The Complex Calculus

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purpose and origins

Much of what I say stems from an in-depth study I made of Gamelin's *Complex Analysis* [G01] in a previous offering of Math 331 at Liberty in 2014-2015. The You Tube videos on Complex Analysis by me in 2015 are tied to that study as are the notes I entitled *Guide to Gamelin*. In those notes I cover is the basic core of undergraduate complex analysis. However, I am now trying to diverge from Gamelin a bit and bring in ideas from other texts such as the excellent text by Saff and Snider, or the text by Matthews and Howell, or Fischer, or Stewart and Tall. These notes will likely reflect reading I've done in all those texts.

My understanding of these topics began with a study of the classic text of Churchill as I took Math 513 at NCSU a few years ago. My advisor Dr. R.O. Fulp taught the course and added much analysis which was not contained in Churchill. I also have learned a great amount from Reinhold Remmert's *Complex Function Theory* [R91]. The history and insight of that book will bring me to say a few dozen things this semester, it's a joy to read, but, it's not a first text in complex analysis so I have not required you obtain a copy. There are about a half-dozen other books I consult for various issues and I will comment on those as we use them.

I've also taught worked through more proofs in calculus I and II which impact this course. I've taken some time to transcribe certain proofs from the real to the complex domain. On the one hand, there ought not be a need since we should have seen these proofs in the real calculus course. But, on the other hand, there is a good chance your calculus course didn't present or test such proof, so the argument may well be new to much of the audience for this course.

Remark: many of the quotes given in this text are from [R91] or [N91] where the original source is cited. I decided to simply cite those volumes rather than add the original literature to the bibliography for several reasons. First, I hope it prompts some of you to read the literature of Remmert. Second, the original documents are hard to find in most libraries. I also obtained a copy of Jeremy Gray's *The Real and the Complex: A History of Analysis in the 19th Century*. I hope to find time to read some of Gray, I know it would add considerable depth to my current understanding of 19th century math history.

For your second read through complex analysis I recommend [R91] and [RR91] or [F09] for the student of pure mathematics. For those with an applied bent, I recommend [A03].

The later chapters of these notes are a first step towards a book I wish to write on $\mathcal{A}$-Calculus. The non-standard material is largely adapted from several papers I have written on $\mathcal{A}$-Calculus in recent years. I once tried to start Math 331 with the $\mathcal{A}$-calculus, but it's a bit much for even overly prepared students. Therefore, I've decided to wait until the end of the course to just give a glimpse into the wild world of hypercomplex analysis. To give a quick snapshot of where my current development has reached, I've basically worked through Calculus I, II, III and much of DEqns over a finite dimensional unital associative algebra. Also, more recently, I've worked some on understanding when it is possible to exchange a system of PDEs for a DEqn over the algebra. Anyway, there is much more to learn and lots of open projects for students who wish to research something which is similar to required course work, yet far from something you can just look-up online. Finally, in terms of complex analysis, the project of $\mathcal{A}$-calculus gives much insight as it shows what aspects of complex analysis are truly special as compared to those features which are common to the general calculus over an associative algebra.

format of this guide

These notes were prepared with LaTeX. I tend to use green for definitions, blue for theorems, red for remarks and black for just about everything else. This is a work in progress, my apologies for mistakes, just email me if one is troubling. I am here to help,

James Cook, August 19, 2023 version 1.0

Notations:

Some of the notations below are from Gamelin, however, others are from [R91] and elsewhere.

Symbol	terminology	Definition in Guide
$\mathbb{C}$	complex numbers	[1.1.1]
$\mathbf{Re}(z)$	real part of z	[1.1.1]
$\mathbf{Im}(z)$	imaginary part of z	[1.1.1]
$\bar{z}$	complex conjugate of z	[1.1.3]
$\|z\|$	modulus or length of z	[1.1.3]
$\mathbb{C}^{\times}$	nonzero complex numbers	[1.1.6]
$\mathbb{C}[z]$	polynomials in z with coefficients in $\mathbb{C}$	[1.1.8]
$\mathbb{R}[z]$	polynomials in z with coefficients in $\mathbb{R}$	
$Arg(z)$	principle argument of z	[1.3.1]
$arg(z)$	set of arguments of z	[1.3.1]
$e^{i\theta}$	imaginary exponential	[1.3.4]
$\|z\|e^{i\theta}$	polar form of z	[1.3.4]
ω	primitive root of unity	[1.3.12]
$\mathbb{C}^*$	extended complex plane	[**??**]
$\mathbb{C}^-$	slit plane $\mathbb{C} - (-\infty, 0]$	[**??**]
$\mathbb{C}^+$	slit plane $\mathbb{C} - [0, \infty)$	[**??**]
$f\|_U$	restriction of f to U	[2.0.1]
$\sqrt[n]{z}$	n-th principal root	[2.6.1]
Arg_α	α-argument of	[1.3.1]
e^z	complex exponential	[2.2.1]
$Log(z)$	principal logarithm	[2.3.1]
$log(z)$	set of logarithms	[2.3.2]
z^α	set of complex powers	[2.4.1]
$\sin(z), \cos(z)$	complex sine and cosine	[2.5.1]
$\sinh(z), \cosh(z)$	complex hyperbolic functions	[2.5.2]
$\tan(z)$	complex tangent	[2.5.3]
$\tanh(z)$	complex hyperbolic tangent	[2.5.3]
$\lim_{n\to\infty} a_n$	limit as $n \to \infty$	[3.2.1]
$\lim_{z\to z_o} f(z)$	limit as $z \to z_o$	[3.2.14]
$C^0(U)$	continuous functions on U	[3.2.16]
$D_\varepsilon(z_o)$	open disk radius ε centered at z_o	[3.1.1]
∂S	boundary of S	[3.1.11]
$[p, q]$	line segment from p to q	[3.1.4]
$f'(z)$	complex derivative	[5.1.1]
J_F	Jacobian matrix of F	[**??**]
$u_x = v_y$ $u_y = -v_x$	CR-equations of $f = u + iv$	[5.3.1]
$\mathcal{O}(C)$	entire functions on $\mathbb{C}$	[5.3.6]
$\mathcal{O}(D)$	holomorphic functions on D	[5.3.11]

You can also use the search function within the pdf-reader.

Contents

Chapter 1

Complex Numbers

In this chapter we examine the complex numbers. The history of complex numbers is fascinating and I only share bits and pieces in these notes. The main goal of this chapter is to learn the basic notation and begin to think in complex notation. We also learn some new insights into basic algebra.

1.1 Algebra and Geometry of Complex Numbers

I set aside the question of *existence* for now. Rest assured there is such a thing as $\mathbb{C}$ which merits all the definitions and constructions shared in this section.

Definition 1.1.1. *Let $a, b, c, d \in \mathbb{R}$. A* **complex number** *is an expressions of the form $a + ib$. By assumption, if $a + ib = c + id$ we have $a = c$ and $b = d$. We define the* **real part** *of $a + ib$ by* $\textbf{Re}(a+ib) = a$ *and the* **imaginary part** *of $a+ib$ by* $\textbf{Im}(a+ib) = b$*. The set of all complex numbers is denoted $\mathbb{C}$. Complex numbers of the form $a + i(0)$ are called* **real** *whereas complex numbers of the form $0+ib$ are called* **imaginary***. The set of imaginary numbers is denoted $i\mathbb{R} = \{iy \mid y \in \mathbb{R}\}$.*

It is customary to write $a + i(0) = a$ and $0 + ib = ib$ as the 0 is superfluous. Furthermore, the notation[1] $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ compactly expresses the fact that each complex number is written as the sum of a real and pure imaginary number. There is also the assumption $\mathbb{R} \cap i\mathbb{R} = \{0\}$. In words, the only complex number which is both real and pure imaginary is 0 itself.

We add and multiply complex numbers in the usual fashion:

Definition 1.1.2. *Let $a, b, c, d \in \mathbb{R}$. We define* **complex addition** *and* **multiplication** *as follows:*

$$(a + ib) + (c + id) = (a + c) + i(b + d) \qquad \& \qquad (a + ib)(c + id) = ac - bd + i(ad + bc).$$

Often the definition is recast in pragmatic terms as $i^2 = -1$ and proceed as usual. Let me remind the reader what is "usual". Addition and multiplication are commutative and obey the usual distributive laws: for $x, y, z \in \mathbb{C}$

$$x + y = y + x, \qquad \& \qquad xy = yx, \qquad \& \qquad x(y + z) = xy + xz,$$

associativity of addition and multiplication can also be derived:

$$(x + y) + z = x + (y + z), \qquad \& \qquad (xy)z = x(yz).$$

[1] see the discussion of $\oplus$ (the direct sum) in my linear algebra notes. Here I view $\mathbb{R} \leq \mathbb{C}$ and $i\mathbb{R} \leq \mathbb{C}$ as independent $\mathbb{R}$-subspaces whose direct sum forms $\mathbb{C}$.

The additive identity is 0 whereas the multiplicative identity is 1, in particular:

$$z + 0 = z \qquad \& \qquad 1 \cdot z = z$$

for all $z \in \mathbb{C}$. Notice, the notation $1z = 1 \cdot z$. Sometimes we like to use a $\cdot$ to emphasize the multiplication, however, usually we just use **juxtaposition** to denote the multiplication. Finally, using the notation of Definition 1.1.2, let us check that $i^2 = ii = (0 + i)(0 + i) = -1$. Take $a = 0, b = 1, c = 0, d = 1$:

$$i^2 = ii = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1.$$

In view of all these properties (which the reader can easily prove follow from Definition 1.1.2) we return to the multiplication of $a + ib$ and $c + id$:

$$\begin{aligned}(a + ib)(c + id) &= a(c + id) + ib(c + id)\\ &= ac + iad + ibc + i^2bd\\ &= ac - bd + i(ad + bc).\end{aligned}$$

Of course, this is precisely the rule we gave in Definition 1.1.2. It is convenient to define the **modulus** and **conjugate** of a complex number before we work on fractions of complex numbers.

Definition 1.1.3. *Let $a, b \in \mathbb{R}$. We define* **complex conjugation** *as follows:*

$$\overline{a + ib} = a - ib.$$

We also define the **modulus** *of $a + ib$ which is denoted $|a + ib|$ where*

$$|a + ib| = \sqrt{a^2 + b^2}.$$

The complex number $a+ib$ is naturally identified[2] with (a, b) and so we have the following geometric interpretations of conjugation and modulus:

(i.) conjugation reflects points over the real axis.

(ii.) modulus of $a + ib$ is the distance from the origin to $a + ib$.

Let us pause to think about the problem of two-dimensional vectors. This gives us another view on the origin of the modulus formula. We call the x-axis the **real axis** as it is formed by complex numbers of the form $z = x$ and the y-axis the **imaginary axis** as it is formed by complex numbers of the form $z = iy$. In fact, we can identify 1 with the unit-vector $(1, 0)$ and i with the unit-vector $(0, 1)$. Thus, 1 and i are orthogonal vectors in the plane and if we think about $z = x + iy$ we can view (x, y) as the coordinates[3] with respect to the basis $\{1, i\}$. Let $w = a+ib$ be another *vector* and note the standard dot-product of such vectors is simply the sum of the products of their horizontal and vertical components:

$$\langle z, w\rangle = xa + yb$$

You can calculate that $\mathbf{Re}(z\overline{w}) = xa + yb$ thus a formula for the dot-product of two-dimensional vectors written in complex notation is just:

$$\langle z, w\rangle = \mathbf{Re}(z\overline{w}).$$

[2]Euler 1749 had this idea, see [N] page 60.

[3]if you've not had linear algebra yet then you may read on without worry

You may also recall from calculus III that the length of a vector $\vec{A}$ is calculated from $\sqrt{\vec{A} \bullet \vec{A}}$. Hence, in our current complex notation the length of the vector z is given by $|z| = \sqrt{\langle z, z\rangle} = \sqrt{z\bar{z}}$. If you are a bit lost, read on for now, we can also simply understand the $|z| = \sqrt{z\bar{z}}$ formula directly:

$$(a+ib)(\overline{a+ib}) = (a+ib)(a-ib) = a^2+b^2 \qquad \Rightarrow \qquad |z| = \sqrt{z\bar{z}}.$$

Properties of conjugation and modulus are fun to work out:

$$\overline{z+w} = \overline{z} + \overline{w} \qquad \& \qquad \overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \& \qquad \overline{\overline{z}} = z \qquad \& \qquad |zw| = |z||w|.$$

We will make use of the following throughout our study:

$$|z+w| \leq |z| + |w|, \qquad |z-w| \geq |z| - |w| \qquad \& \qquad |z| = 0 \;\text{ if and only if }\; z = 0.$$

also, the geometrically obvious:

$$\mathbf{Re}(z) \leq |z| \qquad \& \qquad \mathbf{Im}(z) \leq |z|.$$

We now are ready to work out the formula for the reciprocal of a complex number. Suppose $z \neq 0$ and $z = a + ib$ we want to find $w = c + id$ such that $zw = 1$. In particular:

$$(a+ib)(c+id) = 1 \quad \Rightarrow \quad ac - bd = 1, \;\; \& \;\; ad + bc = 0$$

You can try to solve these directly, but perhaps it will be more instructive[4] to discover the formula for the reciprocal by a formal calculation:

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} \qquad \Rightarrow \qquad \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}.$$

I said *formal* as the calculation in some sense assumes properties which are not yet justified. In any event, it is simple to check that the reciprocal formula is valid: notice, if $z \neq 0$ then $|z| \neq 0$ hence

$$z \cdot \left(\frac{\bar{z}}{|z|^2}\right) = z \cdot \left(\frac{\bar{z}}{|z|^2}\right) = z \cdot \left(\frac{1}{|z|^2} \cdot \bar{z}\right) = \frac{1}{|z|^2}(z\bar{z}) = \frac{1}{|z|^2}|z|^2 = 1.$$

The calculation above proves $z^{-1} = \bar{z}/|z|^2$.

Example 1.1.4.

$$\frac{1}{i} = \frac{-i}{|i|^2} = \frac{-i}{1} = -i.$$

Of course, this can easily be seen from the basic identity $ii = -1$ *which gives* $1/i = -i$.

Example 1.1.5.

$$(1+2i)^{-1} = \frac{1-2i}{|1+2i|^2} = \frac{1-2i}{1+4} = \frac{1-2i}{5}.$$

A more pedantic person would insist you write the standard **Cartesian form** $\frac{1}{5} - i\frac{2}{5}$.

The only complex number which does not have a multiplicative inverse is 0. This is part of the reason that $\mathbb{C}$ forms a **field**. A field is a set which allows addition and multiplication such that the only element without a multiplicative inverse is the additive identity (aka "zero"). There is a more precise definition given in abstract algebra texts, I'll leave that for you to discover. That said, it is perhaps useful to note that $\mathbb{Z}/p\mathbb{Z}$ for p prime, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields. Furthermore, it is sometimes useful to have notation for the set of complex numbers which admit a multicative inverse;

[4]this calculation is how to find $(a+ib)^{-1}$ for explicit examples

Definition 1.1.6. *The group of nonzero complex numbers is denoted* $\mathbb{C}^{\times}$ *where* $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$.

If we envision $\mathbb{C}$ as the plane, this is the plane with the origin removed. For that reason $\mathbb{C}^{\times}$ is also known as the **punctured plane**. The term **group** is again from abstract algebra and it refers to the multiplicative structure paired with $\mathbb{C}^{\times}$. Notice that $\mathbb{C}^{\times}$ is not closed under addition since $z \in \mathbb{C}^{\times}$ implies $-z \in \mathbb{C}^{\times}$ yet $z + (-z) = 0 \notin \mathbb{C}^{\times}$. I merely try to make some connections with your future course work in abstract algebra.

The complex conjugate gives us nice formulas for the real and imaginary parts of $z = x + iy$. Notice that if we add $z = x + iy$ and $\bar{z} = x - iy$ we obtain $z + \bar{z} = 2x$. Likewise, subtraction yields $z - \bar{z} = 2iy$. Thus as (by definition) $x = \mathbf{Re}(z)$ and $y = \mathbf{Im}(z)$ we find:

$$\mathbf{Re}(z) = \frac{1}{2}(z + \bar{z}) \qquad \& \qquad \mathbf{Im}(z) = \frac{1}{2i}(z + \bar{z})$$

In summary, for each $z \in \mathbb{C}$ we have $z = \mathbf{Re}(z) + i\mathbf{Im}(z)$.

Example 1.1.7.

$$|z| = |\mathbf{Re}(z) + i\mathbf{Im}(z)| \leq |\mathbf{Re}(z)| + |i\mathbf{Im}(z)| = |\mathbf{Re}(z)| + |i||\mathbf{Im}(z)| = |\mathbf{Re}(z)| + |\mathbf{Im}(z)|.$$

An important basic type of function in complex function theory is a polynomial. These are sums of power functions. Notice that z^n is defined inductively just as in the real case. In particular, $z^0 = 1$ and $z^n = zz^{n-1}$ for all $n \in \mathbb{N}$. The story of $n \in \mathbb{C}$ waits for a future section.

Definition 1.1.8. *A* **complex polynomial of degree** $n \geq 0$ *is a function of the form:*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_o$$

for $z \in \mathbb{C}$. *The set of all polynomials in* z *is denoted* $\mathbb{C}[z]$.

The theorem which follows makes complex numbers an indispensable tool for polynomial algebra.

Theorem 1.1.9. Fundamental Theorem of Algebra *Every complex polynomial* $p(z) \in \mathbb{C}[z]$ *of degree* $n \geq 1$ *has a factorization*

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $z_1, z_2, \dots, z_k$ *are distinct and* $m_j \geq 1$ *for all* $j \in \mathbb{N}_k$. *Moreover, this factorization is unique upto a permutation of the factors.*

I prefer the statement above (also given on page 4 of Gamelin) to what is sometimes given in other books. The other common version is: *every nonconstant complex polynomial has a zero.* Let us connect this to our version. Recall[5] the **factor theorem** states that if $p(z) \in \mathbb{C}[z]$ with $deg(p(z)) = n \geq 1$ and z_o satisfies $p(z_o) = 0$ then $(z - z_o)$ is a **factor** of $p(z)$. This means there exists $q(z) \in \mathbb{C}[z]$ with $deg(q(z)) = n - 1$ such that $p(z) = (z - z_o)q(z)$. It follows that we may completely factor a polynomial by repeated application of the alternate version of the Fundamental Theorem of Algebra and the factor theorem.

[5] I suppose this was only presented in the case of real polynomials, but it also holds here. See Fraleigh or Dummit and Foote or many other good abstract algebra texts for how to build polynomial algebra from scratch. That is not our current purpose so I resist the temptation.

Example 1.1.10. *Let* $p(z) = (z+1)(z+2-3i)$ *note that* $p(z) = z^2+(3-3i)z-3i$*. This polynomial has zeros of* $z_1 = -1$ *and* $z_2 = -2+3i$*. These are not in a* **conjugate pair** *but this is not surprising as* $p(z) \notin \mathbb{R}[z]$*. The notation* $\mathbb{R}[z]$ *denotes polynomials in* z *with coefficients from* $\mathbb{R}$*.*

Example 1.1.11. *Suppose* $p(z) = (z^2+1)((z-1)^2+9)$*. Notice* $z^2+1 = z^2-i^2 = (z+i)(z-i)$*. We are inspired to do likewise for the first factor which is already in completed-square format:*

$$(z-1)^2+9 = (z-1)^2-9i^2 = (z-1-3i)(z-1+3i).$$

Thus, $p(z) = (z+i)(z-i)(z-1-3i)(z-1+3i)$*. Notice* $p(z) \in \mathbb{R}[z]$ *is clear from the initial formula and we do see the complex zeros of* $p(z)$ *are arranged in conjugate pairs* $\pm i$ *and* $1 \pm 3i$*.*

The example above is no accident: complex algebra sheds light on real examples. Since $\mathbb{R} \subseteq \mathbb{C}$ it follows we may naturally view $\mathbb{R}[z] \subseteq \mathbb{C}[z]$ thus the Fundamental Theorem of Algebra applies to polynomials with real coefficients in this sense: to solve a real problem we enlarge the problem to the corresponding complex problem where we have the mathematical freedom to solve the problem in general. Then, upon finding the answer, we drop back to the reals to present our answer. I invite the reader to derive the Fundamental Theorem of Algebra for $\mathbb{R}[z]$ by applying the Fundamental Theorem of Algebra for $\mathbb{C}[z]$ to the special case of real coefficients. Your derivation should probably begin by showing a complex zero for a polynomial in $\mathbb{R}[z]$ must come with a conjugate zero.

