# Exponential Functions (§5.1)

Let's begin with a short discussion of how interest is earned by a bank account with an initial investment of $P$ dollars.

- Compound Interest: if the rate of yearly interest is $r$ and there are $n$ compounding periods per year then the total balance in the account is "A" after $t$-years,

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

**E179** Suppose we invest \$1000 at a rate of 12% per year compounded monthly. What is the balance after 6 months?

$$A_6 = 1000\left(1 + \frac{0.12}{12}\right)^{12(0.5)}$$

$t = 0.5$ years
$n = 12$ compoundings per year.

$$= 1000(1.01)^6$$

$$= 1061.52 \quad \therefore \quad \underline{A_6 = \$1061.52}$$

Let's break this down month by month

$$A_1 = 1000\left(1 + \frac{0.12}{12}\right)^{12(0.08\overline{33})} = 1000(1.01)^1 \Rightarrow \underline{A_1 = \$1010}.$$

$$A_2 = 1000(1.01)^2 = (1.01)A_1 = 1020.10 \Rightarrow \underline{A_2 = \$1020.10},$$

$$A_3 = (1.01)A_2 = 1030.30 \Rightarrow \underline{A_3 = \$1030.30}.$$

$$A_4 = (1.01)A_3 = 1040.60 \Rightarrow \underline{A_4 = \$1040.60}.$$

$$A_5 = (1.01)A_4 = 1051.01 \Rightarrow \underline{A_5 = \$1051.01},$$

$$A_6 = (1.01)A_5 = 1061.52 \Rightarrow \underline{A_6 = \$1061.52}.$$

See why its called "compound interest".

Remark: turn this around, if you put \$1000 on a credit card with 12% interest then the interest you pay if you paid it back on the 6th month would be \$61.52. (Ignoring many fees etc...)

- **Continuously Compounded Interest**

We take infinitely many compounding periods. Symbolically this means

$$A = \lim_{n \to \infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

The definition of "$\lim_{n\to\infty}$" is made rigorous in calculus and we prove there that this limit yields the <u>exponential function</u> evaluated at $rt$,

$$\boxed{A = Pe^{rt}} \quad \text{Continuous Compounding}$$

$\boxed{E180}$ Same as $\boxed{E179}$ with $P = \$1000$ and $r = 0.12$ but compound continuously instead of monthly.

$$A = 1000\, e^{(0.12)(0.5)} = 1000\, e^{0.06} \cong 1000(1.0618) = \boxed{\$1061.80}$$

which is \$0.28 more then we got with monthly compounding.

## Comments Concerning the Exponential Function

- $f(x) = e^x = \exp(x)$ is <u>the</u> exponential function
- $e \cong 2.718\ldots$ is an irrational number. In fact it is a <u>transcendental</u> number. It can't be defined algebraically.
- $e$ is <u>not</u> a variable. Much like $\pi$ it is shorthand for a known decimal expansion.
- $f(x) = a^x$ for $a > 0$ is <u>an</u> exponential function when $a = e$ this is a special case.

# Exponential Functions with base a

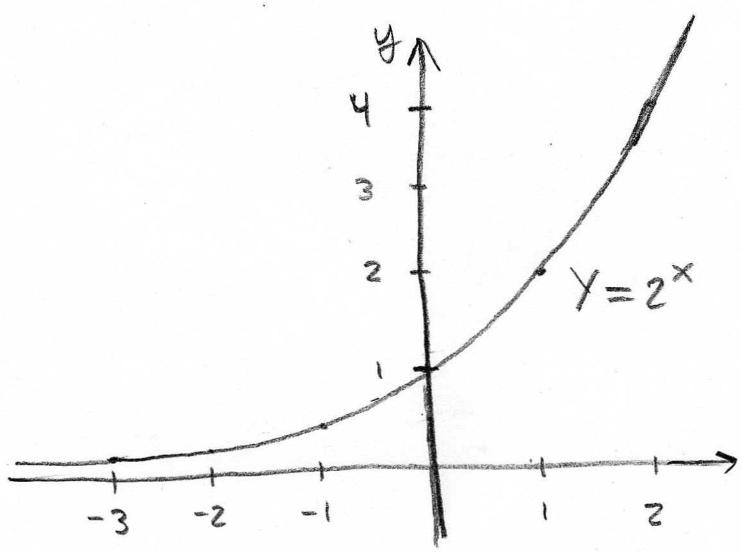
We require $a > 0$ and define

$$\boxed{f(x) = a^x \text{ is an exponential function with base } a \text{ and argument } x}$$

