Properties of Exponentials and Logarithms

$\log_a(xy) = \log_a(x) + \log_a(y)$	$a^{u+v} = a^u a^v$
$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	$a^{u-v} = a^u / a^v$
$\log_a(x^c) = c\,\log_a(x)$	$(a^u)^c = a^{cu}$
$\log_a(a) = 1$	$a^0 = 1$

These formulas hold for $a > 0$, $a \neq 1$ and $x, y > 0$ for the left column. Notice that the left column follows from the rules in the right,

Let $u = \log_a(x)$ and $v = \log_a(y)$ then note,

$$a^{u+v} = a^u a^v$$

$$\Rightarrow \log_a(a^{u+v}) = \log_a(a^u a^v)$$

$$\Rightarrow u+v = \log_a(xy) \quad \left(\text{since } \begin{array}{l} a^u = a^{\log_a(x)} = x \\ a^v = a^{\log_a(y)} = y. \end{array}\right)$$

$$\Rightarrow \boxed{\log_a(x) + \log_a(y) = \log_a(xy)}$$

The proofs of the other properties are similar (see pg. 444 in your text)

E192 Expand the log. as much as possible.

$$\ln\left(x^2\sqrt{x-2}\right) = \ln(x^2) + \ln\left(\sqrt{x-2}\right)$$

$$= \underline{2\ln(x) + \tfrac{1}{2}\ln(x-2)}$$

Remark: in calculus we do a lot of this in the logarithmic differentiation section. These are powerful skills.

E193 Expand the log. below as much as possible.

$$\log_2\left(\frac{16(x^3-7)}{\sqrt{4x}}\right) = \log_2(16) + \log_2(x^3-7) - \log_2\sqrt{4x}$$

$$= \log_2(2^4) + \log_2(x^3-7) - \tfrac{1}{2}\log_2(4x)$$

$$= 4 + \log_2(x^3-7) - \tfrac{1}{2}\log_2(4) - \tfrac{1}{2}\log_2(x)$$

$$= 4 + \log_2(x^3-7) - \tfrac{1}{2}\underbrace{\log_2(2^2)}_{2} - \tfrac{1}{2}\log_2(x)$$

$$= \boxed{3 + \log_2(x^3-7) - \tfrac{1}{2}\log_2(x)}$$

E194. Collect the sum of logs into a single log.

$$? = \ln(x) - 7\ln(\sqrt{x}) + \tfrac{1}{3}\ln(6x) + 8 =$$

$$= \ln(x) - \ln(x^{7/2}) + \ln(\sqrt[3]{6x}) + \underbrace{\ln(e^8)}$$

$$= \boxed{\ln\left[\frac{x\sqrt[3]{6x}\,e^8}{x^{7/2}}\right]}$$

this is just 8 written in a sneaky way.

(this could be simplified further)

E195 Collect the logs below into a single log,

$$\ln(\sqrt{x}) + \ln(x^2) - \ln(x^{3/2}) + \ln(x^2+4) =$$

$$= \tfrac{1}{2}\ln(x) + 2\ln(x) - \tfrac{3}{2}\ln(x) + \ln(x^2+4)$$

$$= 3\ln(x) + \ln(x^2+4)$$

$$= \ln(x^3) + \ln(x^2+4)$$

$$= \boxed{\ln(x^3(x^2+4))}$$

Change of Base Formula

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_b(a)}}$$

In a heuristic sense you can think of the a's as cancelling on the RHS and $\frac{1}{\frac{1}{b}} = b$ so it makes sense. That is not the real reason it works of course. Notice a special case is

$$\boxed{\log_a(x) = \frac{\ln(x)}{\log_a(e)}} \quad \text{or} \quad \boxed{\ln(x) = \frac{\log(x)}{\ln(10)}}$$

Let's see if we can derive this formula from

$$a^x = b^{\log_b(a^x)} = b^{x\log_b(a)}$$

this is the change of exponential formula. We can trade a^x for $b^{x\log_b(a)}$

$$Y = a^x \quad \text{or} \quad Y = b^{x\log_b(a)}$$

$$\log_a(Y) = x \quad \text{or} \quad \log_b(Y) = x\log_b(a)$$

$$\therefore \quad x = \boxed{\log_a(Y) = \frac{\log_b(Y)}{\log_b(a)}}$$

E196 I happen to know $\ln(2) \approx 0.692$,

$$\log_2(e^2) = \frac{\ln(e^2)}{\ln(2)} \cong \frac{2}{0.692} \approx \underline{2.7}$$

$$\begin{array}{r} 2 \\ .692\overline{)2.000} \\ 1\ 584 \\ \hline 4160 \end{array}$$

(need a calculator to do much more here. Or perhaps a slide-rule)

E197 Solve $2e^x = e^{x^2}$.

