Rational Functions and Asymptotes (§4.1)

At this point we know a lot about how to rip apart a polynomial into its basic parts. The next thing to think about is what functions constructed from the quotient of two polynomials look like.

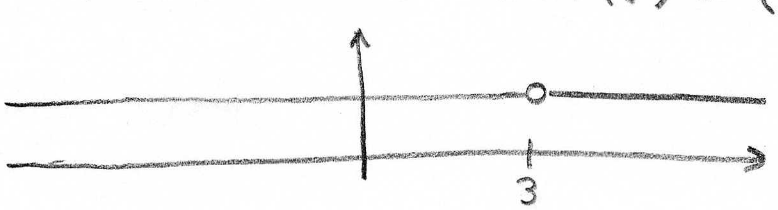
Defn/ A <u>rational</u> <u>function</u> has the form $f(x) = \frac{P(x)}{q(x)}$ where P and q are polynomials. We say that f is a proper rational function if $\deg(P) < \deg(q)$, otherwise we say f is improper. We can use long division to break up f into the sum of a polynomial and a proper rational function.

E168 Let $f(x) = \dfrac{x+1}{x-3}$ this is improper.

Note
$$\begin{array}{r|l} & \;\;1 \\ \hline x-3 & x+1 \\ & x-3 \\ \hline & \;\;-4 \end{array} \Rightarrow f(x) = \underbrace{1}_{\text{polynomial}} - \underbrace{\frac{4}{x-3}}_{\text{proper rational fnct.}}$$

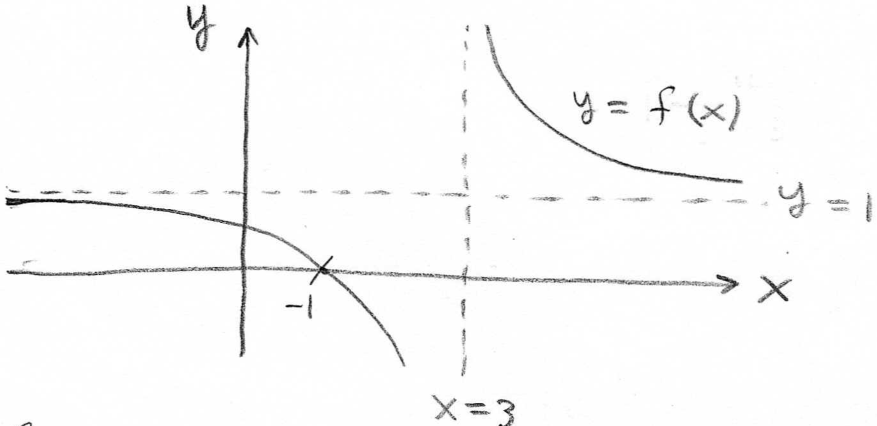
E169 Let $f(x) = \dfrac{x-3}{x-3}$. What is this function's domain?

Well, note $x-3$ cancels with $x-3$ to give $f(x) = 1$. You might be tempted to say $\cancel{\mathrm{dom}(f) = \mathbb{R}}$ <u>BUT</u> we can only cancel $\dfrac{x-3}{x-3} = 1$ if $x \neq 3$. The correct domain is $\mathrm{dom}(f) = (-\infty, 3) \cup (3, \infty)$



this is a <u>"hole in the graph"</u>

$\boxed{E170}$ Graph $y = f(x) = \dfrac{x+1}{x-3} = 1 - \dfrac{4}{x-3}$.

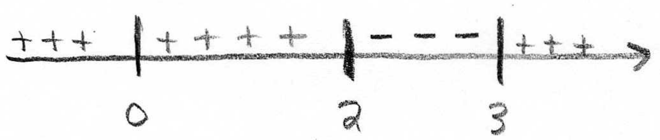


This function has a horizontal asymptote $y=1$ and a vertical asymptote of $x=3$.

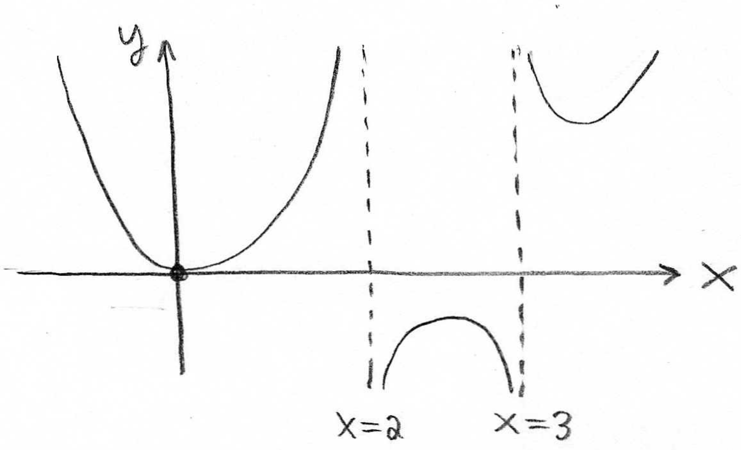
$$f(x) = 0 = \frac{x+1}{x-3} \Rightarrow x+1=0 \Rightarrow \boxed{x=-1 \text{ the only } x\text{-intercept}}$$

$\boxed{E171}$ Graph $y = f(x) = \dfrac{x^4}{(x-3)(x-2)}$. Find critical numbers.

