# COMPLETING SQUARE, FACTORING, QUADRATIC FORMULA



Let $f(x) = ax^2 + bx + c$ where $a \neq 0$.

Observe $f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ and $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

hence $f(x) = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)$. This process of eliminating the linear term by grouping it into a square is called "completing the square". Notice $\frac{-b^2}{4a^2} + \frac{c}{a} = -\left(\frac{b^2 - 4ac}{4a^2}\right)$ thus,

$$f(x) = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right)$$

$$\left\{\begin{array}{l} b^2 - 4ac = \left(\sqrt{b^2 - 4ac}\right)^2 \\ 4a^2 = (2a)^2 \end{array}\right. \quad **$$

$$= a\underbrace{\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)}_{\text{factored! Notice, need } b^2 - 4ac \geq 0 \text{ for } ** \text{ to hold using real numbers.}}$$

Then $f(x) = 0$ implies $x + \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

which gives us that $\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ ← Quadratic Formula

It turns out the derivation above still works when $b^2 - 4ac < 0$ but then the Quadratic Formula gives complex sol$^{\text{n}}$ to $f(x) = 0$.

When $a = 1$ then $f(x) = x^2 + Bx + C$ is easier to think through completing the square,

$$x^2 + \underbrace{Bx}_{\text{half of } B} + C = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2 + C$$

subtract square of whatever we add to $x$ in the square

still here.

Example 1: $x^2 + 6x + 1 = (x+3)^2 - 9 + 1 = \underline{(x+3)^2 - 8}$.

In each example below I factor by completing the square and using the difference of squares formula $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$;

**Example 1:**

$$\begin{aligned} x^2+6x+1 &= (x+3)^2-9+1 && : \text{completed square} \\ &= (x+3)^2-(\sqrt{8})^2 && : 8=(\sqrt{8})^2. \\ &= \boxed{(x+3+\sqrt{8})(x+3-\sqrt{8})} && : \text{factored using} \end{aligned}$$

$\alpha = x+3$ & $\beta=\sqrt{8}$

$\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

**Example 2:**

$$\begin{aligned} x^2+4x+5 &= (x+2)^2 \\ &= \boxed{\underbrace{(x+2)^2+1}_{}} \end{aligned}$$

prime or irreducible over $\mathbb{R}$.

**Example 3:** $x^2+10x+25=(x+5)^2-25+25=\boxed{(x+5)^2}$.

**Remark:** the above three examples illustrate the possible outcome of completing the square. They have $A=1$, but generally,

| Discriminant: $B^2-4AC$ | TYPE OF FACTORING | TYPE OF SOLUTION |
|---|---|---|
| $B^2-4AC>0$ | $(x-r_1)(x-r_2)$ | $x=r_1$ & $x=r_2$ |
| $B^2-4AC=0$ | $(x-r)^2$ | $x=r$ twice |
| $B^2-4AC<0$ | Cannot factor in $\mathbb{R}$ | complex, no real. |

**Example 4:**

$$\begin{aligned} x^2+4x+3 &= (x+2)^2-4+3 \\ &= (x+2)^2-1^2 \\ &= (x+2+1)(x+2-1) \\ &= \underline{(x+3)(x+1)} \end{aligned}$$

$\leftarrow$ maybe we'd just "see" this with our guess & check method. The guess & check method is faster when it works.

**Example 5:** $x^2$ is already done.

**Example 6:** $x^2+3$ is clearly irreducible.

**Remark:** both Example 5 & 6 illustrate a silly case, there is no $x$-term to remove so completing square is not needed.

Example 7:

$$x^2 + 3x + 2 = (x+1)(x+2)$$ ← easiest if you see it, direct guess & check on $1 \cdot 2 = 2$ and $1+2=3$.

Of course, completing square will derive the same,

$$\begin{aligned} x^2 + 3x + 2 &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 2 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{8}{4} \\ &= \left(x + \tfrac{3}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 \\ &= \left(x + \tfrac{3}{2} + \tfrac{1}{2}\right)\left(x + \tfrac{3}{2} - \tfrac{1}{2}\right) \\ &= \underline{(x+2)(x+1)}. \end{aligned}$$

Example 8:

$$\begin{aligned} x^2 + 9x + 2 &= \left(x + \tfrac{9}{2}\right)^2 - \tfrac{81}{4} + \tfrac{8}{4} \\ &= \left(x + \tfrac{9}{2}\right)^2 - \left(\sqrt{\tfrac{73}{4}}\right)^2 \\ &= \underline{\left(x + \tfrac{9}{2} - \tfrac{\sqrt{73}}{2}\right)\left(x + \tfrac{9}{2} + \tfrac{\sqrt{73}}{2}\right)}. \end{aligned}$$

Example 9:

$$x^2 + 15x = x(x+15)$$ ← this is wise way to factor here!

$x^2 + 15x = \left(x + \tfrac{15}{2}\right)^2 - \tfrac{225}{4} = \left(x + \tfrac{15}{2} + \tfrac{15}{2}\right)\left(x + \tfrac{15}{2} - \tfrac{15}{2}\right) = (x+15)x$. But, again, completing square still works!

