A polynomial of degree $n = 1, 2, 3, \ldots$ has the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where $a_n \neq 0$ and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$. We call a_0 the constant term, $y = a_0$ gives the y-intercept of $y = P(x)$. When $a_n > 0$ the graph of $y = P(x)$ looks like the graph of the power function $y = x^n$. When $a_n < 0$ then the graph looks like $y = x^n$ reflected over the x-axis. (see $\boxed{E141}$ on 60)

- We have throughly studied the cases $n = 1$ and $n = 2$

$P(x) = mx + b$ gives $y = P(x)$ a <u>line</u>

$q(x) = ax^2 + bx + c$ gives $y = q(x)$ a <u>parabola</u>

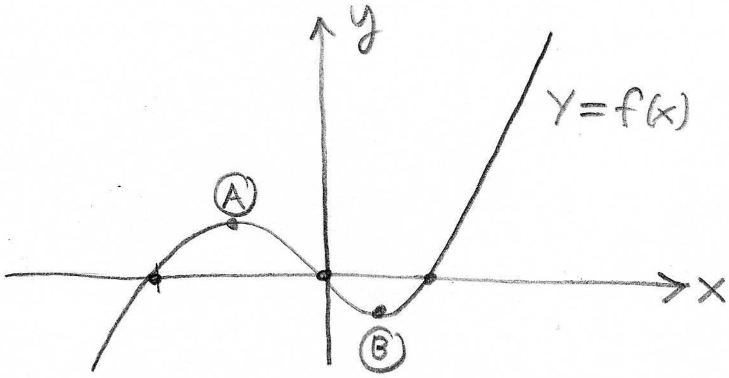
Many nice formulas were given for the cases $n = 1$ and 2. We shift gears a bit now. For $n \geq 3$ we only care about <u>two</u> <u>main things</u>

1.) Is $a_n > 0$ or $a_n < 0$ (opens up or down)

2.) What are the zeros of $P(x)$?
That is what $r_1, r_2, \ldots, r_n$ give
$P(r_1) = 0, P(r_2) = 0, \ldots, P(r_n) = 0.$

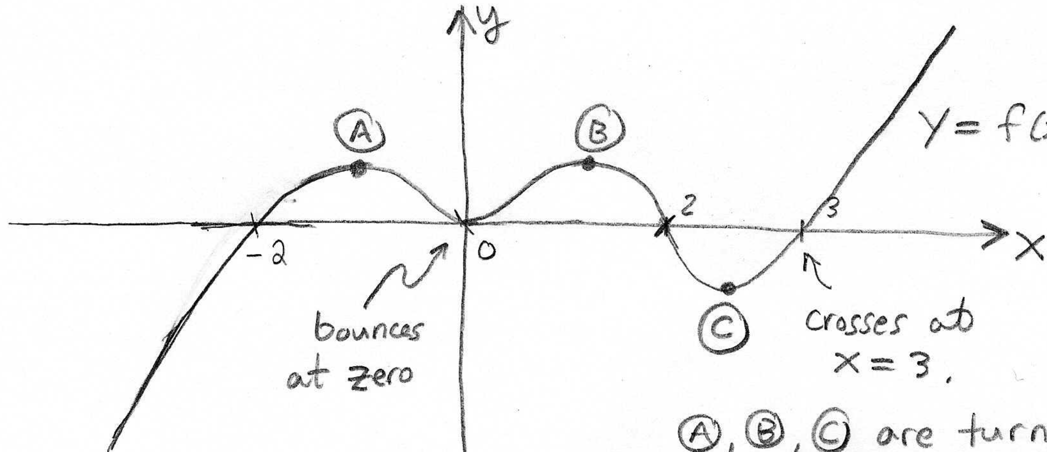
Given these two pieces of information we can then piece to gether the graph's essential shape with a sign chart or a few simple guidelines.