Clearly $x = 0, 2, 3$ are critical #'s. Then the sign chart follows,



$$f(x) = \frac{x^4}{(x-3)(x-2)}$$



Notice $f(x) = 0$ when $x^4 = 0$ thus $x = 0$ is only x-intercept.

We have vertical asymptotes (VA) at $x=2$ and $x=3$
There are no horizontal asymptotes here.

Remark: the zeros of a rational function are at most all the zeros of the numerator.

THE REDUCED FUNCTION CONCEPT

We have seen that the appearance of $(x-a)$ in the denominator of $f(x) = \frac{P(x)}{g(x)}$ may or may not result in a vertical asymptote at $x=a$. Sometimes we just get a hole in the graph. A function which keeps the VA but fills in the holes is called the reduced function and we denote it $f_{red}(x)$.

$$f_{reduced}(x) = \text{"}f(x) \text{ maximally simplified."}$$

Let me give a few examples.

E172

$f(x) = \dfrac{x-2}{(x-2)(x+3)}$ $\text{dom}(f) = \{x \mid x \neq 2 \text{ and } x \neq -3\}$	$f_{reduced}(x) = \dfrac{1}{x+3}$ $\text{dom}(f_{red}) = \{x \mid x \neq -3\}$
$g(x) = \dfrac{x}{x^2(x-1)}$ $\text{dom}(g) = \{x \mid x \neq 0, x \neq 1\}$	$g_{red.}(x) = \dfrac{1}{x(x-1)}$ $\text{dom}(g_{red.}) = \text{dom}(g)$
$h(x) = \dfrac{(x-2)^2}{(x-2)} + \dfrac{1}{x}$ $\text{dom}(h) = \{x \mid x \neq 0, x \neq 2\}$	$h_{red.}(x) = x-2+\dfrac{1}{x}$ $\text{dom}(h_{red}) = \{x \mid x \neq 0\}$

Remark: The function and the reduced function are not technically the same function because they have different domains and different graphs. The subtle point here is that algebraic simplification of function formulas can change the implicit domain for the expression.

THE BEHAVIOUR OF $y = f(x) = \frac{P(x)}{q(x)}$ for $|x| \gg 1$

We seek to describe what the graph looks like for very large $\pm x$ values. ($x \to \infty$ or $x \to -\infty$)

Consider

$$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

$$= \frac{\frac{a_n x^n}{x^m} + \frac{a_{n-1} x^{n-1}}{x^m} + \cdots + \frac{a_1 x}{x^m} + \frac{a_0}{x^m}}{b_m + \frac{b_{m-1}}{x} + \cdots + \frac{b_1 x}{x^m} + \frac{b_0}{x^m}}$$

$$\approx \frac{a_n x^n}{b_m x^m}$$

for $|x| \gg 1$ the other terms are neglible in comparison to this term.

Then we have three cases, for $f(x) = \frac{P(x)}{q(x)}$

1.) $\deg(P) > \deg(q)$ so $n > m$ we find that $\frac{a_n x^n}{b_m x^m} \longrightarrow \pm\infty$ depending on whether we go left or right and also the signs of the leading coefficients a_n & b_m.

2.) $\deg(P) = \deg(q)$ so $n = m$ we find there is a horizontal asymptote $y = \frac{a_n}{b_m}$

3.) $\deg(P) < \deg(q)$ so $n < m$ we find horizontal asymptote of $y = 0$

<u>Remark</u>: the box above concerns the shape of graph $y = \frac{P(x)}{q(x)}$ for $|x| \gg 1$. The behaviour at small x is determined by the zeros, VA's and Holes In the Graph found via <u>factoring</u>.

E173 Graph $y = f(x) = \dfrac{1}{x-2} + \dfrac{1}{x-4}$. Find any horizontal or vertical asymptotes and zeros.