The importance of taking a complex view was supported by Gauss throughout his career. From a letter to Bessel in 1811 [R91](p.1):

> At the very beginning I would ask anyone who wants to introduce a new function into analysis to clarify whether he intends to confine it to real magnitudes [real values of its argument] and regard the imaginary values as just vestigial - or whether he subscribes to my fundamental proposition that in the realm of magnitudes the imaginary ones $a+b\sqrt{-1} = a+bi$ have to be regarded as enjoying equal rights with the real ones. We are not talking about practical utility here; rather analysis is, to my mind, a self-sufficient science. It would lose immeasurably in beauty and symmetry from the rejection of any fictive magnitudes. At each stage truths, which otherwise are quite generally valid, would have to be encumbered with all sorts of qualifications.

Gauss used the complex numbers in his dissertation of 1799 to prove the Fundamental Theorem of Algebra. Gauss offered four distinct proofs over the course of his life. See Chapter 4 of [N91] for a discussion of Gauss' proofs as well as the history of the Fundamental Theorem of Algebra. Many original quotes and sources are contained in that chapter which is authored by Reinhold Remmert.

1.2 On the Existence of Complex Numbers

Euler's work from the eigthteenth century involves much calculation with complex numbers. It was Euler who in 1777 introduced the notation $i = \sqrt{-1}$ to replace $a+b\sqrt{-1}$ with $a+ib$ (see [R91] p. 10). As is often the case in this history of mathematics, we used complex numbers long before we had a formal construction which proved the existence of such numbers. In this subsection I add some background about how to **construct** complex numbers. In truth, my true concept of complex numbers is already given in what was already said in this section in the discussion up to Definition 1.1.3 (after that point I implicitly make use of Model I below). In particular, I would claim a mature viewpoint is that a complex number is defined by it's properties. That said, it is good to give a construction which shows such objects do exist. However, it's also good to realize the construction is not written in stone as it may well be replaced with some *isomorphic* copy. There are three main models:

Model I: complex numbers as points in the plane: Gauss proposed the following construction: $\mathbb{C}_{Gauss} = \mathbb{R}^2$ paired with the multiplication $\star$ and addition rules below:

$$(a,b) + (c,d) = (a+c, b+d) \qquad (a,b) \star (c,d) = (ac - bd, ad + bc)$$

for all $(a,b),(c,d) \in \mathbb{C}_{Gauss}$. What does this have to do with $\sqrt{-1}$? Consider,

$$(1,0) \star (a,b) = (a,b)$$

Thus, multiplication by $(1,0)$ is like multiplying by 1. Also,

$$(0,1) \star (0,1) = (-1,0)$$

It follows that $(0,1)$ is like i. We can define a mapping $\Psi : \mathbb{C}_{Gauss} \to \mathbb{C}$ by $\Psi(a,b) = a + ib$. This mapping has $\Psi(z+w) = \Psi(z) + \Psi(w)$ as well as $\Psi(z \star w) = \Psi(z)\Psi(w)$. We observe that Ψ is a one-one correspondence of $\mathbb{C}_{Gauss}$ and $\mathbb{C}$ which preserves multiplication and addition. Intuitively, the existence of Ψ means that $\mathbb{C}$ and $\mathbb{C}_{Gauss}$ are the *same* object viewed in different notation[6].

Model II: complex numbers as matrices of a special type: perhaps Cayley was the first to [7] propose the following construction:

$$\mathbb{C}_{matrix} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \;\middle|\; a,b \in \mathbb{R} \right\}$$

Addition is matrix addition and we multiply in $\mathbb{C}_{matrix}$ using the standard matrix multiplication:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix}.$$

In matrices, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ serves as the multiplicative identity (it is like 1) whereas the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is analogus to i. Notice, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The mapping $\Phi : \mathbb{C}_{matrix} \to \mathbb{C}$ defined by $\Phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + ib$ is a one-one correspondence for which the algebra of matrices transfers to the algebra of complex numbers.

Model III: complex numbers as an extension field of $\mathbb{R}$: The set of real polynomials in x is denoted $\mathbb{R}[x]$. If we define $\mathbb{C}_{extension} = \mathbb{R}[x]/ < x^2 + 1 >$ then the multiplication and addition in this set is essentially that of polynomials. However, strict polynomial equality is replaced with congruence modulo $x^2 + 1$. Suppose we use $[f(x)]$ to denote the equivalence class of $f(x)$ modulo $x^2 + 1$ then as a point set:

$$[f(x)] = \{f(x) + (x^2+1)h(x) \mid h(x) \in \mathbb{R}[x]\}.$$

[6] the careful reader is here frustrated by the fact I have yet to say what $\mathbb{C}$ is as a point set

[7] I asked this at the math stackexchange site and it appears Cayley knew of these in 1858, see the link for details.

More to the point, $[x^2+1]=[0]$ and $[x^2]=[-1]$. From this it follows:

$$[a+bx][c+dx]=[(a+bx)(c+dx)]=[ac+(ad+bc)x+bdx^2]=[ac-bd+(ad+bc)x].$$

In $\mathbb{C}_{extension}$ the constant polynomial class [1] serves as the multiplicative identity whereas $[x]$ is like i. Furthermore, the mapping $\Xi([a+bx])=a+bi$ gives a one-one correspondence which preserves the addition and multiplication of $\mathbb{C}_{extension}$ to that of $\mathbb{C}$. The technique of field extensions is discussed in some generality in the second course of a typical abstract algebra sequence. Cauchy found this formulation in 1847 see [N91] p. 63.

Conclusion: as point sets $\mathbb{C}_{Gauss}, \mathbb{C}_{matrix}, \mathbb{C}_{extension}$ are not the same. However, each one of these objects provides the algebraic structure which (in my view) defines $\mathbb{C}$. We could use any of them as the complex numbers. For the sake of being concrete, I will by default use $\mathbb{C}=\mathbb{C}_{Gauss}$. But, I hope you can appreciate this is merely a **choice**. But, it's also a good choice since geometrically it is natural to identify the plane with $\mathbb{C}$. You might take a moment to appreciate we face the same foundational issue when we face the question of what is $\mathbb{R}, \mathbb{Q}, \mathbb{N}$ etc. I don't think we ever constructed these in our course work. You have always worked formally in these systems. It sufficed to accept truths about $\mathbb{N}, \mathbb{Q}$ or $\mathbb{R}$, you probably never required your professor to show you such a system could indeed exist. Rest assured, they exist.

Remark: it will be our custom whenever we write $z=x+iy$ it is understood that $x=\mathbf{Re}(z)\in\mathbb{R}$ and $y=\mathbf{Im}(z)\in\mathbb{R}$. If we write $z=x+iy$ and intend $x,y\in\mathbb{C}$ then it will be our custom to make this explicitly known. This will save us a few hundred unecessary utterances in our study.

1.3 Polar Representations

Polar coordinates in the plane are given by $x=r\cos\theta$ and $y=r\sin\theta$ where we define $r=\sqrt{x^2+y^2}$. Observe that $z=x+iy$ and $r=|z|$ hence:

$$z=|z|(\cos\theta+i\sin\theta).$$

The **standard angle** is measured CCW from the positive x-axis. There is considerable freedom in our choice of θ. For example, we identify geometrically $-\pi/2, 3\pi/2, 7\pi/2, \dots$. It is useful to have a notation to express the totality of this ambiguity as well as to remove it by a standard choice:

Definition 1.3.1. *Let $z\in\mathbb{C}$ with $z\neq 0$.* **Principle argument** *of z is the $\theta_o\in(-\pi,\pi]$ for which $z=|z|(\cos\theta_o+i\sin\theta_o)$. We denote the principle argument by $Arg(z)=\theta_o$. The* **argument** *of z is denoted $arg(z)$ which is the* **set** *of all $\theta\in\mathbb{R}$ such that $z=|z|(\cos\theta+i\sin\theta)$. Let α be a real number then define*

$$Arg_\alpha^+(z)=arg(z)\cap(\alpha-2\pi,\alpha] \qquad \& \qquad Arg_\alpha(z)=arg(z)\cap[\alpha,2\pi+\alpha).$$

From basic trigonometry we find: for $z\neq 0$,

$$arg(z)=Arg(z)+2\pi\mathbb{Z}=\{Arg(z)+2\pi k \mid k\in\mathbb{Z}\}.$$

Notice that $arg(z)$ is not a function on $\mathbb{C}$. Instead, $arg(z)$ is a **multiply-valued function**. You should recall a function is, by definition, **single-valued**. In contrast, the Principle argument is a function from the punctured plane $\mathbb{C}^\times=\mathbb{C}-\{0\}$ to $(-\pi,\pi]$. Notice, $Arg_\alpha^+:\mathbb{C}^\times\to(\alpha-2\pi,\alpha]$ and

$Arg_\alpha : \mathbb{C}^\times \to [\alpha, \alpha+2\pi)$. Both Arg_α^+ and Arg_α provide a continuous[8] assignment of standard angle on the plane with the ray at angle α removed. The notation Arg_α^+ was invented for these notes, however the notation Arg_α is also used in Churchill's text and $Arg = Arg_\pi^+$ is a standard notation across many texts. In practice, I usually choose angle $\theta \in [0, 2\pi)$ when choosing a standard angle hence I prefer Arg_0^- as a default. We should choose a **branch** of the argument which best fits whatever application we face.

Example 1.3.2. *Let* $z = 1 - i$ *then* $Arg(z) = -\pi/4$ *and* $arg(z) = \{-\pi/4 + 2\pi k \mid k \in \mathbb{Z}\}$.

Example 1.3.3. *Let* $z = -2 - 3i$. *We can calculate* $\tan^{-1}(-3/-2) \approxeq 0.9828$. *Furthermore, this complex number is found in quadrant III hence the standard angle is approximately* $\theta = 0.9828 + \pi = 4.124$. *In our notation,* $Arg_0^(- 2 - 3i) = 0.9828 + \pi$. *Notice,* $\theta \neq Arg(z)$ *since* $4.124 \notin (-\pi, \pi]$. *We substract* 2π *from* θ *to obtain the approximate value of* $Arg(z)$ *is* -2.159. *To be precise,* $Arg(z) = \tan^{-1}(3/2) - \pi$ *and*

$$arg(z) = \tan^{-1}(3/2) - \pi + 2\pi\mathbb{Z}.$$

At this point it is useful to introduce a notation which simultaneously captures sine and cosine and their appearance in the formulas at the beginning of this section. What follows here is commonly known as **Euler's formula**. Incidentally, it is mentioned in [E91] (page 60) that this formula appeared in Euler's writings in 1749 and the manner in which he wrote about it implicitly indicates that Euler already understood the geometric interpretation of $\mathbb{C}$ as a plane. It fell to nineteenth century mathematicians such as Gauss to clarify and demystify $\mathbb{C}$. It was Gauss who first called $\mathbb{C}$ **complex numbers** in 1831 [E91](page 61). This is what Gauss had to say about the term "imaginary" in a letter from 1831 [E91](page 62)

> It could be said in all this that so long as imaginary quantities were still based on a fiction, they were not, so to say, fully accepted in mathematics but were regarded rather as something to be tolerated; they remained far from being given the same status as real quantities. There is no longer any justification for such discrimination now that the metaphysics of imaginary numbers has been put in a true light and that it has been shown that they have just as good a real objective meaning as the negative numbers.

I only wish the authority of Gauss was properly accepted by current teachers of mathematics. It seems to me that the education of precalculus students concerning complex numbers is far short of where it ought to reach. Trigonometry and two dimensional geometry are both greatly simplified by the use of complex notation.

Definition 1.3.4. *Let* $\theta \in \mathbb{R}$ *and define the* **imaginary exponential** *denoted* $e^{i\theta}$ *by:*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For $z \neq 0$, *if* $z = |z|e^{i\theta}$ *then we say* $|z|e^{i\theta}$ *is a* **polar form** *of* z.

The polar form is not unique unless we restrict the choice of θ.

Example 1.3.5. *Let* $z = -1 + i$ *then* $|z| = \sqrt{2}$ *and* $Arg(z) = \frac{3\pi}{4}$. *Thus,* $-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$.

Example 1.3.6. *If* $z = i$ *then* $|z| = 1$ *and* $Arg(z) = \frac{\pi}{2}$ *hence* $i = e^{i\frac{\pi}{2}}$.

[8] whatever that means...

Properties of the imaginary exponential follow immediately from corresponding properties for sine and cosine. For example, since sine and cosine are never zero at the same angle we know $e^{i\theta} \neq 0$. On the other hand, as $\cos(0) = 1$ and $\sin(0) = 0$ hence $e^0 = \cos(0) + i\sin(0) = 1$ (if this were not the case then the notation of $e^{i\theta}$ would be dangerous in view of what we know for exponentials on $\mathbb{R}$). The imaginary exponential also supports the law of exponents:

$$e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}.$$

This follows from the known adding angle formulas $\cos(\theta+\beta) = \cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)$ and $\sin(\theta+\beta) = \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta)$. However, the imaginary exponential does not behave exactly the same as the real exponentials. It is far from injective[9] In particular, we have 2π-periodicity of the imaginary exponential function: for each $k \in \mathbb{Z}$,

$$e^{i(\theta+2\pi k)} = e^{i\theta}.$$

This follows immediately from the definition of the imaginary exponential and the known trigonometric identities: $\cos(\theta + 2\pi k) = \cos(\theta)$ and $\sin(\theta + 2\pi k) = \cos(\theta)$ for $k \in \mathbb{Z}$. Given the above, we have the following modication of the $1-1$ principle from precalculus:

$$e^{i\theta} = e^{i\beta} \quad \Rightarrow \quad \theta - \beta \in 2\pi\mathbb{Z}.$$

Example 1.3.7. *To solve $e^{3i} = e^{i\theta}$ yields $3 - \theta = 2\pi k$ for some $k \in \mathbb{Z}$. Therefore, the solutions of the given equation are of the form $\theta = 3 - 2\pi k$ for $k \in \mathbb{Z}$.*

In view of the addition rule for complex exponentials the multiplication of complex numbers in polar form is very simple:

Example 1.3.8. *Let $z = re^{i\theta}$ and $w = se^{i\beta}$ then*

$$zw = re^{i\theta}se^{i\beta} = rse^{i(\theta+\beta)}.$$

We learn from the calculation above that the product of two complex numbers has a simple geometric meaning in the polar notation. The magnitude of $|zw| = |z||w|$ and the angle of zw is simply the sum of the angles of the products. To be careful, we can show:

$$arg(zw) = arg(z) + arg(w)$$

where the addition of sets is made in the natural manner[10]:

$$arg(z) + arg(w) = \{\theta' + \beta' \mid \theta' \in arg(z),\ \beta' \in arg(w)\}.$$

If we multiply $z \neq 0$ by $e^{i\beta}$ then we **rotate** $z = |z|e^{i\theta}$ to $ze^{i\beta} = |z|e^{i(\theta+\beta)}$. It follows that multiplication by imaginary exponentials amounts to rotating points in the complex plane.
The formulae below can be derived by an inductive argument and the addition law for imaginary exponentials.

Theorem 1.3.9. de Moivere's formulae *let $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$ then $(e^{i\theta})^n = e^{in\theta}$.*

[9] or 1-1 if you prefer that terminology, the point is multiple inputs give the same output.
[10] Let $S, T \subseteq \mathbb{C}$ and $c \in \mathbb{C}$ then we define

$$cS = \{cs \mid s \in S\} \qquad c + S = \{c + s \mid s \in S\} \qquad S + T = \{s + t \mid s \in S, t \in T\}.$$

Example 1.3.10. *De Moivere gives us* $(e^{i\theta})^2 = e^{2i\theta}$ *but* $e^{i\theta} = \cos\theta + i\sin\theta$ *thus squaring yields:*

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

However, the definition of the imaginary exponential gives $e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$*. Thus,*

$$\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos(2\theta) + i\sin(2\theta).$$

Equating the real and imaginary parts separately yields:

$$\cos^2\theta - \sin^2\theta = \cos(2\theta), \qquad \& \qquad 2\cos\theta\sin\theta = \sin(2\theta).$$

These formulae of de Moivere were discovered between 1707 and 1738 by de Moivere then in 1748 they were recast in our present formalism by Euler [R91] see p. 150. Incidentally, page 149 of [R91] gives a rather careful justification of the polar form of a complex number which is based on the application of *function theory*[11]. I have relied on your previous knowledge of trigonometry which may be very non-rigorous. In fact, I should mention, at the moment $e^{i\theta}$ is simply a convenient notation with nice properties, but, later it will be the inevitable extension of the real exponential to complex values. That mature viewpoint only comes much later as we develop a large part of the theory, so, in the interest of not depriving us of exponentials until that time I follow Gamelin and give a transitional definition. It is important we learn how to calculate with the imaginary exponential as it is ubiquitous in examples throughout our study.

Definition 1.3.11. *Suppose* $n \in \mathbb{N}$ *and* $w, z \in \mathbb{C}$ *such that* $z^n = w$ *then* z *is an* **n-th root of w***. The* **set of all n-th roots** *of* w *is (by default) denoted* $w^{1/n}$*.*

The polar form makes quick work of the algebra here. Suppose $w = \rho e^{i\phi}$ and $z = re^{i\theta}$ such that $z^n = w$ for some $n \in \mathbb{N}$. Observe, $z^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^n e^{in\theta}$ hence we wish to find all solutions of:

$$r^n e^{in\theta} = \rho e^{i\phi} \qquad \star.$$

Take the modulus of the equation above to find $r^n = \rho$ hence $r = \sqrt[n]{\rho}$ where we use the usual notation for the (unique) n-th positive root of $r > 0$. Apply $r = \sqrt[n]{\rho}$ to $\star$ and face what remains:

$$e^{in\theta} = e^{i\phi}.$$

We find $n\theta - \phi \in 2\pi\mathbb{Z}$. Thus, $\theta = \frac{2\pi k + \phi}{n}$ for some $k \in \mathbb{Z}$. At first glance, you might think there are **infinitely** many solutions ! However, it happens[12] as k ranges over $\mathbb{Z}$ notice that $e^{i\theta}$ simply we cycles back to the same solutions over and over. In particular, if we restrict to $k \in \{0, 1, 2, \dots, n-1\}$ it suffices to cover all possible n-th roots of w:

$$(\rho e^{i\phi})^{1/n} = \left\{ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}, \sqrt[n]{\rho} e^{i\frac{2\pi+\phi}{n}}, \dots, \sqrt[n]{\rho} e^{i\frac{2\pi(n-1)+\phi}{n}} \right\} \qquad \star^2.$$

We can clean this up a bit. Note that $\frac{2\pi k+\phi}{n} = \frac{2\pi k}{n} + \frac{\phi}{n}$ hence

$$e^{i\frac{2\pi k+\phi}{n}} = e^{i\left(\frac{2\pi k}{n} + \frac{\phi}{n}\right)} = e^{i\frac{2\pi k}{n}} e^{i\frac{\phi}{n}} = \left(e^{i\frac{2\pi}{n}}\right)^k e^{i\frac{\phi}{n}}$$

The term raised to the k-th power is important. Notice that once we have one element in the set of n-roots then we may **generate** the rest by repeated multiplication by $e^{i\frac{2\pi}{n}}$.

[11] in Remmert's text the term "function theory" means complex function theory

[12] it is very likely I prove this assertion in class via the slick argument found on page 150 of [R91].

Definition 1.3.12. *Suppose $n \in \mathbb{N}$ then $\omega = e^{i\frac{2\pi}{n}}$ is an* **primitive n-th root of unity**. *If $z^n = 1$ then we say z is an* **n-th root of unity**.