Let's study the graph of $y = a^x$ for various $a$,

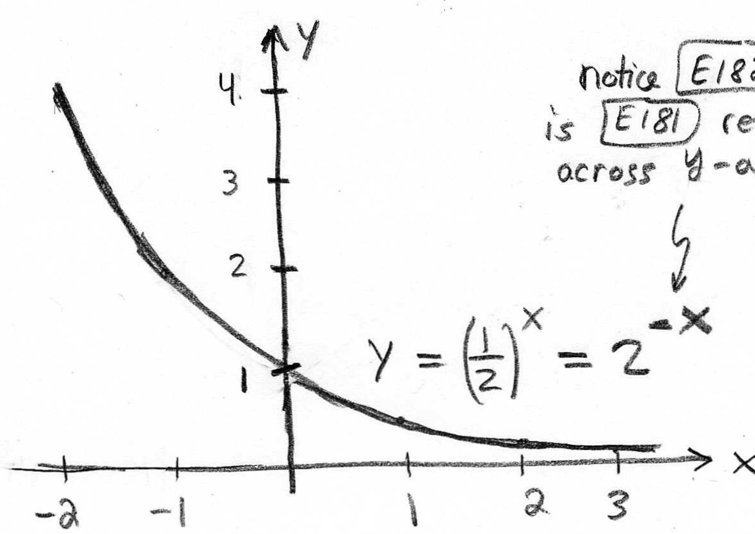
$\boxed{E181}$ Graph $y = 2^x$

| $x$ | $2^x$ |
|---|---|
| $-3$ | $2^{-3} = \frac{1}{8}$ |
| $-2$ | $2^{-2} = 1/4$ |
| $-1$ | $2^{-1} = 1/2$ |
| $0$ | $2^0 = 1$ |
| $1$ | $2^1 = 2$ |
| $2$ | $2^2 = 4$ |




$\boxed{E182}$ Graph $y = \left(\frac{1}{2}\right)^x$

| $x$ | $\left(\frac{1}{2}\right)^x$ |
|---|---|
| $-3$ | $8$ |
| $-2$ | $4$ |
| $-1$ | $2$ |
| $0$ | $1$ |
| $1$ | $1/2$ |
| $2$ | $1/4$ |
| $3$ | $1/8$ |




notice $\boxed{E182}$ is $\boxed{E181}$ reflected across $y$-axis.

**Remark:** Both $\boxed{E181}$ and $\boxed{E182}$ have $y$-intercepts of 1 and horizontal asymptotes of $y = 0$, but to the left for $y = 2^x$ and for the right for $y = 2^{-x}$.

# General Properties of $f(x) = a^x$, $a > 0$

1.) $\text{dom}(f) = \mathbb{R}$

2.) $\text{range}(f) = (0, \infty)$, no x-intercept and $a^x > 0$ (its positive)