$$\ln(2e^x) = \ln(e^{x^2})$$

$$\ln(2) + \ln(e^x) = \ln(e^{x^2})$$

$$\ln(2) + x = x^2$$

$$x^2 - x - \ln(2) = 0$$

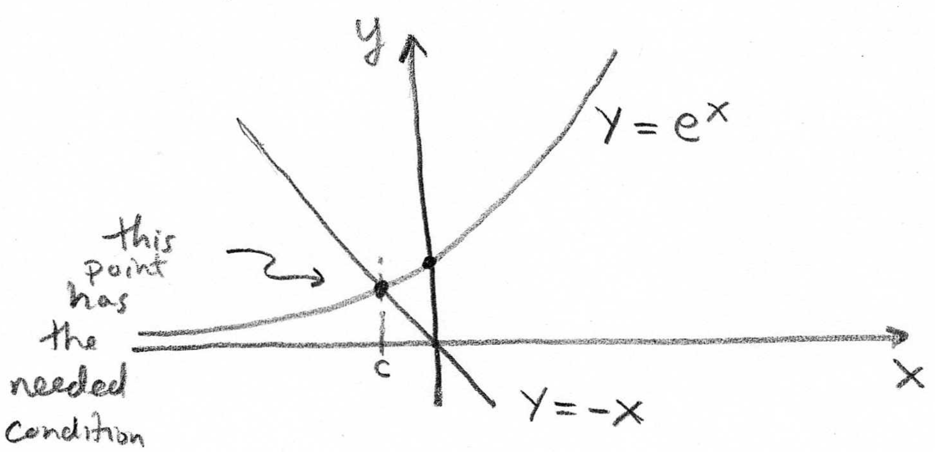
$$\Rightarrow \boxed{x = \frac{1 \pm \sqrt{1 + 4\ln(2)}}{2}}$$

E198 Solve $e^x = -x$.

Let's see if we take the natural log then

$$\ln(e^x) = \ln(-x) \Rightarrow x = \ln(-x)$$

We can play these games all day and we'll never be able to isolate x. We have to settle for an approximate solⁿ in such a case. Graphing helps: find intersection of $y = e^x$ and $y = -x$



$e^c = -c$

(We'd need a graphing calculator to zoom in and get a nice answer)

E199 Solve $e^{2x} = 42$.

$$\ln(e^{2x}) = \ln(42)$$

$$2x = \ln(42)$$

$$\boxed{x = \ln(\sqrt{42})}$$ (see what I did here ?)

E200 Solve $6(2^{3x-1}) - 7 = 9$.

$$6(2^{3x-1}) = 16$$

$$2^{3x-1} = \frac{16}{6} = \frac{8}{3}$$

$$\ln(2^{3x-1}) = \ln(8/3)$$

$$(3x-1)\ln(2) = \ln(8/3)$$

$$3x\ln(2) = \ln(2) + \ln(8/3)$$

$$\Rightarrow \boxed{x = \frac{\ln(8/3) + \ln(2)}{3\ln(2)}}$$

E201 Solve $\ln(x) + \ln(x-1) = \ln(2)$.

Use properties of logs,

$$\ln(x(x-1)) = \ln(2)$$

$$\Rightarrow x(x-1) = 2$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

Notice that $x = -1 < 0$ and $\ln(-1)$ is not a real number thus we discard the $x = -1$ sol$^{\underline{n}}$ since it is not a sol$^{\underline{n}}$ to the given equation. There is one sol$^{\underline{n}}$, $\boxed{x = 2}$

Remark: Should always check answer at end with these problems

$\boxed{E\,202}$ Solve $\log(2x-4) = \log(x+7)$

$$10^{\log(2x-4)} = 10^{\log(x+7)}$$

$$2x - 4 = x + 7$$

$$\boxed{x = 11} \quad \text{(this makes sense in original eq}^{\underline{n}}\text{)}$$

$\boxed{E\,203}$ Solve $\log(x) = \log(2x+3)$.

$$10^{\log(x)} = 10^{\log(2x+3)}$$

$$x = 2x + 3$$

$$-x = 3 \Rightarrow \underline{x = -3}$$

But, notice $\log(-3)$ does not exist (d.n.e.) as a real number, thus there are no sol$^{\underline{n}}$'s. This means the graphs $y = \log(x)$ & $y = \log(2x+3)$ do not intersect.

$\boxed{E\,204}$ Solve $\log(8x) - \log(1+\sqrt{x}) = 2$.

$$\log\left(\frac{8x}{1+\sqrt{x}}\right) = 2 \Rightarrow \frac{8x}{1+\sqrt{x}} = 10^2 = 100$$

Thus, $8x = 100(1+\sqrt{x}) = 100 + 100\sqrt{x}$

$$\Rightarrow \quad 8(\sqrt{x})^2 - 100\sqrt{x} - 100 = 0$$

Quadratic eq$^{\underline{n}}$ in $\sqrt{x}$,

$$\sqrt{x} = \frac{100 \pm \sqrt{(100)^2 + 32(100)}}{16} = \frac{100 \pm \sqrt{13{,}200}}{16}$$

Need $\sqrt{x} \geq 0$ so choose $(+)$, square to get x,

$$\boxed{x = \left(\frac{100 + \sqrt{13{,}200}}{16}\right)^2 \cong 180.4}$$