In terms of the language above, every n-th root of unity can be generated by raising the primitive root to some power between 0 and $n-1$. Returning once more to $\star^2$ we find, using $\omega = e^{i\frac{2\pi}{n}}$:

$$(\rho e^{i\phi})^{1/n} = \left\{ \sqrt[n]{\rho}e^{i\frac{\phi}{n}},\ \sqrt[n]{\rho}e^{i\frac{\phi}{n}}\omega,\ \sqrt[n]{\rho}e^{i\frac{\phi}{n}}\omega^2,\ \dots,\ \sqrt[n]{\rho}e^{i\frac{\phi}{n}}\omega^{n-1} \right\}.$$

We have to be careful with some real notations at this juncture. For example, it is no longer ok to conflate $\sqrt[n]{x}$ and $x^{1/n}$ even if $x \in (0, \infty)$. The quantity $\sqrt[n]{x}$ is, by definition, $w \in \mathbb{R}$ such that $w^n = x$. However, $x^{1/n}$ is a **set** of values ! (unless we specify otherwise for a specific problem)

Example 1.3.13. *The primitive fourth root of unity is $e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = \cos\pi/2 + i\sin\pi/2 = i$. Thus, noting that $1 = 1e^0$ we find:*

$$1^{1/4} = \{1,\ i,\ i^2,\ i^3\} = \{1, i, -1, -i\}$$

Geometrically, these are nicely arranged in perfect symmetry about the unit-circle.

Example 1.3.14. *Building from our work in the last example, it is easy to find $(3+3i)^{1/4}$. Begin by noting $|3+3i| = \sqrt{18}$ and $Arg(3+3i) = \pi/4$ hence $3+3i = \sqrt{18}e^{i\pi/4}$. Thus, note $\sqrt[4]{\sqrt{18}} = \sqrt[8]{18}$*

$$(3+3i)^{1/4} = \{\sqrt[8]{18}e^{i\pi/16}, i\sqrt[8]{18}e^{i\pi/16}, -\sqrt[8]{18}e^{i\pi/16}, -i\sqrt[8]{18}e^{i\pi/16}\}.$$

which could also be expressed as:

$$(3+3i)^{1/4} = \{\sqrt[8]{18}e^{i\pi/16}, \sqrt[8]{18}e^{5i\pi/16}, \sqrt[8]{18}e^{9i\pi/16}, \sqrt[8]{18}e^{13i\pi/16}\}.$$

Example 1.3.15. *$(-1)^{1/5}$ is found by noting $e^{2\pi i/5}$ is the primitive 5-th root of unity and $-1 = e^{i\pi}$ hence*

$$(-1)^{1/5} = \{e^{i\pi/5}, e^{i\pi/5}\omega, e^{i\pi/5}\omega^2, e^{i\pi/5}\omega^3, e^{i\pi/5}\omega^4\}.$$

Add a few fractions and use the 2π-periodicity of the imaginary exponential to see:

$$(-1)^{1/5} = \{e^{i\pi/5}, e^{3\pi i/5}, e^{5\pi i/5}, e^{7\pi i/5}, e^{9\pi i/5}\} = \{e^{i\pi/5}, e^{3\pi i/5}, -1, e^{-3\pi i/5}, e^{-\pi i/5}\}.$$

We can use the example above to factor $p(z) = z^5 + 1$. Notice $p(z) = 0$ implies $z \in (-1)^{1/5}$. Thus, the zeros of p are precisely the fifth roots of -1. This observation and the factor theorem yield:

$$p(z) = (z+1)(z - e^{i\pi/5})(z - e^{-i\pi/5})(z - e^{3i\pi/5})(z - e^{-3i\pi/5}).$$

If you start thinking about the pattern here (it helps to draw a picture which shows how the roots of unity are balanced below and above the x-axis) you can see that the conjugate pair factors for $p(z)$ are connected to that pattern. Furthermore, if you keep digging for patterns in factoring polynomials these appear again whenever it is possible. In particular, if $n \in 1 + 2\mathbb{Z}$ then -1 is a root of unity and all other roots are arranged in conjugate pairs.

The words below are a translation of the words written by Galois the night before he died in a duel at the age of 21:

> Go to the roots of these calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work.

Galois' theory is still interesting. You can read about it in many places. For example, see Chapter 14 of Dummit and Foote's *Abstract Algebra*.

Chapter 2

Functions of a Complex Variable

Let S and T be sets. A **function** from $f : S \to T$ is a single-valued assignment of $f(s) \in T$ for each $s \in S$. This clear definition of function was not clear until the middle of the nineteenth century. It is true that the term originates with Leibniz in 1692 to (roughly) describe magnitudes which depended on the point in question. Then Euler saw fit to call any analytic expression built from variables and some constants a *function*. In other words, Euler essentially defined a function by its formula. However, later, Euler did discuss an idea of an arbitrary function in his study of variational calculus. The clarity to state the modern definition apparently goes to Dirichlet. In 1837 he wrote:

> It is certainly not necessary that the law of dependence of $f(x)$ on x be the same throughout the interval; in fact one need not even think of the dependence as given by explicit mathematical operations.

See [R91] pages 37-38 for more detailed references.

Our goal in this chapter is to provide a catalog of basic functions of a complex variable. These examples are the basic ingredients to build new functions in all the usual ways. New functions from old can be obtained by sums, differences, scalar multiples, products, quotients and the far from trivial process of composition. Thus, while our basic list only goes to about a dozen, once we complete this chapter we have a literally infinite family of examples from which to test our understanding.

Three Questions which Guide our Investigations in this chapter:

(1.) Often in complex analysis we like to use f to denote a function on $\mathbb{C}$. To be more pedantic, f is **function on** $\mathbb{C}$ if there exists $S \subseteq \mathbb{C}$ for which $f : S \to \mathbb{C}$ is a function. Typically we write $f = u + iv$ to indicate that $\mathbf{Re}(f(z)) = u(z)$ and $\mathbf{Im}(f(z)) = v(z)$ for all $z \in S$. If $f = u + iv$ then we say u is the **real component function** of f and v is the **imaginary component function** of f. Collectively, we refer to u, v as the **component functions** of $f = u + iv$. Often we ask students to find the component functions of a given complex function. Generally, finding component functions requires a complete and heathly working knowledge of the identities which are related to the function in question. This will be made clear as we study different examples.

(2.) Another important question for a given complex function f is to determine the modulus of f. Notice $|f| = |u + iv| = \sqrt{u^2 + v^2}$ so are easily able to find the modulus of f if we already have found its component functions. Simplifying and categorizing the behavior of $|f(z)|$ as z varies will play an important role in our study of integration later in this course.

(3.) Finally, we are interested in the question of invertibility. If we're given $f : S \to \mathbb{C}$ then how can we find a suitable inverse function for f ? When is it possible to invert a complex function ? We'll see this is the most challenging of the questions raised in this chapter. In order to invert a complex function we have to understand what is the range of the function as well as how to (possibly) shrink the domain such that function is injective on the restricted domain.

Definition 2.0.1. *Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$ a function. If $U \subseteq S$ then we define the* **restriction of f to U** *to be the function $f|_U : U \to \mathbb{C}$ where $f|_U(z) = f(z)$ for all $z \in U$.*

Typically there are many choices on how to restrict a given function which means in some sense the natural object to view as the inverse of the function is a multiply-valued "function". Customarily we work with functions so it is necessary for us to choose a **branch** of the multiply-valued "function". It's not that complicated if we just keep the goal in mind. The goal is to restrict the function to a domain for which the chosen branch of the multiply-valued inverse "function" takes the restricted domain as its range. This naturally leads to the concept of a *Riemann Surface*, but that is beyond this course.

To understand the range of a complex function it is usually helpful to envision the function as a mapping from the z-plane to the w-plane according to the rule $w = f(z)$. Usually we use u, v as the Cartesian variables in the w-plane so $w = u + iv$. We can't (at least I can't) visualize the graph of a complex function directly since it would require a four dimensional space.

Remark: I would strongly recommend you **not** attempt to memorize *all* the facts presented in this chapter[1]. You need to memorize the definitions and internalize them with some facts which you can check to root out confusion. Ideally, you want to be able to rederive the things which we derive in this chapter. Once you understand the interplay between the different elements then knowledge from one example can often be ported to a different example. For example, sine and cosine and sinh and cosh are really the *same* function. In some sense, they're all just the complex exponential. But, this is a lie. They're not the SAME. I'm just trying to sell you something. Let us cease with these fuzzy pleasantries and get to the examples!

2.1 Polynomial Functions

Definition 2.1.1. *If $a_n, \dots, a_1, a_0 \in \mathbb{C}$ where $a_n \neq 0$ then*

$$f(z) = a_n z^n + \cdots + a_1 z + a_0$$

defines a degree n **polynomial function** *where $dom(f) = \mathbb{C}$.*

Example 2.1.2. *Let $f(z) = 3iz^2 + 2 + 5i$ then as $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ we find*

$$f(z) = 3i(x^2 - y^2 + 2ixy) + 2 + 5i = 2 - 6xy + i[3(x^2 - y^2) + 5]$$

we find f has component functions $u(z) = 2 - 6xy$ and $v(z) = 3(x^2 - y^2) + 5$.

[1] I also strongly recommend you not take the path of memorizing nothing in this chapter at all, there is a middle ground of efficient understanding

Example 2.1.3. *Let $f(z) = z^4$ then using the binomial theorem we calculate:*

$$z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$

therefore, as $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$,

$$f(z) = \underbrace{x^4 - 6x^2y^2 + y^4}_{u(z)} + i[\underbrace{4x^3y - 4xy^3}_{v(z)}]$$

To find the modulus of $f(z)$ we could use $|f| = \sqrt{u^2+v^2}$, but that would be very foolish here. Instead, use properties of the modulus on the given formula:

$$|f(z)| = |z^4| = |z|^4 = |x+iy|^4 = (|x+iy|^2)^2 = (x^2+y^2)^2.$$

We'll return to the problem of studying the square function at the conclusion of this chapter.

2.2 The Exponential Function

In this section we extend our transitional definition for the exponential to complex values. What follows is simply the combination of the real and imaginary exponential functions:

Definition 2.2.1. *The* **complex exponential function** *is defined by $z \mapsto e^z$ where for each $z \in \mathbb{C}$ we define $e^z = e^{\mathbf{Re}(z)}e^{\mathbf{Im}(z)}$. In particular, if $x, y \in \mathbb{R}$ and $z = x + iy$,*

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\left(\cos(y) + i\sin(y)\right).$$

When convenient, we also use the notation $e^z = \exp(z)$ to make the argument of the exponential more readable. [2]. Consider, as $|e^{iy}| = \sqrt{e^{iy}e^{-iy}} = \sqrt{e^0} = 1$ we find

$$|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x.$$

The magnitude of the complex exponential is unbounded as $x \to \infty$ whereas the magnitude approaches zero as $x \to -\infty$. If $z = x+iy$ then $arg(e^{x+iy}) = \{y + 2\pi k \mid k \in \mathbb{Z}\}$. Since $e^{x+iy} = e^x e^{iy}$ it is clear that e^x does not change the direction of e^{x+iy}; $arg(e^{x+iy}) = arg(e^{iy})$.

Observe domain$(e^z) = \mathbb{C}$ however range$(e^z) = \mathbb{C}^\times$ as we know $e^{iy} \neq 0$ for all $y \in \mathbb{R}$. Furthermore, the complex exponential is not injective precisely because the imaginary exponential is not injective. If two complex exponentials agree then their arguments need not be equal. In fact:

$$e^z = e^w \qquad \Leftrightarrow \qquad z - w \in 2\pi i\mathbb{Z}.$$

Moreover, $e^z = 1$ iff $z = 2\pi ik$ for some $k \in \mathbb{Z}$. The complex exponential function is a $2\pi i$-periodic function; $e^{z+2\pi i} = e^z$. We also have

$$e^{z+w} = e^z e^w \qquad \& \qquad (e^z)^{-1} = 1/e^z = e^{-z}.$$

The proof of the addition rule above follows from the usual laws of exponents for the real exponential function as well as the addition rules for cosine and sine which give the addition rule for imaginary

[2]Notice, we have not given a careful definition of e^x here for $x \in \mathbb{R}$. We assume, for now, the reader has some base knowledge from calculus which makes the exponential function at least partly rigorous. Later in this our study we find a definition for the exponential which supercedes the one given here and provides a rigorous underpinning for all these fun facts

exponentials. Of course, $e^ze^{-z} = e^{z-z} = e^0 = 1$ shows $1/e^z = e^{-z}$ but it is also fun to work it out from our previous formula for the reciprocal $1/z = \bar{z}/|z|^2$. We showed $|e^{x+iy}| = e^x$ hence:

$$\frac{1}{e^z} = \frac{e^xe^{-iy}}{(e^x)^2} = e^{-x}e^{-iy} = e^{-(x+iy)} = e^{-z}.$$

As is often the case, the use of x, y notation clutters the argument.

To understand the geometry of $z \mapsto e^z$ we study how the exponential maps the z-plane to the $w = u + iv$-plane where $w = e^z$. Often we look at how lines or circles transform. In this case, lines work well. I'll break into cases to help organize the thought:

(1.) A **vertical line** in the $z = x + iy$-plane has equation $x = x_o$ whereas y is free to range over $\mathbb{R}$. Consider, $e^{x_o+iy} = e^{x_o}e^{iy}$. As y-varies we trace out a circle of radius e^{x_o} in the $w = u+iv$-plane. In particular, it has equation $u^2 + v^2 = (e^{x_o})^2$. Notice that we need only let y range over an interval of length 2π in order to range over the entire circle of radius e^{x_o}.

(2.) A **horizontal line** in the $z = x + iy$-plane has equation $y = y_o$ whereas x is free to range over $\mathbb{R}$. Consider, $e^{x+iy_o} = e^xe^{iy_o}$. As x-varies we trace out a ray at standard angle y_o in the w-plane.

You can see that if $S = \mathbb{R} \times (-\pi, \pi]$ then $e^S = \mathbb{C}^\times$. Likewise, $S_\alpha = \mathbb{R} \times [\alpha, \alpha + 2\pi)$ maps to $\mathbb{C}^\times$ under the complex exponential. In fact, the complex exponential is injective on S_α for any choice of α. This brings us to introduce the inverse function to the complex exponential. Well, to be more precise, a branch of the multiply-valued logarithm is an inverse function to the restriction of of the exponential to an appropriate set such as S or S_α.

2.3 Logarithms

Definition 2.3.1. *The* **principal logarithm** *is defined by $Log(z) = \ln|z| + iArg(z)$ for each $z \in \mathbb{C}^\times$. In particular, for $z = |z|e^{i\theta}$ with $-\pi < \theta \leq \pi$ we define:*

$$Log(x + iy) = \ln|z| + i\theta.$$

We can also simplify the formula by the power property of the real logarithm to

$$Log(x + iy) = \frac{1}{2}\ln(x^2 + y^2) + iArg(x + iy).$$

Notice: we use "ln" for the **real** logarithm function. In contrast, we reserve the notations "*log*" and "*Log*" for complex arguments. Please do not write $\ln(1 + i)$ as in our formalism that is just nonsense. There is a multiply-valued function of which this is just one branch. In particular:

Definition 2.3.2. *The* **logarithm** *is defined by $log(z) = \ln|z| + iarg(z)$ for each $z \in \mathbb{C}^\times$. In particular, for $z = x + iy \neq 0$*

$$log(x + iy) = \left\{\ln\sqrt{x^2 + y^2} + i[Arg(x + iy) + 2\pi k] \mid k \in \mathbb{Z}\right\}.$$

Example 2.3.3. *To calculate $Log(1+i)$ we change to polar form $1+i = \sqrt{2}e^{i\pi/4}$. Thus*

$$Log(1+i) = \ln\sqrt{2} + i\pi/4.$$

Note $arg(1+i) = \pi/4 + 2\pi\mathbb{Z}$ hence

$$log(1+i) = \ln\sqrt{2} + i\pi/4 + 2\pi i\mathbb{Z}.$$

There are many values of the logarithm of $1+i$. For example, $\ln\sqrt{2} + 9i\pi/4$ and $\ln\sqrt{2} - 7i\pi/4$ are also a logarithms of $1+i$. These are the beginnings of the two tails[3] *which Gamelin illustrates on page 22.*

The concept of a split complex plane is important in that while we can define the logarithm on the entire punctured plane, we cannot maintain continuity unless we delete a ray where the angle jumps. This is a fundamental issue we must face and keep in mind carefully both for theory and calculation. Let us introduce some notation[4]. for split complex planes:

Definition 2.3.4. *Let $\mathbb{C}^{\alpha} = \mathbb{C} - \{te^{i\alpha} \mid t \geq 0\}$ denote the* **split complex plane** *with the ray at angle α removed. We also denote the complex plane with the positive real axis removed as $\mathbb{C}^{+} = \mathbb{C}^{0}$ and the plane with the negative real axis removed as $\mathbb{C}^{-} = \mathbb{C}^{\pi}$.*

Alternatively, we can use notation $[0,\infty)$ for the positive real axis and $(-\infty, 0]$ for the negative real axis. This gives $\mathbb{C}^{+} = \mathbb{C} - [0,\infty)$ and $\mathbb{C}^{-} = \mathbb{C} - (-\infty, 0]$.

Definition 2.3.5. *The* **logarithm** *on $\mathbb{C}^{\alpha}$ is defined by $Log_{\alpha}(z) = \ln|z| + iArg_{\alpha}(z)$ for each $z \in \mathbb{C}^{\times}$.*

Notice $Log(-1) = i\pi$ whereas $Log_{-\pi}(-1) = -i\pi$. To be clear, Log_{α} can be defined on $\mathbb{C}^{\times}$, however it is not *continuous*[5] unless we delete the ray at angle α.

Finally, let us examine how the logarithm does provide an inverse for the exponential. If we restrict to a particular branch then the calculation is simple. For example, the principal branch, let $z \in \mathbb{R} \times (-\pi, \pi]$ and consider

$$e^{Log(z)} = e^{\ln|z| + iArg(z)} = e^{\ln|z|}e^{iArg(z)} = |z|e^{iArg(z)} = z.$$

Conversely, for $z = x + iy \in \mathbb{C}^{\times}$ since $e^{x+iy} = e^{x}e^{iy}$,

$$Log(e^{z}) = \ln|e^{x}e^{iy}| + iArg(e^{x}e^{iy}) = \ln(e^{x}) + iy = x + iy = z.$$

The discussion for the multiply valued logarithm requires a bit more care. Let $z \in \mathbb{C}^{\times}$, by definition,

$$log(z) = \{\ln|z| + i(Arg(z) + 2\pi k) \mid k \in \mathbb{Z}\}.$$

Let $w \in \log(z)$ and consider,

$$\begin{aligned} e^{w} &= \exp\left(\ln|z| + i(Arg(z) + 2\pi k)\right) \\ &= \exp\left(\ln|z| + i(Arg(z)\right)) \\ &= \exp(\ln|z|)\exp(i(Arg(z)) \\ &= |z|e^{iArg(z)} \\ &= z. \end{aligned}$$

[3] I can't help but wonder, is there a math with more tails

[4] this is partly inspired by §26 of Brown and Churchill you can borrow from me if you wish

[5] we'll define this soon, but hopefully you have some intuition already

It follows that $e^{\log(z)} = \{z\}$. Sometimes, you see this written as $e^{\log(z)} = z$. if the author is not committed to viewing $\log(z)$ as a set of values. I prefer to use set notation as it is very tempting to use function-theoretic thinking for multiply-valued expressions. For example, a dangerous calculation:

$$1 = -i^2 = -ii = -(-1)^{1/2}(-1)^{1/2} = -((-1)(-1))^{1/2} = -(1)^{1/2} = -1.$$

Wait. This is troubling if we fail to appreciate that $1^{1/2} = \{1, -1\}$. What appears as equality for multiply-valued functions is better understood in terms of inclusion in a set. I will try to be explicit about sets when I use them, but, beware, not all authors share my passion for pedantics.

The trouble arises when we ignore the fact there are multiple values for a complex power function and we try to assume it ought to behave as an honest, single-valued, function.