E149 Graph $y = f(x)$ for $f(x) = x(x-1)(x+2)$.
Notice we have $n = 3$ and zeroes at $x = 0, 1, -2$ thus,



- Polynomials are <u>continuous</u>. This means we can graph them w/o lifting our pencil from the page.
- The points I labled Ⓐ & Ⓑ are called <u>turning points</u>. If you think about it a line has no turning point and a parabola has just <u>one</u> turning point, the vertex.

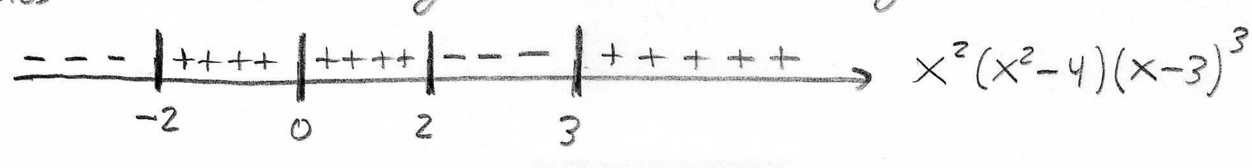
E150 Graph $f(x) = x^2(x^2-4)(x-3)^3$ this is an $n = 2+2+3 = 7^{th}$ order polynomial with $a > 0$ thus for $|x| >> 1$ it looks like $y = x^7$. Notice that $f(x) = x^2(x+4)(x-4)(x-3)^3 \Rightarrow x = 0, -4, 4, 3$ are roots.



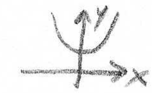
<u>Ⓐ, Ⓑ, Ⓒ are turning points.</u>

I first plot the zeros then I draw a curve that matches $y = x^7$ long term and the sign chart below

(takes thinking) →

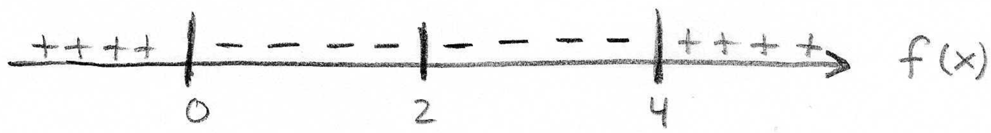


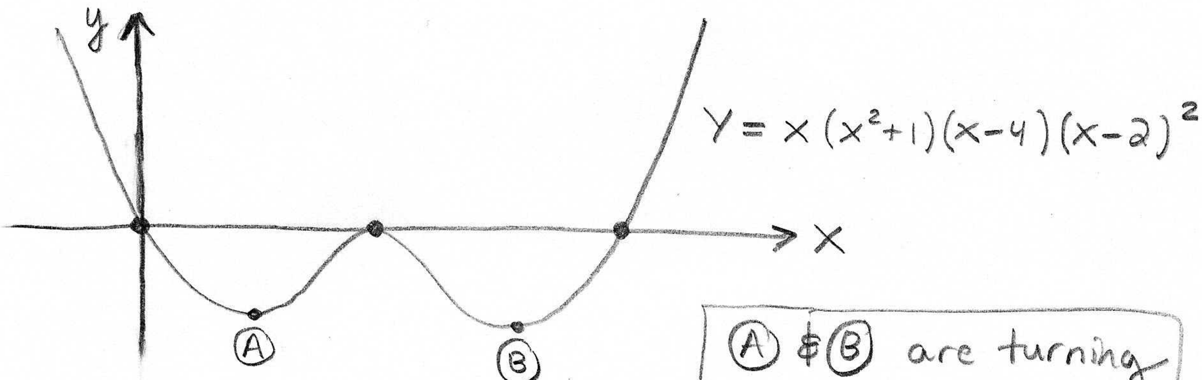
E151 Graph $f(x) = x(x^2+1)(x-4)(x-2)^2$. This is already factored over $\mathbb{R}$ as much as possible. The (x^2+1) factor cannot be reduced further, notice $y = x^2+1$ has no x-intercepts.



