THEORY OF POLYNOMIALS

Up to this point I have been doing the examples with common sense and a few observations about general patterns. We can summarize many of these ideas for future reference and better understanding of the last bunch of examples in retrospect.

Theorem (Real Zeros of Polynomial)

If $f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ is a real polynomial then the following are equivalent

1.) $x = a$ is a zero of the function; $f(a) = 0$.

2.) $x = a$ is a sol^n to the eq^n $f(x) = 0$.

3.) $(x - a)$ is a factor of the polynomial $f(x)$.

4.) $(a, 0)$ is an x-intercept of the graph $y = f(x)$.

We have used all of these in our examples.

Corollary: An n^{th} order polynomial with real coefficients can have at most n-distinct zeros

Proof: (By contradiction) Suppose $\deg(f) = n$ but we have $n+1$ zeroes $r_1, r_2, \ldots, r_n, r_{n+1}$. Then by 3.) of the Theorem we get a factor for each distinct sol^n:

$$f(x) = a\underbrace{(x-r_1)(x-r_2)\cdots(x-r_n)(x-r_{n+1})}_{\text{But this has degree } n+1}$$

this contradicts $\deg(f) = n$.

Therefore, we conclude the Cor. is true. An n^{th} order polynomial has at most n-real roots. //

Now while we can have n-real zeroes for an n^{th} order polynomial, there is no gaurantee that happens. We can have complex roots or repeated real roots.

REPEATED ZEROS: (assume $k \in \mathbb{N}$)
A factor $(x-a)^k$, $k > 1$ corresponds to a repeated zero of multiplicity k. We have observed in many examples
1.) If k is odd the graph crosses at $(a,0)$
2.) If k is even the graph bounces at $(a,0)$.

What about complex roots? Well,

Th$^{\underline{m}}$ (Fundamental Th$^{\underline{m}}$ of Algebra)
Every polynomial of degree n with real (or complex) coefficients can be written as a product of n-linear factors over the complex number system

$$f(x) = a(x-r_1)(x-r_2)\cdots(x-r_{n-1})(x-r_n)$$

here $r_1, r_2, \dots, r_{n-1}, r_n \in \mathbb{C}$ in general

This is equivalent to the seemingly weaker Th$^{\underline{m}}$ on pg. 298.

Observation: If a polynomial has real coefficients then complex roots must come in conjugate pairs
$r = a+ib$ has conjugate $r^* = a - ib$ $\quad (a,b \in \mathbb{R},\ a+ib \in \mathbb{C})$

$$\begin{aligned}(x-r)(x-r^*) &= (x-a-ib)(x-a+ib) \\ &= x^2 - ax + \cancel{ibx} - ax + a^2 - \cancel{iab} - \cancel{ibx} + \cancel{iab} - i^2 b \\ &= \underbrace{x^2 - 2ax + a^2 + b^2}_{\text{real coefficients.}}\end{aligned}$$

If you think about this some more you can see that $r \in \mathbb{C}$ must have $r^* \in \mathbb{C}$ somewhere else in the factorization because otherwise we'd be able to multiply out the factorization and find a complex coefficient in the factorization.

PROPOSITION: Let $P(x)$ be a polynomial with real coefficients, if $r \in \mathbb{C}$ and $P(r) = 0$ then $P(r^*) = 0$. Complex roots must come in conjugate pairs. If we have a factor $(x-(a+ib))$ then there must also be a conjugate factor $(x-(a-ib))$ in the factorization of $P(x)$.

$\boxed{E159}$ Find polynomial with real coefficients such that $P(1) = 0$ and $P(1+i) = 0$.

$$
\begin{aligned}
P(x) &= (x-1)(x-(1+i))\overbrace{(x-(1-i))}^{\text{the conjugate of the given root.}} \\
&= (x-1)(x-1-i)(x-1+i) \\
&= (x-1)(x^2 - x + \cancel{ix} - x + 1 - \cancel{i} - \cancel{ix} + \cancel{i} - i^2) \\
&= (x-1)(x^2 - 2x + 2) \\
&= \underbrace{(x-1)}_{\text{linear factor}}\underbrace{[(x-1)^2 + 1]}_{\text{irreducible quadratic factor}}
\end{aligned}
$$

Remark: when you complete the square on a quadratic if the constant term is negative it factors, if it's positive then the quadratic is irreducible. Compare $\boxed{E159}$ with $(x-1)(x-5) = x^2 - 6x + 5 = \underbrace{(x-3)^2 - 4}_{\text{not irreducible.}}$

CONCLUDING THOUGHT: Every polynomial with real coefficients factors into a product of linear and irreducible quadratic polynomials with real coefficients.