2.4 Power Functions

Definition 2.4.1. *Let $z, p \in \mathbb{C}$ with $z \neq 0$. Define z^p to be the set of values $z^p = exp(plog(z))$.*

In particular,

$$z^p = \{\exp(p[Log(z) + 2\pi ik]) \mid k \in \mathbb{Z}\}.$$

However,

$$\exp(p[Log(z) + 2\pi ik]) = \exp(pLog(z))\exp(2p\pi ik).$$

We have already studied the case $p = 1/n$. In that case $\exp(2p\pi ik) = \exp(2p\pi i/n)$ are the n-roots of unity. In the case $p \in \mathbb{Z}$ the phase factor $\exp(2p\pi ik) = 1$ and $z \mapsto z^p$ is **single-valued** with domain $\mathbb{C}$. Generally, the complex power function is not single-valued unless we make some restriction on the domain.

Example 2.4.2. *Observe that $log(3) = \ln(3) + 2\pi i\mathbb{Z}$ hence:*

$$3^i = e^{i\,log(3)} = e^{i(\ln(3)+2\pi i\mathbb{Z})} = e^{i\ln(3)}e^{-2\pi\mathbb{Z}}.$$

In other words,

$$\begin{aligned} 3^i &= [\cos(\ln(3)) + i\sin(\ln(3))]e^{-2\pi\mathbb{Z}} \\ &= \{[\cos(\ln(3)) + i\sin(\ln(3))]e^{-2\pi k} \mid k \in \mathbb{Z}\}. \end{aligned}$$

In this example, the values fall along the ray at $\theta = \ln(3)$. As $k \to \infty$ the values approach the origin whereas as $k \to -\infty$ the go off to infinity. I suppose we could think of it as two tails, one stretched to ∞ and the other bunched at 0.

On page 25 Gamelin shows a similar result for i^i. However, as was known to Euler [R91] (p. 162), there is a **real** value of i^i. In a letter to Goldbach in 1746, Euler wrote:

> Recently I have found that the expression $(\sqrt{-1})^{\sqrt{-1}}$ has a real value, which in decimal fraction form = 0.2078795763; this seems remarkable to me.

On pages 160-165 of $[R91]$ a nice discussion of the general concept of a logarithm is given. The problem of multiple values is dealt directly with considerable rigor.

2.5 Trigonometric and Hyperbolic Functions

If you've taken calculus with me then you already know that for $\theta \in \mathbb{R}$ the formulas:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \& \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

are of tremendous utility in the derivation of trigonometric identities. They also set the stage for our definitions of sine and cosine on $\mathbb{C}$:

Definition 2.5.1. *Let $z \in \mathbb{C}$. We define:*

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big) \qquad \& \qquad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big)$$

All your favorite algebraic identities from real trigonometry hold here, unless, you are a fan of $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$. Those are not true for the complex sine and cosine. In particular, note:

$$e^{i(x+iy)} = e^{ix}e^{-y} \qquad \& \qquad e^{-i(x+iy)} = e^{-ix}e^{y}$$

Hence,

$$\cos(x+iy) = \frac{1}{2}\big(e^{ix}e^{-y} + e^{-ix}e^{y}\big) \qquad \& \qquad \sin(x+iy) = \frac{1}{2i}\big(e^{ix}e^{-y} - e^{-ix}e^{y}\big)$$

Clearly as $|y| \to \infty$ the moduli of sine and cosine diverge. I present explicit formulas for the moduli of sine and cosine later in terms of the hyperbolic functions.

I usually introduce hyperbolic cosine and sine as the even and odd parts of the exponential function:

$$e^x = \underbrace{\frac{1}{2}\big(e^x + e^{-x}\big)}_{\cosh(x)} + \underbrace{\frac{1}{2}\big(e^x - e^{-x}\big)}_{\sinh(x)}.$$

Once again, the complex hyperbolic functions are merely defined by replacing the real variable x with the complex variable z:

Definition 2.5.2. *Let $z \in \mathbb{C}$. We define:*

$$\cosh z = \frac{1}{2}\big(e^{z} + e^{-z}\big) \qquad \& \qquad \sinh z = \frac{1}{2}\big(e^{z} - e^{-z}\big).$$

The hyperbolic trigonometric functions and the circular trigonometric functions are linked by the following simple identities:

$$\cosh(iz) = \cos(z) \qquad \& \qquad \sinh(iz) = i\sin(z)$$

and

$$\cos(iz) = \cosh(z) \qquad \& \qquad \sin(iz) = i\sinh(z).$$

Return once more to cosine and use the adding angle formula (which holds in the complex domain as the reader is invited to verify)

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

and

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

In view of these identities, we calculate the modulus of sine and cosine directly,

$$|\cos(x+iy)|^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)$$
$$|\sin(x+iy)|^2 = \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y).$$

However, $\cosh^2 y - \sinh^2 y = 1$ hence

$$\begin{aligned}\cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) &= \cos^2(x)[1+\sinh^2(y)] + \sin^2(x)\sinh^2(y)\\ &= \cos^2(x) + [\cos^2(x)+\sin^2(x)]\sinh^2(y)\\ &= \cos^2(x) + \sinh^2(y).\end{aligned}$$

A similar calculation holds for $|\sin(x+iy)|^2$ and we obtain:

$$|\cos(x+iy)|^2 = \cos^2(x) + \sinh^2(y) \qquad \& \qquad |\sin(x+iy)|^2 = \sin^2(x) + \sinh^2(y).$$

Notice, for $y \in \mathbb{R}$, $\sinh(y) = 0$ iff $y = 0$. Therefore, the only way the moduli of sine and cosine can be zero is if $y = 0$. It follows that only zeros of sine and cosine are precisely those with which we are already familar on $\mathbb{R}$. In particular,

$$\sin(\pi\mathbb{Z}) = \{0\} \qquad \& \qquad \cos\left(\frac{2\mathbb{Z}+1}{2}\pi\right) = \{0\}.$$

There are pages and pages of interesting identities to derive for the functions introduced here. However, I resist. In part because they make nice homework/test questions for the students. But, also, in part because a slick result we derive later on forces identities on $\mathbb{R}$ of a particular type to necessarily extend to $\mathbb{C}$.

Definition 2.5.3. *Tangent and hyperbolic tangent are defined in the natural manner:*

$$\tan z = \frac{\sin z}{\cos z} \qquad \& \qquad \tanh z = \frac{\sinh z}{\cosh z}.$$

The domains of tangent and hyperbolic tangent are simply $\mathbb{C}$ with the zeros of the denominator function deleted. In the case of tangent, $\text{domain}(\tan z) = \mathbb{C} - \left(\frac{2\mathbb{Z}+1}{2}\right)\pi$.

Inverse Trigonometric Functions: consider $f(z) = \sin z$ then as $\sin(z+2\pi k) = \sin(z)$ for all $k \in \mathbb{Z}$ we see that the inverse of sine is multiply-valued. If we wish to pick one of those values we should study how to solve $w = \sin z$ for z. Note:

$$2iw = e^{iz} - e^{-iz}$$

multiply by e^{iz} to obtain:

$$2iwe^{iz} = (e^{iz})^2 - 1.$$

Now, substitute $\eta = e^{iz}$ to obtain:

$$2iw\eta = \eta^2 - 1 \qquad \Rightarrow \qquad 0 = \eta^2 - 2iw\eta - 1.$$

Completing the square yields,

$$0 = (\eta - iw)^2 + w^2 - 1 \qquad \Rightarrow \qquad (\eta - iw)^2 = 1 - w^2.$$

Consequently, $\eta - iw \in (1-w^2)^{1/2}$ which in terms of the principal root implies $\eta = iw \pm \sqrt{1-w^2}$. But, $\eta = e^{iz}$ so we find:

$$e^{iz} = iw \pm \sqrt{1-w^2}.$$

There are many solutions to the equation above which are by custom included in the multiply-valued inverse sine mapping below:

$$z = \sin^{-1}(w) = -i\, log(iw \pm \sqrt{1-w^2}).$$

Usually in an application where the above expression was found the context would guide us to choose a particular logarithm. For example, for appropriate w we could study $z = -iLog\left(iw + \sqrt{1-w^2}\right)$ and find $\sin z = w$ for z so-defined.

Once again, the problem of defining an inverse sine **function** requires we reduce the domain of sine to a set which is small enough that sine is injective. The problem of ambiguity in defining an inverse sine function was already present in the context of the real sine function. It is our custom that $\text{range}\sin^{-1} = [-\pi/2, \pi/2]$, but this is just one of an infinitely many choices. Notice the sine function is injective going from any peak to valley of the sine graph. We could just as well have defined $\text{range}\sin^{-1} = [\pi/2, 3\pi/2]$ then inverse sine would be the honest inverse of sine restricted to $[\pi/2, 3\pi/2]$. Why not? To quote my littlest brother when he was little: *cause be why.*

2.6 Root Functions

The title of this section is quite suspicious given our discussion of the n-th roots of unity. We learned that $z^{1/2}$ is not a function because it is double-valued. Therefore, to create a function based on $z^{1/2}$ we must find a method to select one of the values.

Definition 2.6.1. *The* **principal branch** *of the n-th root is defined by:*

$$\sqrt[n]{w} = \sqrt[n]{|w|}e^{i\frac{Arg(w)}{n}}$$

for each $w \in \mathbb{C}^\times$.

Notice that $(\sqrt[n]{w})^n = \left(\sqrt[n]{|w|}e^{i\frac{Arg(w)}{n}}\right)^n = |w|e^{iArg(w)} = w$. Therefore, $f(z) = z^n$ has a local inverse function given by the principal branch. The range of the principal branch function gives the domain on which the principal branch serves as an inverse function. Since $-\pi < Arg(w) \leq \pi$ for $w \in \mathbb{C}^\times$ it follows that $-\pi/n < Arg(w)/n \leq \pi/n$. Thus, the principal branch serves as the inverse function of $f(z) = z^n$ for $z \in \mathbb{C}^\times$ with $-\pi/n < Arg(z) \leq \pi/n$. In general, it will take n-branches to cover the z-plane. We can see those arising from rotating the sector centered about zero by the primitive n-th root. Notice the primitive root of unity in the case of $n = 2$ is just -1 and we obtain the second branch by merely multiplying by -1. This is still true for non-principal branches as I introduce below.

Definition 2.6.2. *The* **lower α-branch** *of the n-th root is defined by:*

$$\mathfrak{B}_\alpha\left(\ \sqrt[n]{w}\ \right) = \sqrt[n]{|w|}e^{i\frac{Arg_\alpha(w)}{n}}$$

for each $w \in \mathbb{C}^\times$. Likewise the **upper α-branch** *of the n-th root is*

$$\mathfrak{B}_{\alpha^+}\left(\ \sqrt[n]{w}\ \right) = \sqrt[n]{|w|}e^{i\frac{Arg_\alpha^+(w)}{n}}$$

Let me run through why $\mathfrak{B}_\alpha(\sqrt[n]{w}) \in w^{1/n}$ for $w \neq 0$. If $w \neq 0$ then we can write $w = |w|e^{iArg_\alpha(w)}$. Thus calculate,

$$\left[\mathfrak{B}_\alpha(\sqrt[n]{w})\right]^n = \left[\sqrt[n]{|w|}e^{i\frac{Arg_\alpha(w)}{n}}\right]^n = \left(\sqrt[n]{|w|}\right)^n \left(e^{i\frac{Arg_\alpha(w)}{n}}\right)^n = |w|e^{iArg_\alpha(w)} = w.$$

Therefore, as all the n-th roots of $w \neq 0$ are given by:

$$w^{1/n} = \{\sqrt[n]{w}, \omega_n \sqrt[n]{w}, \omega_n^2 \sqrt[n]{w}, \dots, \omega_n^{n-1} \sqrt[n]{w}\}$$

it is evident that two branches of $w^{1/n}$ must be related by multiplication by some power of the principle n-th root of unity. That is, there exists $0 \leq j \leq n-1$ for which

$$\mathfrak{B}_\alpha(\sqrt[n]{w}) = \omega^j \mathfrak{B}_\beta(\sqrt[n]{w})$$

where $\omega_n = e^{2\pi i/n}$ for any $\alpha, \beta \in \mathbb{R}$. In the case $n = 2$ we have $\omega_2 = e^{2\pi i/2} = e^{i\pi} = -1$ so any two branches of the square root either agree or are opposite in sign. However, this agreement is point-dependent as the next example exhibits:

Example 2.6.3. *Consider $\sqrt{w} = \sqrt{|w|}e^{iArg(w)/2}$ since $-\pi < Arg(w) \leq \pi$ for $w \in \mathbb{C}^\times$ we find $-\pi/2 < Arg(w)/2 \leq \pi/2$ thus range$(\sqrt{w})$ includes the right half-plane together with the positive imaginary axis. On the other hand, if we use $\mathfrak{B}_0(\sqrt{w}) = \sqrt{|w|}e^{iArg_0(w)/2}$ then for $w \neq 0$ we have $0 \leq Arg_0(w) < 2\pi$ hence $0 \leq Arg_0(w)/2 < \pi$ and we find that range$(\mathfrak{B}_0(\sqrt{w}))$ includes the upper half-plane together with the positive real axis. We can compare the values of $\mathfrak{B}_0(\sqrt{w})$ and $\sqrt{w}$ for any $w \neq 0$. By construction, $(\sqrt{w})^2 = w$ and $(\mathfrak{B}_0(\sqrt{w}))^2 = w$. Thus both $\sqrt{w}$ and $\mathfrak{B}_0(\sqrt{w})$ are elements of $w^{1/2} = \{\sqrt{w}, -\sqrt{w}\}$. The principle square root is easy. The non-principal root requires some analysis which depends on the location of w. Consider that $(-\pi, \pi] \cap [0, 2\pi) = [0, \pi]$ and we note $Arg(w) = Arg_0(w)$ for $w \neq 0$ with $0 \leq Arg(w) \leq \pi$, that is the upper half-plane including the real axis. On the other hand, if $-\pi < Arg(w) < 0$ (that is points below the real axis) then the non-principal argument gives a different angle where $\pi < Arg_0(w) < 0$. Notice $Arg_0(w) - 2\pi = Arg(w)$ for each w below the real axis. Consequently,*

$$\mathfrak{B}_0(\sqrt{w}) = \sqrt{|w|}e^{iArg_0(w)/2} = \sqrt{|w|}e^{i(Arg(w)+2\pi)/2} = -\sqrt{|w|}e^{iArg(w)/2} = -\sqrt{w}.$$

Riemann Surfaces: if we look at all the branches of the n-root then it turns out we can sew them together along the branches to form the Riemann surface $\mathcal{R}$. Imagine replacing the w-plane $\mathbb{C}$ with n-copies of the appropriate slit plane attached to each other along the branch-cuts. This separates the values of $f(z) = z^n$ hence $f : \mathbb{C} \to \mathcal{R}$ is invertible. The idea of replacing the codomain of $\mathbb{C}$ with a Riemann surface constructed by weaving together different branches of the function is a challenging topic in general. I suspect this article by Teleman on Riemann surfaces is a good place to start.

Chapter 3

Topology and Limits

The topology of the complex plane is simply that which we have already studied in the multivariate calculus course. Technically speaking, it is the topology which is naturally induced from the Euclidean metric on the plane. Moreover, the study of sequences and limits in the complex plane is precisely the same as the study of sequences and limits on the Euclidean plane. The essential novelty here is the introduction of complex multiplication, however, that does not draw any especially interesting distinction between the world of real and complex limits. In other words, the study of continuity for functions of a complex variable is essentially the same as the study of continuity for maps on $\mathbb{R}^2$. Sorry if this is boring, but, the odds are high you did not engage these topics at the level of rigor we endeavor in this chapter. Furthermore, I hope the humdrum nature of the current chapter makes what follows in later chapters all the more shocking.

3.1 Topology

Let me apologize at the outset for the dismal lack of pictures in this section. When I lecture this material I will draw many pictures.

Definition 3.1.1. *Let $\varepsilon > 0$ and $z_o \in \mathbb{C}$ then $D_\varepsilon(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < \varepsilon\}$ is the* **open disk of radius ε centered at z_o.** *Let $U \subseteq \mathbb{C}$ then we say $z_o \in U$ is an* **interior point of** *U if there exists $\varepsilon > 0$ for which $D_\varepsilon(z_o) \subseteq U$. If each point in U is an interior point then we say U is an* **open set**.

Heuristically speaking, open sets are those sets whose edges are fuzzy. Alternatively, an open set cannot have a point which is on its boundary because then it is impossible to find an open disk about such a point which is contained within the set.

Definition 3.1.2. *Suppose U is a subset of $\mathbb{C}$ then we denote the* **boundary** *of U by ∂U. We define ∂U to be the set of all points for which there exists an open disk which contains points in U and $\mathbb{C} - U = \{z \in \mathbb{C} \mid z \notin U\}$. If $\partial U \subset U$ then we say U is a closed set.*

You can prove that U is **closed** if and only if $\mathbb{C} - U$ is an open set. Many sets are neither open nor closed. The only subsets of $\mathbb{C}$ which are both open and closed are $\mathbb{C}$ and the emptyset . We can prove the union of any number of open sets is once more open. Likewise, the intersection of finitely many open sets is an open set. Conversely, the finite union and arbitrary intersection of closed sets is once more closed. These claims can be proven by direct argumentation based on the definition of open set given above[1]

[1]the abstract definition of topology for a set X is that $\tau \subseteq \mathcal{P}(X)$ where $X, \in \tau$ and τ is closed under arbitrary

Example 3.1.3. *Suppose U_1, U_2 are open subsets of $\mathbb{C}$. Let $z_o \in U_1 \cap U_2$ then $z_o \in U_1$ and $z_o \in U_2$. Since U_1, U_2 are open we know z_o is an interior point for both sets and thus there exist $\varepsilon_1, \varepsilon_2 > 0$ for which $D_{\varepsilon_1}(z_o) \subseteq U_1$ and $D_{\varepsilon_2}(z_o) \subseteq U_2$. Let $\varepsilon = min(\varepsilon_1, \varepsilon_2)$ and suppose $z \in D_\varepsilon(z_o)$. Observe,*

$$|z - z_o| < \varepsilon \leq \varepsilon_1 \quad \& \quad |z - z_o| < \varepsilon \leq \varepsilon_2$$

Thus $D_\varepsilon(z_o) \subseteq D_{\varepsilon_1}(z_o) \subseteq U_1$ and $D_\varepsilon(z_o) \subseteq D_{\varepsilon_2}(z_o) \subseteq U_2$. Therefore, $D_\varepsilon(z_o) \subseteq U_1 \cap U_2$. Hence z_o is an interior point of $U_1 \cap U_2$ and as z_o was arbitrary we have shown $U_1 \cap U_2$ is an open set.

Definition 3.1.4. *If $a, b \in \mathbb{C}$ then we define the* **directed line segment from** a **to** b *by*

$$[a, b] = \{a + t(b - a) \mid t \in [0, 1]\}$$

This notation is pretty slick as it agrees with interval notation on $\mathbb{R}$ when we think about them as line segments along the real axis of the complex plane. However, certain things I might have called crazy in precalculus now become totally sane. For example, $[4, 3]$ has a precise meaning. I think, to be fair, if you teach precalculus and someone tells you that $[4, 3]$ meant the same set of points, but they prefer to look at them Manga-style then you have to give them credit.

Definition 3.1.5. *A subset U of the complex plane is called* **star shaped** *with* **star center $\mathbf{z_o}$** *if there exists z_o such that each $z \in U$ has $[z_o, z] \subseteq U$.*

A given set may have many star centers[2]. For example, $\mathbb{C}^-$ is star shaped and the only star centers are found on $[0, \infty)$. Likewise, $\mathbb{C}^+$ is star shaped with possible star centers found on $(-\infty, 0]$.

Definition 3.1.6. *A* **polygonal path** *γ from a to b in $\mathbb{C}$ is the union of finitely many line segments which are placed end to end; $\gamma = [a, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-2}, z_{n-1}] \cup [z_{n-1}, b]$.*

It is convenient to defined connectedness in terms of polygonal paths for our context[3]

Definition 3.1.7. *A set $S \subseteq \mathbb{C}$ is* **connected** *iff there exists a polygonal path contained in S between any two points in S. That is for all $a, b \in S$ there exists a polygonal path γ from a to b such that $\gamma \subseteq S$*

Incidentally, the definitions just offered for $\mathbb{C}$ apply equally well to $\mathbb{R}^n$ if we generalize modulus to Euclidean distance between points.