$y = x^2+1$

This is an $n = 6$ order polynomial with $a_6 > 0$.





$\boxed{\text{(A) \& (B) are turning points}}$

<u>Remark</u>: irreducible quadratic factors introduce wiggles in the graph. The sketch above has not included those features. With calculus we could say more without resorting to a graphing calculator.

General Idea to Plot $y = P(x) = a_n x^n + \cdots + a_1 x + a_0$

1.) Check if $a_n > 0$ or $a < 0$. If $a_n > 0$ then keep in mind $y = P(x) \approx x^n$ for $|x| >> 1$ otherwise if $a_n < 0$ need to reflect over x-axis.

2.) Find factorization of $P(x)$. Make sign chart and include all zeros revealed by the factorization. Then graph

- for $(x-a)^2$ or $(x-a)^4$ etc... it bounces at $x=a$.
- for $(x-a)$ or $(x-a)^3$ etc... it crosses at $x=a$.

Then connect the dots.

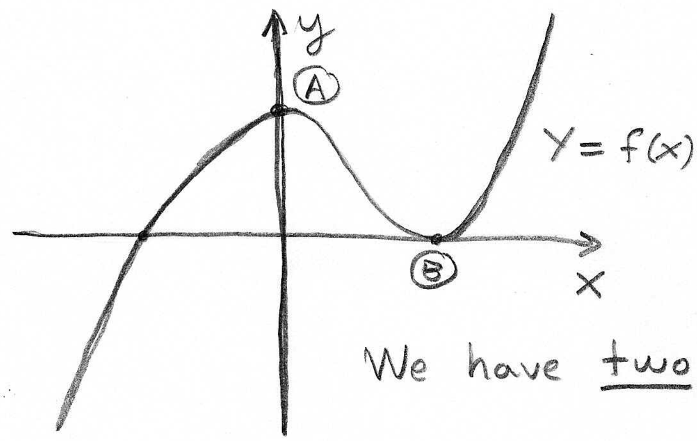
Observation: My examples up to this point were almost factored to begin with. In general we'll need to factor $P(x)$ before we can graph.

$\boxed{\text{E152}}$ Graph $f(x) = 1 - x + x^3 - x^2$. Tilt head and squint ... factor by grouping!

$$
\begin{aligned}
f(x) &= -(x-1) + x^2(x-1) \\
&= (x^2-1)(x-1) \\
&= (x+1)(x-1)(x-1) \\
&= (x+1)(x-1)^2
\end{aligned}
$$

Zeroes at $\underbrace{x=-1}_{\text{will cross (just one factor.)}}$ and $\underbrace{x=1}_{\text{repeated will bounce}}$

Looks like $y = x^3$ for $|x| \gg 1$.



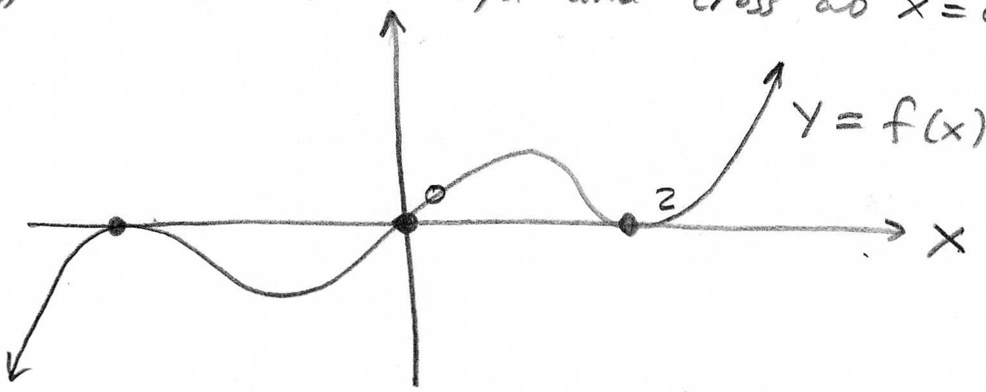
We have two turning points Ⓐ & Ⓑ.

