = 25$$

Thus $(h, k) = (6, 25)$ and $\underline{x = 6 \text{ is the axis of symmetry}}$.
The y-intercept has $x = 0$ it is $\underline{-11 = y}$.



E145 Suppose $f(x) = (x+5)^2 + 6$. Graph the parabola $y = f(x)$ lable the features such as x, y-intercepts, axis of symmety and vertex. This is nice

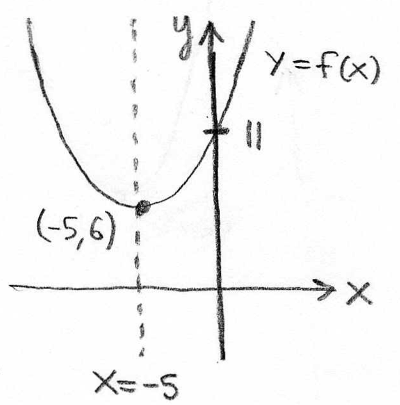
$$f(x) = (x+5)^2 + 6 \quad \text{compared with} \quad a(x-h)^2 + k = y$$

Reveals $\underline{h = -5 \text{ and } k = 6}$. We have many ways to find the x-intercepts.

$$0 = (x+5)^2 + 6 \Rightarrow (x+5)^2 = -6$$

$\Rightarrow$ no real sol^n's !
(cannot have $a^2 < 0$ for $a \in \mathbb{R}$)
$\underline{\text{no such } x \in \mathbb{R}.}$

Clearly $a = 1$ so the parabola opens up.
$\underline{\text{y-intercept}}$? That's where $\underline{x = 0}$ thus $f(0) = (0+5)^2 + 6 = 31$
Now we can sketch using all this information,



E146 Write the formula for the function $f(x)$ whose graph $y = f(x)$ is a parabola with vertex $(1,1)$ and y-intercept 2. The "$\underline{\text{standard form}}$" of a quadratic is $y = a(x-h)^2 + k$ thus,

$$y = a(x-1)^2 + 1 \qquad \underline{\text{But, what is a ?}}$$

We also were given $f(0) = 2$ thus

$$2 = a(0-1)^2 + 1 = a + 1 \Rightarrow \underline{a = 1} \quad \therefore \boxed{f(x) = (x-1)^2 + 1}$$

E147 Find quadratic function f with zeroes $x=-4$ and $x=0$ such that the vertex of $y=f(x)$ is at $(-2,2)$.

- We were given $f(-4)=0$ and $f(0)=0$ thus

$$f(x) = a(x+4)(x-0) \Rightarrow f(x) = ax(x+4)$$

- We also know the vertex $(h,k)=(-2,2)$. How can we use this information? Let's rewrite $f(x)$

$$\begin{aligned} f(x) &= ax(x+4) \\ &= ax^2+4ax \\ &= a(x^2+4x) \\ &= a((x+2)^2-4) \quad \text{completed square.} \\ &= a(x+2)^2-4a \end{aligned}$$

Now compare with $f(x)=a(x+2)^2+2$ to see we must have $-4a=2 \therefore \underline{a=-1/2}$.

$$\boxed{f(x) = \frac{-1}{2}(x+2)^2+2}$$

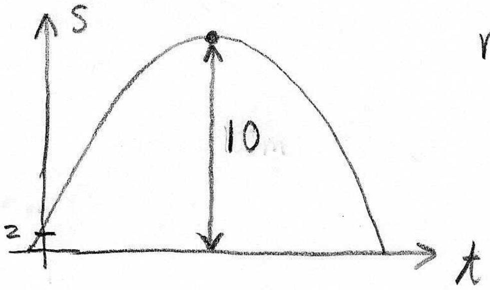
Geometric Definition of Parabola: I defined things for us by equations because its convenient. It should be mentioned that a parabola has a purely geometric description as well. A parabola is the collection of all points equidistant from a line called the directerix and a point (not on the directerix) called the focus. The directerix for E142 - E147 etc... will be a horizontal line. In general the parabola could be tilted, but the equations get more complicated.

E148 It can be shown that if the acceleration a is constant,

$$\boxed{s = s_o + v_o t + \frac{1}{2} a t^2}$$

is the position s of an object launched with velocity v_o at position s_o when $t = 0$.

Application: if a ball is to reach a height of 10m? then how fast must it be thrown to reach that height? We know $g = 9.8\,m/s^2$ and $a = -9.8\,\frac{m}{s^2}$. Assume the person throws the ball from $s_o = 2m$.



max. height occurs at the vertex of the parabola in the ts-plane

The graph above is $s = s_o + v_o t + \frac{1}{2}at^2$ but we know $s_o = 2$ and $a = -9.8$ thus,

$$s = 2 + v_o t - 4.9t^2$$

$$\Rightarrow \quad a = -4.9, \quad b = v_o, \quad c = 2$$

$(s = at^2 + bt + c$ is parabola$)$

We wish to find v_o. Notice since the vertex has $k = 10$ we can figure out v_o from

$$k = c - \frac{b^2}{4a} \Rightarrow 10 = 2 - \frac{v_o^2}{4(-4.9)}$$

$$8 = \frac{v_o^2}{2(9.8)}$$

$$v_o^2 = 8(2)(9.8) = 16(9.8)$$

$$v_o = \pm\sqrt{16(9.8)}$$

choose (+) for physical reasons.

Thus $\boxed{v_o \approx 13\,\frac{m}{s}}$