Definition 3.1.8. *An open connected set is called a* **domain**. *We say R is a* **region** *if $R = D \cup S$ where D is a domain D and $S \subseteq \partial D$.*

The concept of a domain is most commonly found in the remainder of our study. You should take note of its meaning as it will not be emphasized every time it is used later.

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unions and finite intersections of the sets within τ. Moreover, a subset of X is said to be open if it is in the topology τ. The special case we consider here is an example of a metric topology. Metric topologies are among the most natural as they are based on geometric intuition. We have offered an entire course on Topology in recent years for interested math majors. Inquire if interested.

[2] if a person knew something about this activity called basketball there must be team-specific jokes to make here

[3] See the end of this section for a bit of a digression on the topological definition of connectedness. Moreover, notice that in the context of an open set it is fairly obvious that if we can find a polygonal path connecting any pair of points then we can equally well find a polygonal path comprised of only horizontal and vertical line-segments. Certain proofs are much easier if we approach the criterion of connectedness from the viewpoint of polygonal paths with vertical and horizontal legs.

Definition 3.1.9. *A subset $U \subseteq \mathbb{C}$ is* **bounded** *if there exists $M > 0$ and $z_o \in U$ for which $U \subseteq D_M(z_o)$. If $U \subseteq \mathbb{C}$ is both closed and bounded then we say U is* **compact**.

I should mention the definition of compact given here is not a primary definition, when you study topology or real analysis you will learn a more fundamental characterization of compactness. We may combine terms in reasonable ways. For example, a domain which is also star shaped is called a **star shaped domain**. A region which is also compact is a **compact region**.

The theorem which follows is interesting because it connect a algebraic condition $\nabla h = 0$ with a topological trait of connectedness. Recall that $h : \mathbb{R}^2 \to \mathbb{R}$ is **continuously differentiable** if each of the partial derivatives of h is continuous. We need this condition to avoid pathological issues which arise from merely assuming the partial derivatives exist. In the real case, the existence of the partial derivatives does not imply their continuity. We'll see something different for $\mathbb{C}$ as we study complex differentiability.

Theorem 3.1.10. *If $h(x, y)$ is a continuously differentiable function on a domain D such that $\nabla h = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right\rangle = 0$ on D then h is constant.*

Proof: Let $p, q \in D$. As D is connected there exists a polygonal path γ from p to q. Let $p_1, p_2, \dots, p_n$ be the points at which the line segments comprising γ are joined. In particular, γ_1 is a path from p to p_1 and we parametrize the path such that $\text{dom}(\gamma_1) = [0, 1]$. By the chain rule,

$$\frac{d}{dt}(h(\gamma_1(t))) = \nabla h(\gamma_1(t))) \bullet \frac{d\gamma_1(t))}{dt}$$

however, $\gamma_1(t) \in D$ for each t hence $\nabla h(\gamma_1(t))) = 0$. Consequently,

$$\frac{d}{dt}(h(\gamma_1(t))) = 0$$

It follows from calculus that $h(\gamma_1(0)) = h(\gamma_1(1))$ hence $h(p) = h(p_1)$. But, we can repeat this argument to show $h(p_2) = h(p_3)$ and so forth and we arrive at:

$$h(p) = h(p_1) = h(p_2) = \cdots = h(p_n) = h(q).$$

But, p, q were arbitrary thus h is constant on D. $\square$

We might use these terms correctly even without a formal definition. But, in math, we should use terms with precision in as much as is possible.

Definition 3.1.11. *We say $y \in \mathbb{C}$ is a* **limit point** *of S iff every open disk centered at y contains points in $S - \{y\}$. We say $y \in S$ is an* **isolated point** *or* **exterior point** *of S if there exist open disks about y which do not contain other points in S. The set of all interior points of S is denoted int(S) and is called the* **interior of** *S. Likewise the set of all exterior points for S is denoted ext(S) and is called the* **exterior** *of S. The* **closure** *of S is defined to be $\overline{S} = S \cup \{y \in \mathbb{C} \mid y \text{ a limit point of } S\}$.*

Example 3.1.12. *Consider $S = D_R(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < R\}$ then the boundary of S is the circle $\partial S = \{z \in \mathbb{C} \mid |z - z_o| = R\}$. Also, int$(S) = S$ which is to say every point in S is an interior point. That is, an open disk is an open set. Finally, the exterior of the disk is given by ext$(S) = \{z \in \mathbb{C} \mid |z - z_o| > R\}$. The collection of all points outside a given circle is known as an* **outer annulus**.

Annuli play an important role in one of the the final story arcs in this course. We will think of outer annuli as open disks about ∞. More on that when the time is right.

3.1.1 topological definition of connectedness

Definition 3.1.13. *An open $S \subseteq \mathbb{C}$ is said to be* **(topologically) connected set** *if there does not exist a pair of non-empty open subsets U, V of S for which $U \cap V =$ and $U \cup V = S$.*

Such a pair of sets is called a **separation** of S. If S has a separation then S is a **disconnected** set. If a set is not connected then it is disconnected. We usually find it far more convenient to use judge connectedness by checking for a strong form of path-connectedness; a space is path-connected if any pair of points in the space can be connected by a continuous path within the space.

Theorem 3.1.14. *Let $S \subseteq \mathbb{C}$ be open. If for any $p, q \in S$ there exist a collection of n line-segments $[z_0, z_1], [z_1, z_2], \dots, [z_{n-1}, z_n]$ where $z_0 = p$ and $z_n = q$ and $[z_{j-1}, z_j] \subseteq S$ for each $j = 1, 2, \dots, n$ then S is topologically connected.*

Proof Sketch: suppose S is open and any pair of points is connected by a finite union of linked line segments. If U, V is a separation of S and we pick $p \in U$ and $q \in V$ then we have $[z_0, z_1], [z_1, z_2], \dots, [z_{n-1}, z_n]$ where $z_0 = p$ and $z_n = q$ and $[z_{j-1}, z_j] \subseteq S$ for each $j = 1, 2, \dots, n$. However, this produces a contradiction since we can prove line-segments are topologically connected sets and the finite union of topologically connected sets is topologically connected. Notice the intersection of the separation of S with the union of the line segments produces a separation of the set of line-segments. □

Alternatively, since the set of line-segments begins in U and ends in V there must be some line-segment which travels from U to V and the fact that U and V are disjoint will force the line-segment connecting U and V to be likewise disjoint. But, that is absurd. Line segments cannot be separated into a pair of disjoint sets whose union forms the whole line segment.

Remark 3.1.15. *My apologies if I spend much classtime on this subsection. I'm writing it here to get it out of my system before class. This discussion belongs better to a topology course. Ask if interested.*

3.2 Sequences and Limits

Portions of this section are for breadth and my main intention is to till the soil of your mind for the Math 431 course. It is likely I project much of this section rather than right it out in its entirety. Notice I provide proofs for the limit laws at the conclusion of this section. Our main logical takeaway from this section is simply that limits in the complex domain work just as they did in the real case. However, beware we lose the squeeze theorem for complex-valued sequences. If we need to use the squeeze theorem then we're forced to make an argument which focuses on squeezing the modulus of the complex value.

A function $n \mapsto a_n$ from $\mathbb{N}$ to $\mathbb{C}$ is a **sequence** of complex numbers. Sometimes we think of a sequence as an ordered list; $\{a_n\} = \{a_1, a_2, \dots\}$. We assume the domain of sequences in this section is $\mathbb{N}$ but this is not an essential constraint, we could just as well study sequences with domain $\{k, k+1, \dots\}$ for some $k \in \mathbb{Z}$.

Definition 3.2.1. Sequential Limit: *Let a_n be a complex sequence and $a \in \mathbb{C}$. We say $a_n \to a$ iff for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$. In this case we write*

$$\lim_{n \to \infty} a_n = a.$$

Essentially, the idea is that the sequence clusters around L as we go far out in the list.

Definition 3.2.2. Bounded Sequence: *Suppose $R > 0$ and $|a_n| < R$ for all $n \in \mathbb{N}$ then $\{a_n\}$ is a* **bounded sequence**

The condition $|a_n| < R$ implies a_n is in the disk of radius R centered at the origin.

Theorem 3.2.3. Convergent Sequence Properties: *A convergent sequence is bounded. Furthermore, if $s_n \to s$ and $t_n \to t$ then*

(a.) $s_n + t_n \to s + t$

(b.) $s_n t_n \to st$

(c.) $s_n/t_n \to s/t$ *provided* $t \neq 0$.

The proof of the theorem above mirrors the proof you would give for real sequences.

Theorem 3.2.4. in-between theorem: *If $r_n \leq s_n \leq t_n$, and if $r_n \to L$ and $t_n \to L$ then $s_n \to L$.*

The theorem above is for real sequences. We have no[4] **order relations** on $\mathbb{C}$. Recall, by definition, monotonic sequences s_n are either always decreasing ($s_{n+1} \leq s_n$) or always increasing ($s_{n+1} \geq s_n$). The **completeness**, roughly the idea that $\mathbb{R}$ has no *holes*, is captured by the following theorem:

Theorem 3.2.5. *A bounded monotone sequence of real numbers coverges.*

The existence of a limit can be captured by the limit inferior and the limit superior. These are in turn defined in terms of **subsequences**.

Definition 3.2.6. *Let $\{a_n\}$ be a sequence. We define a* **subsequence** *of $\{a_n\}$ to be a sequence of the form $\{a_{n_j}\}$ where $j \mapsto n_j \in \mathbb{N}$ is a strictly increasing function of j.*

Standard examples of subsequences of $\{a_j\}$ are given by $\{a_{2j}\}$ or $\{a_{2j-1}\}$.

Example 3.2.7. *If $a_j = (-1)^j$ then $a_{2j} = 1$ whereas $a_{2j-1} = -1$. In this example, the even subsequence and the odd sequence both converge. However, $\lim a_j$ does not exist.*

Apparently, considering just one subsequence is insufficient to gain much insight. On the other hand, if we consider all possible subsequences then it is possible to say something definitive.

Definition 3.2.8. *Let $\{a_n\}$ be a sequence. We define $\text{limsup}(a_n)$ to be the upper bound of all possible subsequential limits. That is, if $\{a_{n_j}\}$ is a subsequence which converges to t (we allow $t = \infty$) then $t \leq \text{limsup}(a_n)$. Likewise, we define $\text{liminf}(a_n)$ to be the lower bound (possibly $-\infty$) of all possible subsequential limits of $\{a_n\}$.*

Theorem 3.2.9. *The sequence $a_n \to L \in \mathbb{R}$ if and only iff $\text{limsup}(a_n) = \text{liminf}(a_n) = L \in \mathbb{R}$.*

The concepts above are not available directly on $\mathbb{C}$ as there is no clear definition of an increasing or decreasing complex number. However, we do have many other theorems for complex sequences which we had before for $\mathbb{R}$. In the context of advanced calculus, I call the following the *vector limit theorem*. It says: *the limit of a vector-valued sequence is the vector of the limits of the component sequences*. Here we just have two components, the real part and the imaginary part.

[4] to be fair, you can order $\mathbb{C}$, but the order is not consistent with the algebraic structure. See this answer

Theorem 3.2.10. *Suppose $z_n = x_n + iy_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ and $z = x + iy \in \mathbb{C}$. The sequence $z_n \to z$ if and only iff both $x_n \to x$ and $y_n \to y$.*

Proof Sketch: Notice that if $x_n \to x$ and $y_n \to y$ then it is an immediate consequence of Theorem 3.2.3 that $x_n + iy_n \to x + iy$. Conversely, suppose $z_n = x_n + iy_n \to z$. We wish to prove that $x_n \to x = \mathbf{Re}(z)$ and $y_n \to y = \mathbf{Im}(z)$. The inequalities below are crucial:

$$|x_n - x| \leq |z_n - z| \qquad \& \qquad |y_n - y| \leq |z_n - z|$$

Let $\varepsilon > 0$. Since $z_n \to z$ we are free to select $N \in \mathbb{N}$ such that for $n \geq N$ we have $|z_n - z| < \varepsilon$. But, then it follows $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ by the crucial inequalities. Hence $x_n \to x$ and $y_n \to y$. $\square$

Definition 3.2.11. *We say a sequence $\{a_n\}$ is* **Cauchy** *if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ for which $N < m < n$ implies $|a_m - a_n| < \varepsilon$.*

A Cauchy sequence is one where $a_m - a_n$ tend to zero in the tail of the sequence. At first glance, this hardly seems like an improvement on the definition of convergence, yet, in practice, so many proofs elegantly filter through the Cauchy criterion. In any space, if a sequence converges then it is Cauchy. However, the converse only holds for special spaces which are called **complete**.

Definition 3.2.12. *A space is complete if every Cauchy sequence converges.*

The content of the theorem below is that $\mathbb{C}$ is complete.

Theorem 3.2.13. *A complex sequence converges iff it is a Cauchy sequence.*

Real numbers as also complete. This is an essential difference between the rational and the real numbers. There are certainly sequences of rational numbers whose limit is irrational. For example, the sequence of partial sums from the $p = 2$ series $\{1, 1 + 1/4, 1 + 1/4 + 1/9, \dots\}$ has rational elements yet limits to $\pi^2/6$. This was shown by Euler in 1734 as is discussed on page 333 of [R91]. The process of adjoining all limits of Cauchy sequences to a space is known as **completing a space**. In particular, the completion of $\mathbb{Q}$ is $\mathbb{R}$. Ideally, you will obtain a deeper appreciation of Cauchy sequences and completion when you study real analysis. That said, if you are willing to accept the truth that $\mathbb{R}$ is complete it is not much more trouble to show $\mathbb{R}^n$ is complete. I plan to guide you through the proof for $\mathbb{C}$ in your homework.

Analysis with sequences is discussed at length in our real analysis course. On the other hand, what follows is the natural extsension of the $(\varepsilon\delta)$-definition to our current context[5]. In what follows we assume $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ is a function and $L \in \mathbb{C}$.

Definition 3.2.14. *Let z_o be a limit point of the domain of the function f. We say $\lim_{z \to z_o} f(z) = L$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $z \in \mathbb{C}$ with $0 < |z - z_o| < \delta$ implies $|f(z) - L| < \varepsilon$.*

We also write $f(z) \to L$ as $z \to z_o$ when the limit exists.

Theorem 3.2.15. *Suppose $\lim_{z \to z_o} f(z), \lim_{z \to z_o} g(z) \in \mathbb{C}$ then*

(a.) $\lim_{z \to z_o} [f(z) + g(z)] = \lim_{z \to z_o} f(z) + \lim_{z \to z_o} g(z)$

[5] in fact, we can also give this definition in a vector space with a norm, if such a space is complete then we call it a Banach space

(b.) $\lim_{z \to z_o} [f(z)g(z)] = \lim_{z \to z_o} f(z) \cdot \lim_{z \to z_o} g(z)$

(c.) $\lim_{z \to z_o} [cf(z)] = c \lim_{z \to z_o} f(z)$

(d.) $\lim_{z \to z_o} \left[\frac{f(z)}{g(z)}\right] = \frac{\lim_{z \to z_o} f(z)}{\lim_{z \to z_o} g(z)}$

where in the last property we assume $\lim_{z \to z_o} g(z) \neq 0$.

Once more, the proof of this theorem mirrors the proof which was given in the calculus of $\mathbb{R}$. One simply replaces absolute value with modulus and the same arguments go through. If you would like to see explicit arguments you can take a look at my calculus I lecture notes (for free !). The other way to prove these is to use Lemma 3.2.17 and apply it to Theorem 3.2.3.

Definition 3.2.16. *If* $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ *is a function* $z_o \in dom(f)$ *such that* $\lim_{z \to z_o} f(z) = f(z_o)$ *then* f *is* **continuous at** $\mathbf{z}_o$. *Iif* f *is continuous at each point in* $U \subseteq dom(f)$ *then we say* f *is* **continuous on U**. *When* f *is continuous on* $dom(f)$ *we say* f *is* **continuous**. *The set of all continuous functions on* $U \subseteq \mathbb{C}$ *is denoted* $C^0(U)$.

The definition above gives continuity at a point, continuity on a set and finally continuitiy of the function itself. In view of Theorem 3.2.15 we may immediately conclude that if f, g are continuous then $f+g, fg, cf$ and f/g are continuous provided $g \neq 0$. The conclusion holds at a point, on a common subset of the domains of f, g and finally on the domains of the new functions $f+g, fg, cf, f/g$.

The lemma below connects the sequential and $\varepsilon\delta$-definitions of the limit. In words, if every sequence z_n converging to z_o gives sequences of values $f(z_n)$ converging to L then $f(z) \to L$ as $z \to z_o$.

Lemma 3.2.17. $\lim_{z \to z_o} f(z) = L$ *iff whenever* $z_n \to z_o$ *it implies* $f(z_n) \to L$.

Proof: this is a biconditional claim. I'll to prove half of the lemma. You can prove the interesting part for some bonus points.

($\Rightarrow$) Suppose $\lim_{z \to z_o} f(z) = L$. Also, let z_n be a sequence of complex numbers which converges to z_o. Let $\varepsilon > 0$. Notice, as $f(z) \to L$ we may choose $\delta_\varepsilon > 0$ for which $0 < |z - z_o| < \delta_\varepsilon$ implies $|f(z) - L| < \varepsilon$. Furthemore, as $z_n \to z_o$ we can choose $M_{\delta_\varepsilon} \in \mathbb{N}$ such that $n > M_{\delta_\varepsilon}$ implies $|z_n - z_o| < \delta_\varepsilon$. Finally, consider if $n > M_{\delta_\varepsilon}$ then $|z_n - z_o| < \delta_\varepsilon$ hence $|f(z_n) - L| < \varepsilon$. Thus $f(z_n) \to L$.

($\Leftarrow$) left to reader. See this answer to the corresponding question in the real case. □

The paragraph on page 37 repeated below is **very** important to the remainder of the text. He often uses this simple principle to avoid writing a complete (and obvious) proof. His refusal to write the full proof is typical of analysts' practice. In fact, this textbook was recommended to me by a research mathematician whose work is primarily analytic. Rigor should not be mistaken for the only true path. It is merely the path we teach you before you are ready for other more intuitive paths. You might read Terry Tao's excellent article on the different postures we strike as our mathematical education progresses. See *There's more to mathematics than rigour and proofs* from Tao's blog. From page 36 of Gamelin,

> "A useful strategy for showing that $f(z)$ is continuous at z_o is to obtain an estimate of the form $|f(z) - f(z_o)| \leq C|z - z_o|$ for z near z_o. This guarantees that $|f(z) - f(z_o)| < \varepsilon$ whenever $|z - z_o| < \varepsilon/C$, so that we can take $\delta = \varepsilon/C$ in the formal definition of limit"

It is also worth repeating Gamelin's follow-up example here:

Example 3.2.18. *The inequalities*

$$|\mathbf{Re}(z - z_o)| \leq |z - z_o|, \quad |\mathbf{Im}(z - z_o)| \leq |z - z_o|, \quad \& \quad ||z| - |z_o|| \leq |z - z_o|$$

indicate that $\mathbf{Re}, \mathbf{Im}$ *and modulus are continuous at* z_o.

It may be useful to mention a result which is nearly a Corollary to Theorem 3.2.10:

Theorem 3.2.19. *If* $f = u + iv$ *is a complex function with limit point* $z_o = x_o + iy_o$ *then* $\lim_{z \to z_o} f(z) = L_1 + iL_2$ *if and only if* $\lim_{z \to z_o} u(z) = L_1$ *and* $\lim_{z \to z_o} v(z) = L_2$. *We assume* u, v *are real-valued and* $L_1, L_2 \in \mathbb{R}$.