Clearly we have VA at $x = 2$ and $x = 4$. Find zeros,

$$\frac{1}{x-2} + \frac{1}{x-4} = 0$$

$$\Rightarrow \frac{1}{x-2} = \frac{-1}{x-4}$$

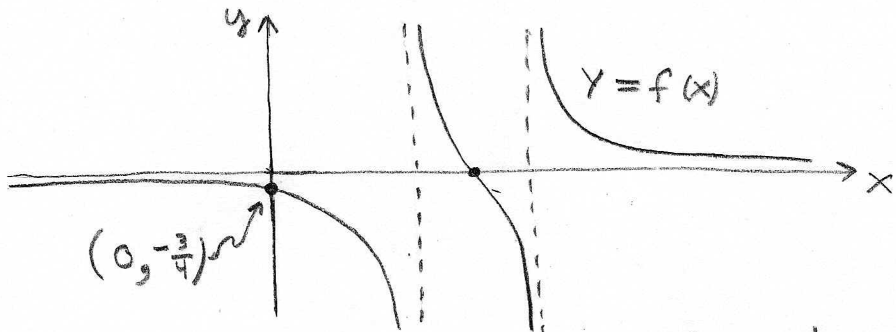
$$\Rightarrow x - 2 = -(x-4)$$

$$2x = 2 + 4 = 6$$

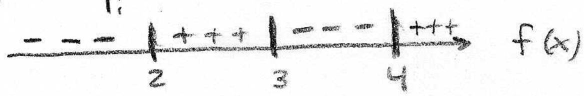
this step is not sensible for $x = 2$ or $x = 4$ but we know those are VAs so there's no danger.

$$\Rightarrow \underline{x = 3 \quad \text{the } x\text{-intercept}}.$$

From the discussion on (85) we know there is a horizontal asymptote $y = 0$, both pieces tend to zero for $|x| \gg 1$.



I used a sign chart



E174 Find zeros of $f(x) = \dfrac{x^3-1}{x^2+6}$. Notice as always $f(x) = 0$ only for numerator $= 0$,

$$x^3 - 1 = 0$$

$$x^3 = 1 \Rightarrow \boxed{x = 1}$$

Notice $x^2 + 6 \neq 0$ for all x thus this is not a hole in the graph.

E175 Find zeros of $f(x) = \dfrac{x(x+3)}{x^2+x}$. The possible zeros come from $x(x+3) = 0 \Rightarrow \underline{x = 0}$ or $\underline{x = -3}$. But, the denominator $x^2 + x = x(x+1) = 0$ for $x = 0$ or $x = -1$ thus $x = 0$ is a hole in the graph and $\boxed{x = -3 \text{ is only genuine zero}}$.

$\boxed{E176}$ Find zeros (if any) of $f(x) = 2 + \frac{5}{x^2+2}$

Intuitively I'd guess there are no zeros here since both terms look positive so no cancellation is possible. Let's see how the algebra works out here,

$$f(x) = 0 = 2 + \frac{5}{x^2+2} \Rightarrow -2 = \frac{5}{x^2+2}$$

$$\Rightarrow -2(x^2+2) = 5$$

$$\Rightarrow -2x^2 - 4 = 5$$

$$\Rightarrow x^2 = -9/2$$

$$\Rightarrow x = \pm\sqrt{-9/2} = \pm \frac{3i}{\sqrt{2}}$$

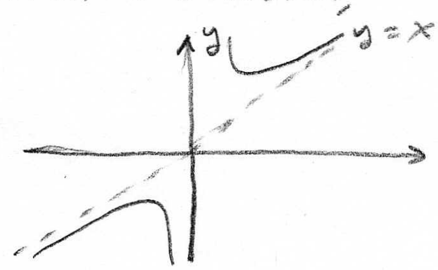
no real sol$^{\underline{n}}$.

$\Rightarrow$ no x-intercept.

Slant Asymptotes and More

I'll just give a few examples w/o much discussion,

$\boxed{E177}$ $f(x) = \frac{x^2+3}{x} \longrightarrow f_{|x|>>1} \approx \frac{x^2}{x} = x$



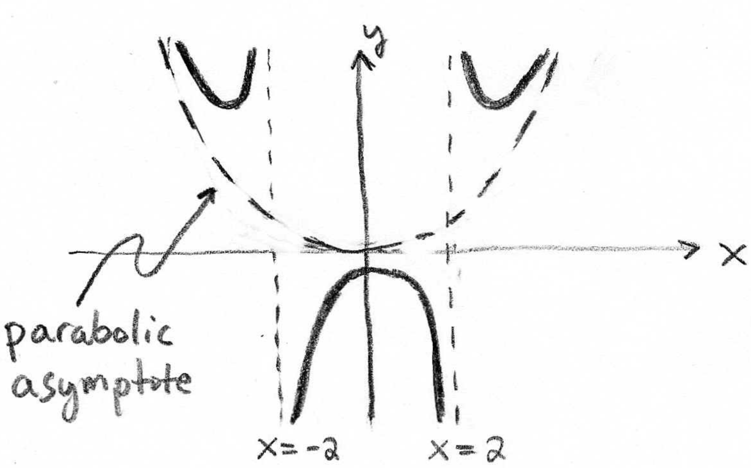
This function approaches $y = x$ for $x >> 1$. In other words $f(x) \cong x$ when $x >> 1$ where 3 is neglible compared to x^2.

$\boxed{E178}$ $f(x) = \frac{x^4+x^2+1}{(x-2)(x+2)}$; observe, no zeros and for $|x| >> 1$ we'll find

$$f(x) \cong \frac{x^4}{(x)(x)} = x^2$$

we do have VA at $x = \pm 2$

Also note $f(0) = \frac{1}{-2(2)} = \frac{-1}{4}$



Remark: We could have cubic or even quartic asymptotes. Don't worry I only expect you to graph H.A.