Example 10:

$$\begin{aligned} x^2 - 11x + 2 &= \left(x - \tfrac{11}{2}\right)^2 - \tfrac{121}{4} + \tfrac{8}{4} \\ &= \left(x - \tfrac{11}{2}\right)^2 - \tfrac{113}{4} \\ &= \left(x - \tfrac{11}{2}\right)^2 - \left(\tfrac{\sqrt{113}}{2}\right)^2 \\ &= \underline{\left(x - \tfrac{11}{2} + \tfrac{\sqrt{113}}{2}\right)\left(x - \tfrac{11}{2} - \tfrac{\sqrt{113}}{2}\right)}. \end{aligned}$$

## How to solve $f(x) = 0$ ?

If $f(x) = (x - r_1)(x - r_2)$ then $x - r_1 = 0$ or $x - r_2 = 0$ thus $\underline{x = r_1 \text{ or } r_2}$

Since we factored $f(x)$ for examples 1 – 10 we can solve $f(x) = 0$ easily,

Example 1:

$$x^2 + 6x + 1 = (x + 3 + \sqrt{8})(x + 3 - \sqrt{8}) = 0 \Rightarrow \boxed{x = -3 \pm \sqrt{8}}.$$

Example 2:

$$x^2 + 4x + 5 = (x+2)^2 + 1 = 0 \Rightarrow \boxed{\text{no real sol}^{\text{n}}}.$$

Example 3:

$$x^2 + 10x + 25 = (x+5)^2 = 0 \Rightarrow \boxed{x = -5 \text{ twice}}.$$

Example 4:

$$x^2 + 4x + 3 = (x+3)(x+1) = 0 \Rightarrow \boxed{x = -3 \text{ or } x = -1}.$$

Example 5:

$$x^2 = 0 \Rightarrow \boxed{x = 0 \text{ twice}}.$$

Example 6:

$$x^2 + 3 = 0 \Rightarrow \boxed{\text{no real sol}^{\text{n}}}.$$

Example 7:

$$x^2 + 3x + 2 = (x+2)(x+1) = 0 \Rightarrow \boxed{x = -2 \text{ or } x = -1}.$$

Example 8:

$$x^2 + 9x + 2 = \left(x + \tfrac{9}{2} - \tfrac{\sqrt{73}}{2}\right)\left(x + \tfrac{9}{2} + \tfrac{\sqrt{73}}{2}\right) = 0 \Rightarrow \boxed{x = \frac{-9 \pm \sqrt{73}}{2}}.$$

Example 9:

$$x^2 + 15x = x(x+15) = 0 \Rightarrow \boxed{x = 0 \text{ or } x = -15}$$

Example 10:

$$x^2 - 11x + 2 = \left(x - \tfrac{11}{2} + \tfrac{\sqrt{113}}{2}\right)\left(x - \tfrac{11}{2} - \tfrac{\sqrt{113}}{2}\right) = 0$$

$$\Rightarrow \boxed{x = \frac{11 \pm \sqrt{113}}{2}}.$$

**Remark:** we don't need the quadratic formula if we can do the math shown above! However, there is another way ↴

## How to solve $f(x)=0$ given $f(x)$ as completed square?

I'll allow for complex solutions this time, not on Test 2,

**Example 1:** $x^2+6x+1 = \underbrace{(x+3)^2-8=0}$

$$(x+3)^2 = 8 \Rightarrow x+3 = \pm\sqrt{8}$$
$$\therefore \boxed{x = -3 \pm \sqrt{8}}$$

**Example 2:** $x^2+4x+5 = \underbrace{(x+2)^2+1=0}$

$$(x+2)^2 = -1 \Rightarrow x+2 = \pm\sqrt{-1} = \pm i$$
$$\therefore \boxed{x = -2 \pm i}$$

**Remark:** the sol$^{n}$ of $x^2+4x+5=0$ comes in a complex conjugate pair, this always happens for irreducible quadratics with real coefficients.

**Example 3:** $x^2+10x+25 = (x+5)^2 = 0 \Rightarrow x+5=0 \Rightarrow \boxed{x=-5}$

**Example 4:** $x^2+4x+3 = (x+2)^2-1 = 0 \Rightarrow (x+2)^2 = 1 \Rightarrow x+2 = \pm\sqrt{1} = \pm 1$

$$\therefore \boxed{x = -2 \pm 1}$$

**Example 5:** $x^2 = 0 \Rightarrow \boxed{x=0}$

**Example 6:** $x^2+3=0 \Rightarrow x^2 = -3 \Rightarrow x = \pm\sqrt{-3} = \pm\sqrt{-1}\sqrt{3} = \pm i\sqrt{3}$

$$\therefore \boxed{x = \pm i\sqrt{3}}$$

**Example 7:**

$$x^2+3x+2 = \underbrace{\left(x+\tfrac{3}{2}\right)^2 - \tfrac{1}{4} = 0}$$
$$\left(x+\tfrac{3}{2}\right)^2 = \tfrac{1}{4} \Rightarrow x+\tfrac{3}{2} = \pm\sqrt{\tfrac{1}{4}} = \pm\tfrac{1}{2}$$
$$x = -\tfrac{3}{2} \pm \tfrac{1}{2} \Rightarrow \boxed{x=-1 \text{ or } x=-2}$$

**Example 8:**

$$x^2+9x+2 = \left(x+\tfrac{9}{2}\right)^2 - \tfrac{73}{4} = 0 \Rightarrow x+\tfrac{9}{2} = \pm\sqrt{\tfrac{73}{4}}$$
$$\therefore \boxed{x = \frac{-9 \pm \sqrt{73}}{2}}$$