Proof: I will prove converse direction. Assume $\lim_{z \to z_o} u(z) = L_1$ and $\lim_{z \to z_o} v(z) = L_2$. Let $\varepsilon > 0$ and select δ_1, δ_2 for which $0 < |z - z_o| < \delta_1$ implies $|u(z) - L_1| < \varepsilon/2$ and $0 < |z - z_o| < \delta_2$ implies $|v(z) - L_2| < \varepsilon/2$. Set $\delta = min(\delta_1, \delta_2)$ and suppose $0 < |z - z_o| < \delta$. Let $L = L_1 + iL_2$ and observe

$$\begin{aligned} |f(z) - L| &= |u(z) - L_1 + i(v(z) - L_2)| \\ &\leq |u(z) - L_1| + |i(v(z) - L_2)| \\ &\leq |u(z) - L_1| + |i||v(z) - L_2| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $f(z) \to L_1 + iL_2$ as $z \to z_o$. I leave the forward direction for homework! $\square$

3.3 limit laws

We assume $a \in \mathbb{C}$ and f, g are functions on $\mathbb{C}$ with limit point a throughout this section unless otherwise explicitly stated. Let us begin by proving a limit has a single value.

Proposition 3.3.1. *limit is unique.*

> If $\lim\limits_{z \to a} f(z) = L_1$ and $\lim\limits_{z \to a} f(z) = L_1$ then $L_1 = L_2$.

Proof: let $\varepsilon > 0$. Suppose $\lim\limits_{z \to a} f(z) = L_1$ and $\lim\limits_{z \to a} f(z) = L_2$. Choose $\delta_1 > 0$ for which $0 < |z - a| < \delta_1$ implies $|f(z) - L_1| < \varepsilon/2$. Likewise, choose $\delta_2 > 0$ for which $0 < |z - a| < \delta_2$ implies $|f(z) - L_2| < \varepsilon/2$. Let $\delta = min(\delta_1, \delta_2)$ and suppose $0 < |z - a| < \delta \leq \delta_1, \delta_2$ hence

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(z) + f(z) - L_2| \\ &\leq |L_1 - f(z)| + |f(z) - L_2| \\ &= |f(z) - L_1| + |f(z) - L_2| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{3.1}$$

Thus $|L_1 - L_2| < \varepsilon$ for arbitary $\varepsilon > 0$ and that implies $|L_1 - L_2| = 0$ hence $L_1 = L_2$. $\square$

It is amusing that the proof of Proposition 3.3.4 rests on nearly the same calculation as the uniqueness result above.

Proposition 3.3.2. *limit of identity function.*

$$\lim_{z \to a} z = a.$$

Proof: Fix $a \in \mathbb{R}$. Let $f(z) = z$ for all $z \in \mathbb{C}$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $0 < |z - a| < \delta$ then $|f(z) - a| = |z - a| < \varepsilon$ thus $\lim_{z \to a} f(z) = a$ which is to say $\lim_{z \to a} z = a$. $\square$

Proposition 3.3.3. *limit of constant function.*

$$\lim_{z \to a} c = c.$$

Proof: Fix $a \in \mathbb{R}$ and define $f(z) = c$ for all $z \in \mathbb{C}$. Suppose $\varepsilon > 0$ and choose $\delta = 42$. If $z \in \mathbb{C}$ with $0 < |z - a| < 42$ then $|f(z) - c| = |c - c| = 0 < \varepsilon$ thus $\lim_{z \to a} f(z) = c$ by the definition of the limit. Thus $\lim_{z \to a} c = c$. $\square$

The choice of $\delta = 42$ in the above proof is just silly. You could choose any positive number.

Proposition 3.3.4. *additivity of the limit.*

Suppose $\lim_{z \to a} f(z) = L_f \in \mathbb{C}$ and $\lim_{z \to a} g(x) = L_g \in \mathbb{C}$ then

$$\lim_{z \to a} [f(z) + g(z)] = \lim_{z \to a} f(z) + \lim_{z \to a} g(z).$$

Proof: we are given that $\lim\limits_{z \to a} f(z) = L_f$ and $\lim\limits_{z \to a} g(z) = L_g$. Let $\varepsilon > 0$ and choose $\delta_f > 0$ such that $0 < |z - a| < \delta_f$ implies $|f(z) - L_f| < \frac{\varepsilon}{2}$. Likewise, choose $\delta_g > 0$ for which $0 < |z - a| < \delta_g$ implies $|g(z) - L_g| < \frac{\varepsilon}{2}$. Let $\delta = min(\delta_f, \delta_g)$ then $\delta \leq \delta_f$ and $\delta \leq \delta_g$. Suppose $z \in \mathbb{C}$ and $0 < |z - a| < \delta$ then $|f(z) - L_f| < \frac{\varepsilon}{2}$ and $|g(z) - L_g| < \frac{\varepsilon}{2}$. Consider that

$$\begin{aligned} |f(z) + g(z) - (L_f + L_g)| &= |f(z) - L_f + g(z) - L_g| \\ &\leq |f(z) - L_f| + |g(z) - L_g| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{3.2}$$

Therefore, by the definition of the limit, $\lim\limits_{z \to a} [f(x) + g(x)] = \lim\limits_{z \to a} f(z) + \lim\limits_{z \to a} g(z)$. $\square$.

Proposition 3.3.5. *homogeneity of the limit.*

Suppose $c \in \mathbb{C}$ and $\lim_{z \to a} f(z) = L \in \mathbb{C}$ then $\lim\limits_{z \to a} cf(z) = c \lim\limits_{z \to a} f(z)$.

Proof: Suppose $c \in \mathbb{C}$ and $\lim_{z \to a} f(z) = L \in \mathbb{C}$. Let $\varepsilon > 0$. If $c \neq 0$ then choose $\delta > 0$ for which $0 < |z - a| < \delta$ implies $|f(z) - L| < \frac{\varepsilon}{|c|}$. Observe

$$|cf(z) - cL| = |c||f(z) - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon \tag{3.3}$$

If $c = 0$ then $|cf(z) - cL| = 0 < \varepsilon$ for all $z \in dom(f)$. Thus, by the definition of the limit, $\lim\limits_{z \to a} cf(z) = c \lim\limits_{z \to a} f(z)$. $\square$

I often collectively refer to the previous two theorems as the *linearity* of the limit. In calculus we will learn that most major constructions obey the linearity rules. We can also extend the rules to give the limit law for a finite linear combination of convergent functions.

Proposition 3.3.6. *limit of linear combination of convergent functions.*

Suppose $a \in \mathbb{C}$ and $f_i(z) \to L_i \in \mathbb{C}$ as $z \to a$ for $i = 1, 2, \dots, n$. Then,

$$\lim_{z \to a} (c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)) = c_1 \lim_{z \to a} f_1(z) + c_2 \lim_{z \to a} f_2(z) + \cdots + c_n \lim_{z \to a} f_n(z).$$

Proof: Suppose $f_i(z) \to L_i \in \mathbb{C}$ as $z \to a$ for $i = 1, 2, \dots, n$. We claim

$$\lim_{z \to a} (c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)) = c_1 \lim_{z \to a} f_1(z) + c_2 \lim_{z \to a} f_2(z) + \cdots + c_n \lim_{z \to a} f_n(z)$$

for all $n \in \mathbb{N}$. We will prove this claim by induction on n. Notice the claim is true for $n = 1$ since Proposition 3.3.5 provides that $\lim_{z \to a}(c_1 f_1(z)) = c_1 \lim_{z \to a} f_1(z)$. Inductively suppose the claim is true for some $n \in \mathbb{N}$. Consider the linear combination of $n + 1$ functions,

$$\begin{aligned} \lim_{z \to a}\big(c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z) + c_{n+1} f_{n+1}(z)\big) &= \\ &= \lim_{z \to a}\big(c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)\big) + \lim_{z \to a}\big(c_{n+1} f_{n+1}(z)\big) \quad (3.4) \\ &= c_1 \lim_{z \to a} f_1(z) + c_2 \lim_{z \to a} f_2(z) + \cdots + c_n \lim_{z \to a} f_n(z) + c_{n+1} \lim_{z \to a} f_{n+1}(z) \quad (3.5) \end{aligned}$$

We used Proposition 3.3.4 for Equation 3.4 and we applied the induction hypothesis and Proposition 3.3.5 for Equation 3.5. Thus we have shown the claim holds for $n + 1$ and it follows the result is true for all $n \in \mathbb{N}$ by induction on n. $\square$

Proposition 3.3.7. *limit of product is product of limits.*

If $\lim_{z \to a} f(z) = L_f \in \mathbb{C}$ and $\lim_{z \to a} g(z) = L_g \in \mathbb{C}$ then

$$\lim_{z \to a} [f(z)g(z)] = \left(\lim_{z \to a} f(z)\right)\left(\lim_{z \to a} g(z)\right).$$

Preparing for Proof: Consider that we wish to find $\delta > 0$ that forces $z \in B_\delta(a)_o$ to satisfy

$$|f(z)g(z) - L_f L_g| < \varepsilon \quad (3.6)$$

we have control over $|f(z) - L_f|$ and $|g(z) - L_g|$. If we can somehow factor these out then we have something to work with. Add and subtract $L_f g(z)$ towards that goal:

$$\begin{aligned} |f(z)g(z) - L_f L_g| &= |f(z)g(z) - L_f g(z) + L_f g(z) - L_f L_g| \quad (3.7) \\ &\leq |f(z) - L_f||g(z)| + |L_f||g(z) - L_g| \end{aligned}$$

Proof: let $\varepsilon > 0$ and suppose $f(z) \to L_f$ and $g(z) \to L_g$ as $z \to a$. Observe we may select positive constants δ_1, δ_2 and δ_3 for which:

(i.) $0 < |z - a| < \delta_1$ implies $|f(z) - L_f| < \dfrac{\varepsilon}{2(1 + |L_g|)}$,

(ii.) $0 < |z - a| < \delta_2$ implies $|g(z) - L_g| < \dfrac{\varepsilon}{2(1 + |L_f|)}$,

(iii.) $0 < |z - a| < \delta_3$ implies $|g(z) - L_g| < 1$.

Observe, from (iii.) we also have the bound below:

$$|g(z)| = |g(z) - L_g + L_g| \leq |g(z) - L_g| + |L_g| < 1 + |L_g| \tag{3.8}$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and suppose $0 < |z - a| < \delta$ thus (i.), (ii.) and (iii.) hold true and $|g(z)| < 1 + |L_g|$. Thus calculate:

$$\begin{aligned} |f(z)g(z) - L_f L_g| &= |f(z)g(z) - L_f g(z) + L_f g(z) - L_f L_g| \\ &\leq |f(z) - L_f||g(z)| + |L_f||g(z) - L_g| \\ &\leq \frac{\varepsilon}{2(1+|L_g|)}(1+|L_g|) + |L_f|\frac{\varepsilon}{2(1+|L_f|)} \\ &< \varepsilon \end{aligned} \tag{3.9}$$

where the last inequality stems from the observation that $|L_f|/(1 + |L_f|) < 1$. Therefore, we have shown $f(z)g(z) \to L_f L_g$ as $z \to a$ and this completes the proof. $\square$

The proof given above is fairly standard. I found the argument in this Wikibook.

Proposition 3.3.8. *power function limit (for powers $n \in \mathbb{N}$).*

> Let $a \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, $\lim_{z \to a} z^n = a^n$.

Proof: is by induction on n. Observe $n = 0$ is true by Proposition 3.3.3. Inductively suppose $\lim\limits_{z \to a} z^n = a^n$ for some $n \in \mathbb{N}$. Consider the $(n + 1)$ case,

$$\lim_{x \to a} z^{n+1} = \lim_{z \to a} z^n z = \left(\lim_{z \to a} z^n\right)\left(\lim_{z \to a} z\right) = a^n a = a^{n+1}$$

where I used the Proposition 3.3.7 based on the induction hypothesis and Proposition 3.3.2. We find the statement true for n implies it is likewise true for $n + 1$ hence the theorem is true for all $n \in \mathbb{N}$ by proof by mathematical induction. $\square$

Proposition 3.3.9. *polynomial function limit.*

> Suppose $c_n, \dots, c_1, c_0 \in \mathbb{R}$ and $p(z) = c_n z^n + \cdots + c_1 z + c_0$ then $\lim\limits_{z \to a} p(z) = p(a)$.

Proof: by Proposition 3.3.8 we note $f_i(z) = z^i$ has $\lim\limits_{z \to a} f_i(z) = a^i$ for $i = 0, 1, 2, \dots, n$. Changing numbering slightly on Proposition 3.3.6 with $f_i(z) = z^i$ for $i = 0, 1, \dots, n$ we obtain:

$$\lim_{z \to a} (c_n z^n + \cdots + c_1 z + c_0) = c_n a^n + \cdots + c_1 a + c_0 = p(a). \quad \square$$

Proposition 3.3.10. *limit of composite. Suppose f has limit point a and g has limit point L_1,*

If $\lim_{z \to a} f(z) = L_1$ and $\lim_{y \to L_1} g(y) = L_2$ then $\lim_{z\to a} g(f(z)) = L_2$.

Proof: let $\varepsilon > 0$. Since $\lim_{y\to L_1} g(y) = L_2$ we may choose $\delta_g > 0$ such that $0 < |y - L_1| < \delta_g$ implies $|g(y) - L_2| < \varepsilon$. Likewise, since $\lim_{z\to a} f(z) = L_1$ we may select $\delta_f > 0$ for which $0 < |z-a| < \delta_f$ implies $|f(z) - L_1| < \delta_g$. Suppose $0 < |z - a| < \delta_f$ and let $y = f(z)$ then $|y - L_1| = |f(z) - L_1| < \delta_g$ hence $|g(y) - L_2| < \varepsilon$. Thus $|g(f(z)) - L_2| < \varepsilon$. Therefore, by definition of limit, $\lim_{z\to a} g(f(z)) = L_2$. $\square$

This proposition can be written without use of L_1 and L_2 but the statement is a bit clunky:

$$\lim_{z\to a} [g(f(z))] \;=\; \lim_{y\to \lim_{z\to a} f(z)} [g(y)]\,. \tag{3.10}$$

Notice the proof and application of the composite limit rule both rest on the substitution $y = f(z)$. When we make the subsitution of $y = f(z)$ we have to swap $f(z)$ for y as we trade $g(f(z))$ for $g(y)$. Likewise, we exchange $z \to a$ for the corresponding limit in y of $y \to \lim_{z\to a} f(z)$.

Proposition 3.3.11. *reciprocal function limit.*

If $a \neq 0$ then $\lim_{z\to a} \frac{1}{z} = \frac{1}{a}$.

Proof: since the proof I have for the real case seems to rely on the ordering of real numbers, I decided to go a rather different path for this proof. Consider,

$$f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

thus identify $f = u + iv$ where $u(z) = \dfrac{x}{x^2+y^2}$ and $v(z) = \dfrac{-y}{x^2+y^2}$. If $z_o = x_o + iy_o \neq 0$ then $x_o^2 + y_o^2 \neq 0$ thus it is clear from multivariate calculus that

$$\lim_{z\to z_o} u(z) = \frac{x_o}{x_o^2 + y_o^2} \qquad \& \qquad \lim_{z\to z_o} v(z) = \frac{-y_o}{x_o^2 + y_o^2}$$

Therefore, by Theorem 3.2.19,

$$\lim_{z\to z_o} \frac{1}{z} = \frac{x_o}{x_o^2+y_o^2} - i\frac{y_o}{x_o^2+y_o^2} = \frac{x_o - iy_o}{x_o^2+y_o^2} = \frac{1}{x_o+iy_o} = \frac{1}{z_o}. \;\square$$

$\square$

Proposition 3.3.12. *limit of quotient is quotient of limits.*

Suppose $\lim_{z\to a} f(z) = L_f \in \mathbb{C}$ and $\lim_{z\to a} g(z) = L_g \in \mathbb{C}$ with $L_g \neq 0$ then

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}.$$

Proof: Let $h(y) = \frac{1}{y}$ and note Proposition 3.3.11 provides $\lim_{y\to L_g} h(y) = \frac{1}{L_g}$ since $L_g \neq 0$. Furthermore, by Proposition 3.3.10 we find the limit of the composite function $h(g(z)) = \frac{1}{g(z)}$ is given by $\lim_{y\to L_g} h(y) = \frac{1}{L_g}$. Proposition 3.3.7 completes the proof since:

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \lim_{z\to a}\left[f(z)\cdot\frac{1}{g(z)}\right] = \left(\lim_{z\to a} f(z)\right)\left(\lim_{z\to a}\frac{1}{g(z)}\right) = L_f \cdot \frac{1}{L_g} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}. \;\square$$

Chapter 4

Real Differential Calculus

This chapter contains a condensed introduction to the theory of real differentiation. I think the treatment I give in Advanced Calculus[1] is better since it embraces and uses normed linear spaces and some abstract linear algebra. That said, about half the audience of this course has no such background so I will focus our attention here on functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The linear algebra needed is simply basic facts about linear transformations, matrix multiplication and the use of the standard basis for calculation. In short, the same corner of matrix theory which are already encountered in the previous chapter.

In particular, for $F : \mathbb{R}^n \to \mathbb{R}^m$ we define the *differential* at p for F to be the linear transformation $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which best approximates the change in F near p. We quantify *best* by insisting dF_p satisfy the *Frechet quotient*. In contrast to first semester calculus, this definition only *implicitly* defines the differential. Since dF_p is a linear transformation its action can be expressed in terms of matrix multiplication by the standard matrix; $dF_p(h) = J_F(p)h$ where $J_F(p)$ is the *Jacobian matrix* of F. The Jacobian matrix is the standard matrix of the differential; $[dF_p] = J_F(p)$. The partial derivative with respect to the j-th Cartesian coordinate is given by $dF_p(e_j) = \partial_j F(p)$ thus $J_F = [\partial_1 F|\cdots|\partial_n F]$.

It turns out $p \mapsto dF_p$ is continuous[2] provided all the partial derivatives of F are continuous near p[3]. Furthermore, we will show that continuous differentiability of partial derivative functions at p implies differentiability of a function at p. The theorem that continuously differentiable implies differentiable is one of the cornerstone theorems of Advanced Calculus. I will go over a two-dimensional version of it in class, but I include the n-dimensional proof here for the sake of completeness.

Finally, I share a few basic theorems of general calculus on $\mathbb{R}^n$. Linearity, chain and a rather general product rule justify much of the differential calculus you ever saw in previous coursework.

4.1 Partial Derivatives

This is an easy section. Partial differentiation is simply the process of studying the change in a map where all but one of the variables is held fixed. I'll give a slightly more general definition than we need here. Let me use the notation $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$ and $e_n = (0, \dots, 0, 1)$

[1] I hope we can offer this course regularly sometime soon, ask if interested.
[2] in a sense which we'd rather not explain here
[3] which is easy enough to understand in terms of limits we've already discussed

for the **standard basis** of **unit-vectors** in $\mathbb{R}^n$.

Definition 4.1.1. *Suppose that $U \subseteq \mathbb{R}^n$ is open and $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function. Define*

$$\frac{\partial F}{\partial x_j}(a) = \lim_{t \to 0} \frac{F(a + te_j) - F(a)}{t}$$

to be the partial derivative of F with respect to x_j if the limit above exists for $j = 1, 2, \dots, n$.

You encountered such partial derivatives in Calculus III when you calculated the partial derivative vectors of the parametrization map of a surface.

Example 4.1.2. *If $F(s,t) = (s^2 + t^3, 2st, s + 3t)$ then $\frac{\partial F}{\partial s} = (2s, 2t, 1)$ and $\frac{\partial F}{\partial t} = (3t^2, 2s, 3)$.*

The partial derivative of a vector-valued function is once more a vector-valued function. Let $F = (F_1, F_2, \dots, F_m)$ denote a function from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. We call $F_j : U \to \mathbb{R}$ the ***j*-th component function** of F for $j = 1, 2, \dots, m$. There is a simple relation between the partial derivatives of the component functions of a vector-valued function and the partial derivative of the map as a whole:

Theorem 4.1.3. *If $F = (F_1, \dots, F_m)$ then*

$$\frac{\partial F}{\partial x_j} = \left(\frac{\partial F_1}{\partial x_j}, \frac{\partial F_2}{\partial x_j}, \dots, \frac{\partial F_m}{\partial x_j} \right)$$

and the partial derivative of F exists iff the partial derivative of F_j exists for all $j = 1, 2, \dots, m$.

