Guided Factorization:

Our goal is to factor some given polynomial with the help of some additional information. Often it's easy to find one or two roots by graphing or guessing. Then once you have a part of the puzzle you can fit together the rest with the help of "undetermined coefficients". Let me illustrate,

E160 $f(x) = x^3 + 3x^2 - x - 3$ note $f(1) = 0$. Factor the polynomial.

Since $f(1) = 0 \Rightarrow f(x) = (x-1)(Ax^2 + Bx + C)$ for some $A, B, C \in \mathbb{R}$. The question now is how do we determine the undetermined coefficients A, B, C?

$$\begin{aligned} f(x) &= (x-1)(Ax^2 + Bx + C) \\ &= Ax^3 + Bx^2 + Cx - Ax^2 - Bx - C \\ &= Ax^3 + (B-A)x^2 + (C-B)x - C = x^3 + 3x^2 - x - 3 \end{aligned}$$

Now compare coefficients of the LHS and RHS. We find,

$x^3 \rfloor$ $A = 1$

$x^2 \rfloor$ $B - A = 3 \Rightarrow B = 3 + A = 3 + 1 \Rightarrow B = 4$

$x \rfloor$ $C - B = -1 \Rightarrow C = B - 1 = 4 - 1 \Rightarrow C = 3$

$x^0 \rfloor$ $-C = -3 \Rightarrow C = 3 \leftarrow$ good, this checks with

Thus $f(x) = (x-1)(x^2 + 4x + 3)$. The quadratic is always factored easily. We have many options here factoring is clear; $x^2 + 4x + 3 = (x+1)(x+3)$.

$$\boxed{f(x) = (x-1)(x+1)(x+3)}$$

$\boxed{\text{E161}}$ Factor $f(x) = x^4 + 7x^3 + 7x^2 + 7x + 6$ given that $f(-1) = 0$ and $f(-6) = 0$.

From the given zeros we deduce that there exist $A, B, C \in \mathbb{R}$ such that: (we must preserve overall degree of $n = 4$)

$$
\begin{aligned}
f(x) &= (x+1)(x+6)(Ax^2 + Bx + C) \\
&= (x^2 + 7x + 6)(Ax^2 + Bx + C) \\
&= Ax^4 + Bx^3 + Cx^2 + 7Ax^3 + 7Bx^2 + 7Cx + 6Ax^2 + 6Bx + 6C \\
&= Ax^4 + (B + 7A)x^3 + (C + 7B + 6A)x^2 + (7C + 6B)x + 6C
\end{aligned}
$$

This has to equal $f(x) = x^4 + 7x^3 + 7x^2 + 7x + 6$. Compare coefficients to find,

$\underline{x^4}$) $A = 1$

$\underline{x^3}$) $B + 7A = 7 \Rightarrow B = 0$

$\underline{x^2}$) $C + 7B + 6A = 7$

$\underline{x}$) $7C + 6B = 7$

$\underline{x^0}$) $6C = 6 \Rightarrow C = 1$

Thus, $\boxed{f(x) = (x+1)(x+6)(x^2+1)}$ this time we cannot factor the quadratic further.

Remark: this idea of guided factorization is probably not in your text. This is my way of looking at things. We'll soon see how long division will allow us to avoid these undetermined coefficient calculations. I like my way of course, otherwise why would I show you?

LONG DIVISION AND FACTORING (§3.3)

THE DIVISION ALGORITHM

If $f(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$ and $\deg(d) < \deg(f)$ then there exist <u>unique</u> polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

and either $r(x) = 0$ or $\deg(r) < \deg(d)$. When $r(x) = 0$ we say $d(x)$ is a <u>factor</u> of $f(x)$.

Long division is the nuts & bolts of this claim. Let's see how it works.

E162 First, let's do some elementary school math,

$$\begin{array}{r|l} & \;\;110 \\ \hline 13 & 1432 \\ & \underline{13} \\ & 13 \\ & \underline{13} \\ & \boxed{2} \end{array}$$

remainder.

$$\Rightarrow \boxed{\frac{1432}{13} = 110 + \frac{2}{13}}$$

E163 Let $f(x) = x^2 + 3x + 1$ divide by $x + 2$

$$\begin{array}{r|l} & x \\ \hline x+2 & x^2 + 3x + 1 \\ & \underline{x^2 + 2x} \\ & \; x + 1 \end{array}$$

$r(x)$

the remainder

$$\underbrace{\frac{x^2 + 3x + 1}{x+1} = x + \frac{x+2}{x+1}}$$

this also shows $x^2 + 3x + 1 = x(x+1) + x + 2$

as you can see $(x+1)$ is not a factor of $x^2 + 3x + 1$ the remainder $r(x) = x + 1$ is evidence of this fact.