Remark: We can see a pattern. For an n^{th} order polynomial graph we can have at most $n-1$ turning points. These are interesting points because they give local min/max for $y = P(x)$. In math 131 or 126 we learn how to find them via calculus.

E153 Graph $f(x) = x[x^2 + x - 6]^2$. Notice that $x^2 + x - 6 = (x+3)(x-2)$ thus,

$$f(x) = x[(x+3)(x-2)]^2$$
$$= \underline{x(x+3)^2(x-2)^2}, \quad \underline{\text{zeros at } x = 0, -3, 2}$$

This looks like $y = x^5$ for $|x| >> 1$. Thus, noting it'll bounce at $x = -3, 2$ and cross at $x = 0$ we graph,



Notice, I can skip the sign chart if I want, but the sign chart would give a good redundancy to the calculation

E154 Build a polynomial with zeros at $x = -2, -1, 0$ and 7 such that the graph $y = P(x)$ bounces at $x = -2$ and 0 and crosses at $x = -1$ and 7. Also we want a leading coefficient of 13.

$$\boxed{P(x) = 13(x+2)^2(x+1)x^2(x-7)}$$

E155 Given $f(x)$ is a polynomial such that $f(2) = 6$ construct a new polynomial $g(x)$ such that g has a zero at $x = 2$.

$$g(x) = f(x) - 6 \quad \text{(think graphically)}$$

This has $g(2) = f(2) - 6 = 6 - 6 = 0$.

$\boxed{E156}$ Find a polynomial with zeroes at $x = -3, -1, 2$ such that the polynomial's graph crosses the x-axis at -1 and 2 and bounces off the x-axis at -3. In addition, we wish the polynomial has y-intercept 36.

The smallest degree polynomial that works here is

$$f(x) = a(x+1)(x-2)(x+3)^2$$

We want $f(0) = 36$ thus

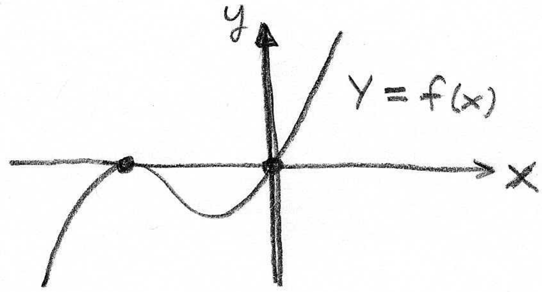
$$36 = a(0+1)(0-2)(0+3)^2 = -18a \Rightarrow \underline{a = -2}$$

$$\therefore \boxed{f(x) = -2(x+1)(x-2)(x+3)^2}$$

$\boxed{E157}$ Factor $f(x) = x^3 + 8x^2 + 16x$ and graph $y = f(x)$.

Notice
$$\begin{aligned} f(x) &= x^3 + 8x^2 + 16x \\ &= x(x^2 + 8x + 16) \\ &= x(x+4)^2 \end{aligned}$$
, we have $f(x) = 0$ for $x = 0$ and $x = -4$ (twice)

("multiplicity two")



$\boxed{E158}$ How many intersection points can there be between a parabola and a cubic?

Let's think about this,

$$f(x) = ax^3 + bx^2 + cx + d$$
$$g(x) = a_2x^2 + a_1x + a_0$$

Intersection points have $f(x) = g(x) \Rightarrow f(x) - g(x) = 0$

the sol^n's are zeroes of the cubic; $ax^3 + bx^2 + cx + d - a_2x^2 - a_1x - a_0 = 0$

A cubic eq^n has at most 3 real solutions. $\boxed{\text{Thus three is the most}}$