Proof: I will omit this proof, it's not especially interesting to this course. $\square$

Finally, let us appreciate the definition of partial derivatives for $f : U \subseteq \mathbb{C} \to \mathbb{C}$ as that is our main application. In our usual complex notation, $(x, y) = x + iy$ and $F = (F_1, F_2) = F_1 + iF_2$, but we typically use $f = u + iv$ to denote the fact that the complex map f has real component functions u and v. Observe:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \& \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(u + iv) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Example 4.1.4. *Recall $f(z) = e^z = e^x \cos y + ie^x \sin y$. Thus*

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y) + i\frac{\partial}{\partial x}(e^x \sin y) = e^x \cos y + ie^x \sin y = e^z,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y) + i\frac{\partial}{\partial y}(e^x \sin y) = -e^x \sin y + ie^x \cos y = ie^z.$$

Example 4.1.5. *Let $f(z) = z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$. Thus*

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) + i\frac{\partial}{\partial x}(2xy) = 2x + i(2y) = 2z,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) + i\frac{\partial}{\partial y}(2xy) = -2y + i(2x) = 2iz.$$

There would seem to be a pattern in the examples above. We find the tools to understand why this is happening when we study differentiation with respect to a complex variable in the next chapter. Calculus for partial derivatives of complex-valued functions is rather natural:

Theorem 4.1.6. *Suppose $f, g : \mathbb{C} \to \mathbb{C}$ have partial derivatives which exist then*

(1.) $\partial_x(f \pm g) = \partial_x f \pm \partial_x g$ *and* $\partial_y(f \pm g) = \partial_y f \pm \partial_y g$,

(2.) $\partial_x(fg) = (\partial_x f)g + f(\partial_x g)$ *and* $\partial_y(fg) = (\partial_y f)g + f(\partial_y g)$

Proof: the proof of the product rule (2.) is most interesting here. Suppose $f = u+iv$ and $g = a+ib$ have partial derivatives which exist. Observe,

$$fg = (u + iv)(a + ib) = ua - vb + i[ub + va]$$

Partial differentiate the RHS of the equation above with respect to x and make a four-fold application of the product rule:

$$\begin{aligned}
\partial_x(fg) &= (\partial_x u)a + u\partial_x a - (\partial_x v)b - v\partial_x b + i[(\partial_x u)b + u\partial_x b + (\partial_x v)a + v\partial_x a] \\
&= (\partial_x u + i\partial_x v)(a + ib) + (u + iv)(\partial_x a + i\partial_x b) \qquad (\text{ noting } i^2 = -1) \\
&= (\partial_x f)g + f(\partial_x g)
\end{aligned}$$

The proof for the partial derivative with respect to y follows the same pattern. $\square$

What about the composition of complex functions ? Let's play a bit with another one of our basic complex functions.

Example 4.1.7. *Consider $f(z) = \sin(z)$. Use the adding angles formula for sine and the definitions of hyperbolic sine and cosine to derive the component functions for sine,*

$$\sin(z) = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy) = \sin x \cosh y + i \cos x \sinh y.$$

On the other hand,

$$\cos(z) = \cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y.$$

Observe that $\partial_x \sin z = \cos x \cosh y - i \sin x \sinh y = \cos z$ and $\partial_x \cos z = -\sin z$ (check it out). Now, let's experiment with a composition of sine with $z^2 = x^2 - y^2 + 2ixy$. Observe

$$\sin(z^2) = \sin(x^2 - y^2)\cosh(2xy) + i\cos(x^2 - y^2)\sinh(2xy).$$

Calculate the partial derivative with respect to x,

$$\begin{aligned}
\partial_x \sin(z^2) &= 2x\cos(x^2 - y^2)\cosh(2xy) + 2y\sin(x^2 - y^2)\sinh(2xy) \\
&\qquad + i\left[-2x\sin(x^2 - y^2)\sinh(2xy) + 2y\cos(x^2 - y^2)\cosh(2xy)\right] \\
&= 2x\left[-i\sin(x^2 - y^2)\sinh(2xy) + \cos(x^2 - y^2)\cosh(2xy)\right] \\
&\qquad + 2y\left[\sin(x^2 - y^2)\sinh(2xy) + i\cos(x^2 - y^2)\cosh(2xy)\right] \\
&= 2(x + iy)\left[\cos(x^2 - y^2)\cosh(2xy) - i\sin(x^2 - y^2)\sinh(2xy)\right] \\
&= 2z\cos(z^2).
\end{aligned}$$

It would seem there is hope for a chain rule. However, not every compostion goes so nicely. There is a particularly nice pattern to the functions we've studied in this section.

We've seen the apparent chain-rule of $\partial_x(f(g(z))) = (\partial_x f)(g(z))\partial_x g$ work out if you study the pattern of the preceding examples in this section. It's actually a fortunate accident of the formulas we've thus far studied. The chain-rule of advanced calculus for $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ composed yields

$$\frac{\partial(F \circ G)}{\partial x}(p) = \frac{\partial F}{\partial x}(G(p))\frac{\partial G_1}{\partial x}(p) + \frac{\partial F}{\partial y}(G(p))\frac{\partial G_2}{\partial x}(p)$$

Example 4.1.8. *Consider $f(z) = |z|^2 = x^2 + y^2$ and $g(z) = z^2 = x^2 - y^2 + 2ixy$. Then $f(g(z)) = (x^2 - y^2)^2 + 4x^2y^2$ yields $\partial_x(f(g(z))) = 4x(x^2 - y^2) + 8xy^2$. On the other hand, $\partial_x f = 2x$ thus*

$$\frac{\partial f}{\partial x}(g(z))\frac{\partial g}{\partial x} = 2(x^2 - y^2)(2x + 2iy) \neq \frac{\partial(f \circ g)}{\partial x}(z).$$

In contrast, the chain rule of advanced calculus can be verified here. In our current notation, $g_1(z) = x^2 - y^2$ whereas $g_2(z) = 2xy$. Calculate, $\partial_y f = 2y$ thus

$$\frac{\partial f}{\partial x}(g(z))\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial y}(g(z))\frac{\partial g_2}{\partial x} = 2(x^2 - y^2)(2x) + 2(2xy)(2y) = 4x(x^2 - y^2) + 8xy^2 = \frac{\partial(f \circ g)}{\partial x}(z).$$

Once we have completed the study of complex differentiability in the next chapter the results of this section will be completely unsurprising.

4.2 Frechet Derivative

The definition below says that $\triangle F = F(a + h) - F(a) \cong dF_a(h)$ when h is close to zero. I should mention, going forward in this course, when I say a function is real differentiable I mean that it is real differentiable in the Frechet sense defined below:

Definition 4.2.1. *Suppose that $U \subseteq \mathbb{R}^n$ is open and $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function the we say that F is* **differentiable** *at $a \in U$ iff there exists a linear mapping $dF_a : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to 0}\left[\frac{F(a + h) - F(a) - dF_a(h)}{||h||}\right] = 0.$$

In such a case we call the linear mapping dF_a the **differential at** *a.*

Partial derivatives are defined in the usual fashion. If $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we define, for such points $a \in dom(F)$ as the limit exists,

$$\frac{\partial F}{\partial x_i}(a) = \lim_{h \to 0}\frac{F(a + he_i) - F(a)}{h}.$$

Here $e_i \in \mathbb{R}^n$ has all components 0 except for the i-th component which is 1. The connection of the differential to partial derivatives is a bit subtle. On the one hand, if the differential exists then partial derivatives exist and allow a nice formula for the differential.

Theorem 4.2.2. *If $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at p then the partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 1, 2, \dots, n$ all exist at p and $dF_p(h) = J_F(p)h$ where, suppressing p, $J_F = \left[\frac{\partial F}{\partial x_1}\middle|\frac{\partial F}{\partial x_2}\middle|\cdots\middle|\frac{\partial F}{\partial x_n}\right]$.*

In contrast, it is possible for J_F to exist at p and yet dF_p fails to exist! We explore some of the nefarious ways this may occur and then offer a remedy in the following subsection. But first let me share some explicit examples of the Jacobian matrix.

You may recall the notation from calculus III at this point, omitting the a-dependence,

$$\nabla F_j = grad(F_j) = \left[\ \partial_1 F_j,\ \partial_2 F_j,\ \cdots,\ \partial_n F_j\ \right]^T$$

So if the derivative exists we can write it in terms of a stack of gradient vectors of the component functions: (I used a transpose to write the stack side-ways),

$$F' = \left[\nabla F_1 | \nabla F_2 | \cdots | \nabla F_m\right]^T$$

Finally, just to collect everything together,

$$F' = \begin{bmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 F_m & \partial_2 F_m & \cdots & \partial_n F_m \end{bmatrix} = \left[\partial_1 F \mid \partial_2 F \mid \cdots \mid \partial_n F\ \right] = \begin{bmatrix} (\nabla F_1)^T \\ \hline (\nabla F_2)^T \\ \hline \vdots \\ \hline (\nabla F_m)^T \end{bmatrix}$$

4.2.1 Examples of Jacobian Matrices

Example 4.2.3. *Suppose* $f(z) = z^2$ *defines a mapping on* $\mathbb{R}^2 = \mathbb{C}$. *If* $z = x + iy$ *then* $z^2 = x^2 - y^2 + 2xyi$. *Real notation for* f *reads* $f(x, y) = (x^2 - y^2, 2xy)$ *thus*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

Example 4.2.4. *If* $f(x, y) = (x^2 + y^2, 2xy)$ *then*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}.$$

Example 4.2.5. *Let* $f(z) = e^z$ *where* $z = x + iy \in \mathbb{C}$. *Recall* $e^z = e^x(\cos y + i \sin y)$ *hence* $f(x, y) = (e^x \cos y, e^x \sin y)$ *and*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

Example 4.2.6. *Let* $f(z) = 3z + \bar{z}$ *where* $z = x + iy$ *and* $\bar{z} = x - iy$ *in* $\mathbb{C}$. *In real notation,* $f(x, y) = (4x, 2y)$ *thus* $J_f = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

The examples which follow may help you understand the Jacobian matrix in better depth. Often we understand math more completely when we study more than our mere area of concentration. This is why school math teachers should study math far beyond highschool algebra. Abstraction leads to mastery when properly appreciated.

Example 4.2.7. *Let* $f(t) = (t, t^2, t^3)$ *then* $f'(t) = (1, 2t, 3t^2)$. *In this case we have*

$$f'(t) = [df_t] = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

Example 4.2.8. *Let $f(\vec{x},\vec{y}) = \vec{x}\cdot\vec{y}$ be a mapping from $\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1,x_2,x_3,y_1,y_2,y_3)$ thus $f(\vec{x},\vec{y}) = x_1y_1+x_2y_2+x_3y_3$. Calculate,*

$$[df_{(\vec{x},\vec{y})}] = \nabla f(\vec{x},\vec{y})^T = [y_1,y_2,y_3,x_1,x_2,x_3]$$

Example 4.2.9. *Suppose $F(x,y) = (x^2+y^2, xy, x+y)$ we find the Jacobian is a 3×2 matrix:*

$$J_F = \left[\frac{\partial F}{\partial x}\bigg|\frac{\partial F}{\partial y}\right] = \begin{bmatrix} 2x & 2y \\ y & x \\ 1 & 1 \end{bmatrix}.$$

Example 4.2.10. *When other variables are used we still follow the same pattern. Suppose that $F(r,\theta) = (r\cos\theta, r\sin\theta)$. We calculate,*

$$J_F = [\partial_r F|\partial_\theta F] = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Example 4.2.11. *Let $f(x,y,z) = (x+y, y+z, x+z, xyz)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ yz & xz & xy \end{bmatrix}$$

Example 4.2.12. *Let $f(x,y,z) = xyz$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

Example 4.2.13. *Let $f(x,y,z) = (xyz, 1-x-y)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ -1 & -1 & 0 \end{bmatrix}$$

4.2.2 Frechet derivative's relation to the usual derivative of a function

The discussion below connects the difference quotient definition for the derivative with the Frechet quotient introduced above. This subsection can be skipped in a first reading.

Example 4.2.14. *Suppose $f : dom(f) \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at x. It follows that there exists a linear function $df_x : \mathbb{R}\to\mathbb{R}$ such that $\lim_{h\to 0}\frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0$. Note that*

$$\lim_{h\to 0}\frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0 \quad\Leftrightarrow\quad \lim_{h\to 0^\pm}\frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0.$$

In the left limit $h\to 0^-$ we have $h<0$ hence $|h| = -h$. On the other hand, in the right limit $h\to 0^+$ we have $h>0$ hence $|h| = h$. Thus, differentiability suggests that $\lim_{h\to 0^\pm}\frac{f(x+h)-f(x)-df_x(h)}{\pm h} = 0$. But we can pull the minus out of the left limit to obtain $\lim_{h\to 0^-}\frac{f(x+h)-f(x)-df_x(h)}{h} = 0$. Therefore, after an algebra step, we find:

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} - \frac{df_x(h)}{h}\right] = 0.$$

Linearity of $df_x : \mathbb{R} \to \mathbb{R}$ *implies there exists* $m \in \mathbb{R}^{1\times 1} = \mathbb{R}$ *such that* $df_x(h) = mh$. *Observe that*

$$\lim_{h\to 0} \frac{df_x(h)}{h} = \lim_{h\to 0} \frac{mh}{h} = m.$$

It is a simple exercise to show that if $\lim(A - B) = 0$ *and* $\lim(B)$ *exists then* $\lim(A)$ *exists and* $\lim(A) = \lim(B)$. *Identify* $A = \frac{f(x+h)-f(x)}{h}$ *and* $B = \frac{df_x(h)}{h}$. *Therefore,*

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, we find the 1×1 *matrix* m *of the differential is precisely* $f'(x)$ *as we defined it via a difference quotient in first semester calculus. In summary, we find* $\boxed{df_x(h) = f'(x)h}$.

4.3 Continuous Differentiability

We have noted that differentiablility on some set U implies all sorts of nice formulas in terms of the partial derivatives. Curiously the converse is not quite so simple. It is possible for the partial derivatives to exist on some set and yet the mapping may fail to be differentiable. We need an extra topological condition on the partial derivatives if we are to avoid certain pathological[4] examples.

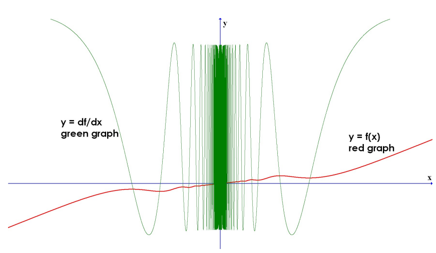
Example 4.3.1. *I found this example in Hubbard's advanced calculus text(see Ex. 1.9.4, pg. 123). It is a source of endless odd examples, notation and bizarre quotes. Let* $f(x) = 0$ *and*

$$f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$$

for all $x \neq 0$. *I can be shown that the derivative* $f'(0) = 1/2$. *Moreover, we can show that* $f'(x)$ *exists for all* $x \neq 0$, *we can calculate:*

$$f'(x) = \frac{1}{2} + 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

Notice that $dom(f') = \mathbb{R}$. *Note then that the tangent line at* $(0, 0)$ *is* $y = x/2$.



You might be tempted to say then that this function is increasing at a rate of 1/2 for x *near zero. But this claim would be false since you can see that* $f'(x)$ *oscillates wildly without end near zero. We have a tangent line at* $(0, 0)$ *with positive slope for a function which is not increasing at* $(0, 0)$ *(recall that increasing is a concept we must define in a open interval to be careful). This sort of thing cannot happen if the derivative is continuous near the point in question.*

[4]"pathological" as in, "your clothes are so pathological, where'd you get them?"

The one-dimensional case is really quite special, even though we had discontinuity of the derivative we still had a well-defined tangent line to the point. However, many interesting theorems in calculus of one-variable require the function to be continuously differentiable near the point of interest. For example, to apply the 2nd-derivative test we need to find a point where the first derivative is zero and the second derivative exists. We cannot hope to compute $f''(x_o)$ unless f' is continuous at x_o. The next example is *sick.*

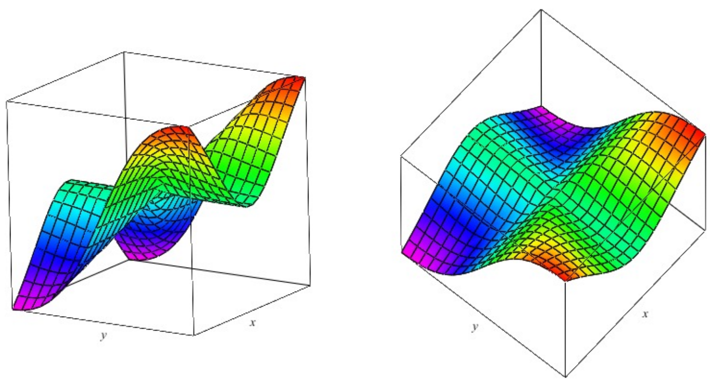
Example 4.3.2. *Let us define* $f(0,0) = 0$ *and*

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

for all $(x,y) \neq (0,0)$ *in* $\mathbb{R}^2$*. It can be shown that* f *is continuous at* $(0,0)$*. Moreover, since* $f(x,0) = f(0,y) = 0$ *for all* x *and all* y *it follows that* f *vanishes identically along the coordinate axis. Thus the rate of change in the* e_1 *or* e_2 *directions is zero. We can calculate that*

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2} \qquad \textit{and} \qquad \frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

If you examine the plot of $z = f(x,y)$ *you can see why the tangent plane does not exist at* $(0,0,0)$*.*



Notice the sides of the box in the picture are parallel to the x *and* y *axes so the path considered below would fall on a diagonal slice of these boxes*[5]*. Consider the path to the origin* $t \mapsto (t,t)$ *gives* $f_x(t,t) = 2t^4/(t^2+t^2)^2 = 1/2$ *hence* $f_x(x,y) \to 1/2$ *along the path* $t \mapsto (t,t)$*, but* $f_x(0,0) = 0$ *hence the partial derivative* f_x *is not continuous at* $(0,0)$*. In this example, the discontinuity of the partial derivatives makes the tangent plane fail to exist.*

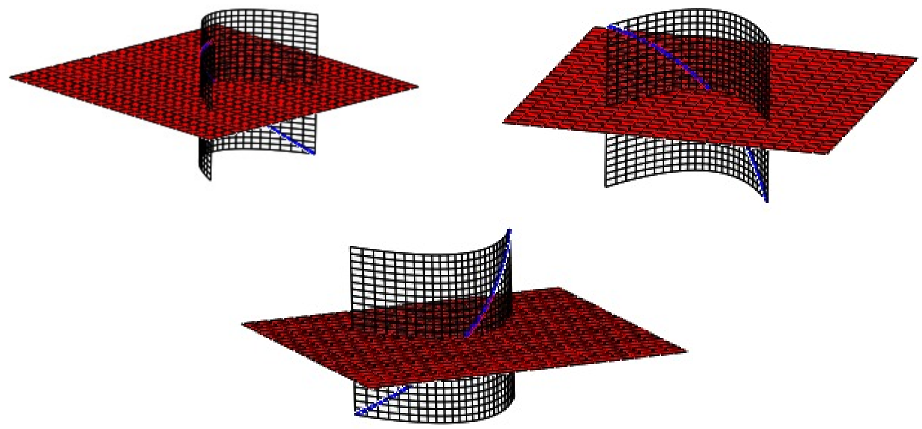
One might be tempted to suppose that if a function is continuous at a given point and if all the possible directional derivatives exist then differentiability should follow. It turns out this is not sufficient since continuity of the function does not imply some continuity along the partial derivatives. For example:

Example 4.3.3. *Let us define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(x,y) = 0$ *for* $y \neq x^2$ *and* $f(x,x^2) = x$*. I invite the reader to verify that this function is continuous at the origin. Moreover, consider the directional derivatives at* $(0,0)$*. We calculate, if* $v = \langle a,b \rangle$

$$D_v f(0,0) = \lim_{h\to 0} \frac{f(0+hv) - f(0)}{h} = \lim_{h\to 0} \frac{f(ah,bh)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

[5] the argument to follow stands alone, you don't need to understand the picture to understand the math here, but it's nice if you do

To see why $f(ah, bh) = 0$, consider the intersection of $\vec{r}(h) = (ha, hb)$ and $y = x^2$ the intersection is found at $hb = (ha)^2$ hence, noting $h = 0$ is not of interest in the limit, $b = ha^2$. If $a = 0$ then clearly (ah, bh) only falls on $y = x^2$ at $(0, 0)$. If $a \neq 0$ then the solution $h = b/a^2$ gives $f(ha, hb) = ha$ a nontrivial value. However, as $h \to 0$ we eventually reach values close enough to $(0, 0)$ that $f(ah, bh) = 0$. Hence we find **all** *directional derivatives exist and are zero at $(0, 0)$. Let's examine the graph of this example to see how this happened. The pictures below graph the xy-plane as red and the nontrivial values of f as a blue curve. The union of these forms the graph $z = f(x, y)$.*



Clearly, f is continuous at $(0, 0)$ as I invited you to prove. Moreover, clearly $z = f(x, y)$ cannot be well-approximated by a tangent plane at $(0, 0, 0)$. If we capture the xy-plane then we lose the blue curve of the graph. On the other hand, if we use a tilted plane then we lose the xy-plane part of the graph.