$\boxed{E164}$ Let's try $\boxed{E160}$ in light of long division,

$$
\begin{array}{r l}
 & x^2 + 4x + 3 \\
x-1 \,\big) & x^3 + 3x^2 - x - 3 \\
 & \underline{x^3 - x^2} \\
 & \quad 4x^2 - x \\
 & \quad \underline{4x^2 - 4x} \\
 & \qquad 3x - 3 \\
 & \qquad \underline{3x - 3} \\
 & \qquad \boxed{0} = r(x)
\end{array}
$$

We find $\dfrac{x^3+3x^2-x-3}{x-1} = x^2+4x+3$

Therefore, $\boxed{x^3+3x^2-x-3 = (x-1)(x^2+4x+3) = (x-1)(x+3)(x+1)}$

$\boxed{E165}$ Let's try $\boxed{E161}$ in light of long division,

$$
\begin{array}{r l}
 & x^2 + 1 \\
x^2+7x+6 \,\big) & x^4 + 7x^3 + 7x^2 + 7x + 6 \\
 & \underline{x^4 + 7x^3 + 6x^2} \\
 & \qquad\qquad x^2 + 7x + 6 \\
 & \qquad\qquad \underline{x^2 + 7x + 6} \\
 & \qquad\qquad\qquad \boxed{0} \leftarrow \underline{\text{remainder}}\ \underline{\text{zero}}
\end{array}
$$

We find, $\dfrac{x^4+7x^3+7x^2+7x+6}{x^2+7x+6} = x^2+1$

Therefore, $x^4+7x^3+7x^2+7x+6 = (x^2+7x+6)(x^2+1)$

$$= \boxed{(x+1)(x+6)(x^2+1)}$$

<u>Remark</u>: I'm not covering "synthetic" division since long division does all we need and is more general any way. We also skip Rational Root's Th$^{\underline{m}}$ and Descartes Rule for signs.

Long Division Examples

E166

$$
\begin{array}{r}
x^2 + 3x - 2 \\
x^2+2 \,\overline{)\, x^4 + 3x^3 + x - 3} \\
x^4 + 2x^2 \qquad\quad \\
\hline
3x^3 - 2x^2 + x \\
3x^3 + 6x \\
\hline
-2x^2 - 5x \\
-2x^2 - 4 \\
\hline
\boxed{-5x + 4}
\end{array}
\leftarrow \text{remainder}
$$

$$
\boxed{\frac{x^4+3x^3+x-3}{x^2+2} = x^2+3x-2+\left(\frac{-5x+4}{x^2+2}\right)}
$$

E167 Let $f(x) = x^3+9x^2+6x-13$. Is $(x+7)$ a factor of $f(x)$?

$$
\begin{array}{r}
x^2 + 2x - 8 \\
x+7 \,\overline{)\, x^3 + 9x^2 + 6x - 13} \\
x^3 + 7x^2 \qquad\qquad \\
\hline
2x^2 + 6x \\
2x^2 + 14x \\
\hline
-8x - 13 \\
-8x - 56 \\
\hline
\boxed{43}
\end{array}
\leftarrow \text{nope, } \underline{(x+7) \text{ not a factor of } f(x).}
$$

$$
\frac{x^3+9x^2+6x-13}{x+7} = x^2+2x-8 + \frac{43}{x+7}
$$

Curious, if $f(-7) \neq 0$ then what is it here?

$$
\begin{aligned}
f(-7) &= (-7)^3 + 9(49) - 42 - 13 \\
&= -7(49) + 9(49) - 55 \\
&= 2(49) - 55 \\
&= 98 - 55 \\
&= \boxed{43}
\end{aligned}
$$

Coincidence? Nope

$\boxed{\text{Th}^{\underline{m}} \text{ (Remainder Theorem) If } f(x) \text{ is divided by } x-k \text{ then the remainder } r = f(k).}$