3.) $f(0) = a^0 = 1$

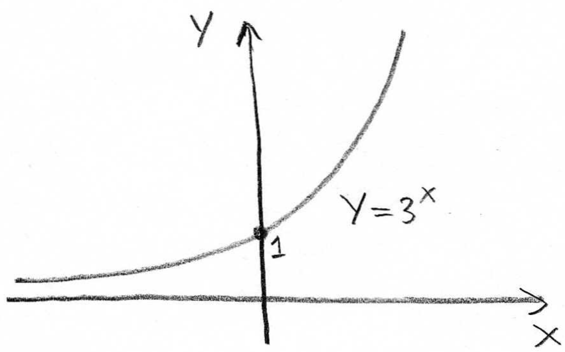
4.) horizontal asymptote of $y = 0$ for

   i.) $x >> 1$ for $a < 1$

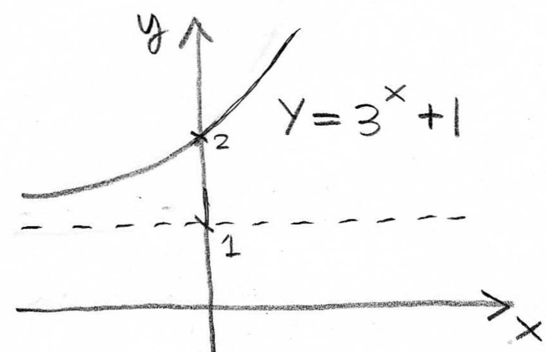
   ii.) $x << 1$ for $a > 1$

$\boxed{E183}$ We can shift $Y = a^x$ just like any other graph.

Consider $Y = 3^x$

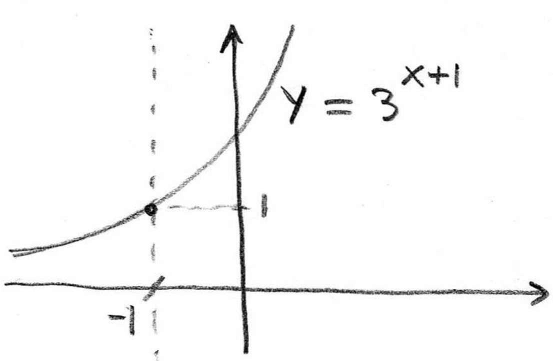



$\xrightarrow{\text{shift vertically}}$




(H.A. of $Y = 1$ for $x << 1$)

$\downarrow$ shift left




(H.A. of $y = 0$ for $x << 1$)

Remark: If we stretch $Y = 3^x$ vertically by a factor of 3 then

$$Y_{stretch} = 3\,3^x = 3^{x+1}$$

So its the same as shifting left by 1 unit for base $a = 3$.

$\boxed{E184}$ Let $f(x) = 7^{-x}$. What is the base of this exponential function? Also calculate $f(2)$ and $f(-3)$

$$f(x) = 7^{-x} = \frac{1}{7^x} = \left(\frac{1}{7}\right)^x, \quad \boxed{\text{base} = \frac{1}{7}}$$

$$f(2) = \frac{1}{7^2} = \boxed{\frac{1}{49} = f(2)}$$

$$f(-3) = 7^{-(-3)} = 7^3 = \boxed{343 = f(-3)}$$

<u>One-one Property for $f(x) = a^x$, $a \neq 1$</u>

$$a^x = a^y \Rightarrow a^x/a^y = 1 \Rightarrow a^{x-y} = 1$$
$$\Rightarrow x - y = 0$$

$$\boxed{\text{Idea: } a^x = a^y \iff x = y}$$

$\boxed{E185}$ Solve the exponential eq$^{\underline{n}}$ $2^x = 8$.

Notice $8 = 2^3$ thus

$$2^x = 2^3 \Rightarrow \boxed{x = 3}$$

$\boxed{E186}$ Solve $\frac{1}{64} = 2^{y+3}$.

Notice $\frac{1}{64} = \frac{1}{2^6} = 2^{-6}$ thus $2^{-6} = 2^{y+3}$

$$\Rightarrow -6 = y + 3$$
$$\Rightarrow \boxed{y = -9}$$

$\boxed{E187}$ Solve $3^x = 9^{x^2+1}$ if possible.

Notice $9 = 3^2$ thus $9^{x^2+1} = (3^2)^{x^2+1} = 3^{2x^2+2}$

thus $3^x = 3^{2x^2+2}$

$$\Rightarrow x = 2x^2 + 2$$
$$\Rightarrow 2x^2 - x + 2 = 0$$
$$\Rightarrow x = \frac{1 \pm \sqrt{1-16}}{4} \in \mathbb{C}, \quad \boxed{\text{no real sol}^{\underline{n}}\text{s}}$$