The moral of the story in the last two examples is simply that derivatives at a point, or even all directional derivatives at a point do not necessarily tell you much about the function near the point. This much is clear: something else is required if the differential is to have meaning which extends beyond one point in a nice way. Therefore, we consider the following:

It would seem the trouble has something to do with discontinuity in the derivative[6].

Definition 4.3.4.

A mapping $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuously differentiable** at $a \in U$ iff the partial derivative mappings $D_j F$ exist on an open set containing a and are continuous at a.

The import of the theorem below is that we can build the tangent plane from the Jacobian matrix provided the partial derivatives exist near the point of tangency and are continuous at the point of tangency. This is a very nice result because the concept of the linear mapping is quite abstract but partial differentiation of a given mapping is often easy. The proof that follows here is found in many texts, in particular see C.H. Edwards *Advanced Calculus of Several Variables* on pages 72-73. I will probably give a simplified two-dimensional version of this proof in lecture.

[6] see commentary near Equations 2.18 and 2.19 in my Advanced Calculus notes for why the term continuously differentiable is quite natural

Theorem 4.3.5.

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable at a then F is differentiable at a

Proof: We give a proof for $m = 1$ since the result then extends to $m > 1$ by the vector limit theorem. Consider $a+h$ sufficiently close to a that all the partial derivatives of F exist. Furthermore, consider going from a to $a+h$ by traversing a hyper-parallel-piped travelling n-perpendicular paths:

$$\underbrace{a}_{p_o} \to \underbrace{a + h_1e_1}_{p_1} \to \underbrace{a + h_1e_1 + h_2e_2}_{p_2} \to \cdots \underbrace{a + h_1e_1 + \cdots + h_ne_n}_{p_n} = a + h.$$

Let us denote $p_j = a + b_j$ where clearly b_j ranges from $b_o = 0$ to $b_n = h$ and $b_j = \sum_{i=1}^{j} h_ie_i$. Notice that the difference between p_j and p_{j-1} is given by:

$$p_j - p_{j-1} = a + \sum_{i=1}^{j} h_ie_i - a - \sum_{i=1}^{j-1} h_ie_i = h_je_j$$

Consider then the following identity,

$$F(a + h) - F(a) = F(p_n) - F(p_{n-1}) + F(p_{n-1}) - F(p_{n-2}) + \cdots + F(p_1) - F(p_o)$$

This is to say the change in F from $p_o = a$ to $p_n = a + h$ can be expressed as a sum of the changes along the n-steps. Furthermore, if we consider the difference $F(p_j) - F(p_{j-1})$ you can see that only the j-th component of the argument of F changes. Since the j-th partial derivative exists on the interval for h_j considered by construction we can apply the mean value theorem to locate c_j such that:

$$h_j\partial_jF(p_{j-1} + c_je_j) = F(p_j) - F(p_{j-1})$$

Therefore, using the mean value theorem for each interval, we select $c_1, \dots c_n$ with:

$$F(a + h) - F(a) = \sum_{j=1}^{n} h_j\partial_jF(p_{j-1} + c_je_j)$$

It follows we should propose L to satisfy the definition of Frechet differentation as follows:

$$L(h) = \sum_{j=1}^{n} h_j\partial_jF(a)$$

It is clear that L is linear (in fact, perhaps you recognize this as $L(h) = (\nabla F)(a) \bullet h$). Let us prepare to study the Frechet quotient,

$$\begin{aligned} F(a + h) - F(a) - L(h) &= \sum_{j=1}^{n} h_j\partial_jF(p_{j-1} + c_je_j) - \sum_{j=1}^{n} h_j\partial_jF(a) \\ &= \sum_{j=1}^{n} h_j \underbrace{\left[\partial_jF(p_{j-1} + c_je_j) - \partial_jF(a)\right]}_{g_j(h)} \end{aligned}$$

Observe that $p_{j-1} + c_je_j \to a$ as $h \to 0$. Thus, $g_j(h) \to 0$ by the continuity of the partial derivatives at $x = a$. Finally, consider the Frechet quotient:

$$\lim_{h\to 0} \frac{F(a + h) - F(a) - L(h)}{||h||} = \lim_{h\to 0} \frac{\sum_j h_jg_j(h)}{||h||} = \lim_{h\to 0} \sum_j \frac{h_j}{||h||} g_j(h)$$

Notice $|h_j| \leq ||h||$ hence $\left|\frac{h_j}{||h||}\right| \leq 1$ and

$$0 \leq \left|\frac{h_j}{||h||} g_j(h)\right| \leq |g_j(h)|$$

Apply the squeeze theorem to deduce each term in the sum $\star$ limits to zero. Consquently, $L(h)$ satisfies the Frechet quotient and we have shown that F is differentiable at $x = a$ and the differential is expressed in terms of partial derivatives as expected; $dF_x(h) = \sum_{j=1}^{n} h_j \partial_j F(a)$ $\square$.

4.4 Linear Algebra of the Plane

Let us discuss linear algebra for the plane. I'll begin with the matrix-column product:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

In my notation, $\begin{bmatrix} x \\ y \end{bmatrix} = (x, y)$ for ease of typesetting. Define $e_1 = (1, 0)$ and $e_2 = (0, 1)$ these form the **standard basis** for $\mathbb{R}^2$. Notice:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = Col_1(A) \qquad \& \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = Col_2(A)$$

where I have introduced the matrix variable $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the column notation. To summarize, to find the j-th column of a matrix we simply multiply by the j-th standard basis vector:

$$Col_1(A) = Ae_1 \qquad \& \qquad Col_2(A) = Ae_2.$$

We should also appreciate the matrix column product is really a **linear combination** of the columns in the following sense:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}.$$

Concisely, $A(x, y) = xCol_1(A) + yCol_2(A)$. Next, let us define a linear transformation on $\mathbb{R}^2$, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function for which $T(v + w) = T(v) + T(w)$ and $T(cv) = cT(v)$ for all $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^2$ then T is a **linear transformation**. Matrices and linear transformations are essentially interchangeable. Consider the following calculation:

$$\begin{aligned} T(x, y) &= T(x(1, 0) + y(0, 1)) \\ &= T(xe_1 + ye_2) \\ &= xT(e_1) + yT(e_2) \\ &= [T(e_1)|T(e_2)] \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

where $[T(e_1)|T(e_2)]$ is the 2×2 matrix formed by **concatenating** the column vectors $T(e_1)$ and $T(e_2)$. Let us make a formal definition on this point:

Definition 4.4.1.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation then it has **standard matrix** denoted $[T]$ which is the 2×2 matrix given by $[T(e_1)|T(e_2)]$.

Let me give a couple examples to clear up any mystery.

Example 4.4.2. *If* $T(x, y) = (3x - y, 2x + 7y)$ *then* $T(1, 0) = (3, 2)$ *and* $T(0, 1) = (-1, 7)$ *thus*

$$[T] = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}$$

Remember, my notation is a bit sneaky, I use $(1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *etc.*

Example 4.4.3. *If* $S(x, y) = (4x + 5y, 6y)$ *then* $S(1, 0) = (4, 0)$ *and* $S(0, 1) = (5, 6)$ *thus*

$$[S] = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}.$$

We can compose linear transformations and the result is a new linear transformation. Let us compose the linear transformations in the previous pair of examples and see how that works.

Example 4.4.4. *Once more define* $S(x, y) = (4x + 5y, 6y)$ *and* $T(x, y) = (3x - y, 2x + 7y)$,

$$\begin{aligned} (S \circ T)(x, y) = S(T(x, y)) &= S(3x - y, 2x + 7y) \\ &= (4(3x - y) + 5(2x + 7y), 6(2x + 7y)) \\ &= (22x + 31y, 12x + 42y). \end{aligned}$$

likewise,

$$\begin{aligned} (T \circ S)(x, y) = T(S(x, y)) &= T(4x + 5y, 6y) \\ &= (3(4x + 5y) - (6y), 2(4x + 5y) + 7(6y)) \\ &= (12x + 9y, 8x + 52y). \end{aligned}$$

Notice that $[S \circ T] = \begin{bmatrix} 22 & 31 \\ 12 & 42 \end{bmatrix}$ *whereas* $[T \circ S] = \begin{bmatrix} 12 & 9 \\ 8 & 52 \end{bmatrix}$. *Apparently composition of linear maps need not commute.*

Definition 4.4.5.

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ then we define the matrix AB by

$$(AB)_{ij} = \sum_{k=1}^{2} A_{ik} B_{kj} = Row_i(A) \bullet Col_j(B) \qquad \text{for } 1 \leq i, j \leq 2.$$

Equivalently, if A is a 2×2 matrix and $B = [B_1|B_2]$ where B_1, B_2 are columns of the 2×2 matrix B then $AB = A[B_1|B_2] = [AB_1|AB_2]$. I usually give this identity as a homework when I teach linear algebra. It's very important to understand how matrices apply to systems of differential equations.

Example 4.4.6. *Let* $A = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$. *Calculate,*

$$AB = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} (3,-1)\bullet(4,0) & (3,-1)\bullet(5,6) \\ (2,7)\bullet(4,0) & (2,7)\bullet(5,6) \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 8 & 52 \end{bmatrix}$$

On the other hand,

$$BA = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} (4,5)\bullet(3,2) & (4,5)\bullet(-1,7) \\ (0,6)\bullet(3,2) & (0,6)\bullet(-1,7) \end{bmatrix} = \begin{bmatrix} 22 & 31 \\ 12 & 42 \end{bmatrix}$$

These are the matrices from Example 4.4.4 and we see $[S \circ T] = [S][T]$ *and* $[T \circ S] = [T][S]$.

In fact, we defined matrix multiplication as we did in order that the matrix of a composite be the product of the standard matrices of the composed maps. If we define the sum, difference and scalar multiple of linear transformations S, T by the usual point-wise rules then $S + T$ and $S - T$ and cT are linear maps where let $c \in \mathbb{R}$. In summary:

Theorem 4.4.7.

If $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations and $c \in \mathbb{R}$ then $S \pm T$, cS and $S \circ T$ are linear transformations with

$$[S + T] = [S] + [T] \quad \& \quad [S - T] = [S] - [T] \quad \& \quad [cS] = c[S] \quad \& \quad [S \circ T] = [S][T].$$

Proof: I'll only prove the most interesting one:

$$\begin{aligned} [S \circ T] &= [(S \circ T)(e_1)|(S \circ T)(e_2)] \\ &= [S(T(e_1))|S(T(e_2))] \\ &= [[S]T(e_1)|[S]T(e_2)] \\ &= [S][T(e_1)|T(e_2)] \\ &= [S][T]. \end{aligned}$$

The proofs for $S \pm T$ and cT are similar, but easier. □

4.4.1 What about complex numbers ?

We can show that multiplication by a complex number $\alpha = a + ib$ naturally induces a linear transformation on $\mathbb{R}^2$. Recall once more our usual notation $(x, y) = x + iy$ which means we are identifying $1 = (1, 0)$ and $i = (0, 1)$. Define

$$L_\alpha(v) = \alpha v$$

If $c \in \mathbb{R}$ and $v \in \mathbb{R}^2$ then $L_\alpha(cv) = \alpha(cv) = c\alpha v = cL_\alpha(v)$. Likewise, for $v, w \in \mathbb{R}^2$ notice $L_\alpha(v + w) = \alpha(v + w) = \alpha v + \alpha w = L_\alpha(v) + L_\alpha(w)$. Therefore, $L_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation for any $\alpha \in \mathbb{C}$. We should record this notation for future reference:

Definition 4.4.8.

Let $\alpha \in \mathbb{C}$ then we define $L_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by complex multiplication by α; $L_\alpha(v) = \alpha v$ for all $v \in \mathbb{R}^2$. Let T be a linear transformation on $\mathbb{R}^2$, if there exists $\alpha \in \mathbb{C}$ for which $T = L_\alpha$ then we say T is **complex linear**.

Are all linear transformations on $\mathbb{R}^2$ complex linear ? What is special about complex linear maps? The following theorem encapsulates the essential data about complex linear maps on the plane:

Theorem 4.4.9.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation then the following are equivalent:

(1.) T is complex linear,

(2.) $T(cv) = cT(v)$ for all $c \in \mathbb{C}$ and $v \in \mathbb{R}^2$,

(3.) the standard matrix of T has the form $[T] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $T(1) = a+ib$.

(4.) the standard matrix of $[T] = [z|iz]$ where $z = T(1)$.

Proof: notice the equivalence of (3.) and (4.) is immediate once we make the necessary notational identifications of $\begin{bmatrix} a \\ b \end{bmatrix} = a + ib$ since $i(a+ib) = ia - b = \begin{bmatrix} -b \\ a \end{bmatrix}$. Hence (3.) is equivalent to (4.) as we know $1 = e_1$ and $T(e_1)$ gives the first column of the standard matrix.

Suppose (1.) is true; suppose $T = L_\alpha$ for some $\alpha \in \mathbb{C}$. Let $c \in \mathbb{C}$ and $v \in \mathbb{R}^2$ and calculate:

$$T(cv) = L_\alpha(cv) = \alpha(cv) = c\alpha v = cL_\alpha(v) = cT(v). \tag{4.1}$$

Conversely, suppose (2.) is true. Let $v \in \mathbb{R}^2 = \mathbb{C}$ and notice $v = v(1)$ and calculate,

$$T(v) = T(v(1)) = vT(1) = L_{T(1)}(v).$$

Thus $T = L_{T(1)}$ which means $T = L_\alpha$ where $\alpha = T(1)$. Thus T is complex linear and we've shown (1.) and (2.) are equivalent.

Next, suppose (2.) is true and let $a, b \in \mathbb{R}$ such that $T(1) = a + ib$. Recall the standard matrix of T is given by $[T(e_1)|T(e_2)]$. Since $e_1 = 1$ and $e_2 = i$ in our context, by assumption of (2.), we find $T(i) = T(i(1)) = iT(1)$ hence

$$[T] = [T(1)|iT(1)] = [a+ib|i(a+ib)] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Conversely, if $[T] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ then we calculate

$$T(x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = (ax - by) + i(bx + ay) = (a+ib)(x+iy)$$

thus $T = L_{a+ib}$ and it follows from Equation 4.1 that $T(cv) = cT(v)$ for all $c \in \mathbb{C}$ and $v \in \mathbb{R}^2$. $\square$

The proof above is not intended to be logically minimal. I'm trying to show how we can go from any one of the calculational perspectives to another. It turns out this simple linear algebra is key to understanding the structure of the complex derivative in the next chapter.

Finally, let me complete this brief tour of linear algebra with the explicit isomorphism which links complex numbers and their real 2×2 matrices representation.

Theorem 4.4.10. *The regular representation of* $\mathbb{C}$

Let $\mathbf{M}(x+iy) = \left[\begin{array}{cc} x & -y \\ y & x \end{array} \right]$ then $\mathbf{M}(1) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ and

$$\mathbf{M}(zw) = \mathbf{M}(z)\mathbf{M}(w) \qquad \& \qquad \mathbf{M}(z+w) = \mathbf{M}(z) + \mathbf{M}(w).$$

Furthermore, $det(\mathbf{M}(z)) = |z|^2$ and $(\mathbf{M}(z))^T = \mathbf{M}(\overline{z})$ and for $z \neq 0$, $\mathbf{M}(z^{-1}) = (\mathbf{M}(z))^{-1}$.

Proof: these seem like good homework exercises. I will focus on the statement about inverses since perhaps there exists a student who does not yet know the well-known formula for the 2×2 inverse,

$$\left[\begin{array}{cc} a & b \\ c & d \end{array} \right]^{-1} = \frac{1}{ad-bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array} \right]$$

Applying this to $\mathbf{M}(x+iy)$ we find

$$(\mathbf{M}(x+iy))^{-1} = \frac{1}{x^2+y^2} \left[\begin{array}{cc} x & y \\ -y & x \end{array} \right] = \mathbf{M}\left(\frac{x-iy}{x^2+y^2} \right) = \mathbf{M}(z^{-1}).$$

I leave the other assertions to the interested reader. $\square$

So, in summary, a linear map on $\mathbb{R}^2$ is a complex linear map if and only if it has a standard matrix which is in the regular representation of the complex number system.

Chapter 5

Complex Differentiability

In this chapter we explore four seemingly different definitions of the complex derivative[1]. I would wager the difference quotient definition is most popular, however each of the remaining definitions serves to add insight and clarify certain proofs and discussions. Ironically, the difference quotient definition is the one definition which does not nicely generalize to the calculus of a unital associative algebra. To summarize what we learn here briefly: the rules of complex calculus closely mirror that of the usual real calculus in terms of rules like the product, quotient and chain rule. However, not all functions on the plane are complex differentiable. In fact, most real differentiable maps on the plane do not have a complex derivative which exists. Complex differentiability of a map on the plane is a very strong condition. Understanding the difference between real differentiability and complex differentiability is one of the central goals of this course. The structure of a complex differentiable maps, or *holomorphic* map on a domain has many facets. In this chapter we only seek to answer the basic question of when a map on the plane is complex differentiable. The beautiful features of such maps are investigated in future chapters.

5.1 difference quotient definition

If you recall the definition of derivative from Calculus I then this definition is totally unsurprising. I decided to go ahead and share the terminology *holomorphic* and *entire* from the outset. Both of those terms are just ways of expressing a function is complex differentiable on a subset of the complex plane.

Definition 5.1.1. *If* $\lim_{z \to z_o} \frac{f(z)-f(z_o)}{z-z_o}$ *exists then we say* f *is* **complex differentiable at** $\mathbf{z}_o$ *and we denote* $f'(z_o) = \lim_{z \to z_o} \frac{f(z)-f(z_o)}{z-z_o}$. *Furthermore, the mapping* $z \mapsto f'(z)$ *is the* **complex derivative of f** *with domain formed by all such* z *as* $f'(z)$ *exists.*

[1]If you're interested, I can show how one of these approaches readily allows generalization of complex analysis to other algebras beyond $\mathbb{C}$. For example, without much more work, we can begin to calculate derivatives with respect to the hyperbolic variables built over the hyperbolic numbers $\mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. That is not part of the required content of this course, but, it seems to be an open area where a student might take a stab at some math research. In 2012-2013, W. Spencer Leslie, Minh L. Nguyen, and Bailu Zhang worked with me to produce *Laplace Equations for Real Semisimple Associative Algebras of Dimension 2, 3 or 4* published in the 2013 report **Topics from the 8th Annual UNCG Regional Mathematics and Statistics Conference**. Later still I worked out much of Calculus II with Daniel Freese and Differential Equations with Nathan BeDell. Khang Nguyen improved our basic estimates for norms systematically accross an interesting collection of algebras. In short, I have been blessed to work with some very creative students in the past and I'm usually interested in exploring new generalizations to the $\mathcal{A}$-calculus.

Given the complete parallel of limit laws over $\mathbb{C}$ as compared to limit laws over $\mathbb{R}$ we can replicate all the standard proofs for the linearity, product, quotient and chain rules. I'll forego all but a token example or two since I have proofs given for those properties in a formalism you probably have not before encountered. Arguably a better formalism, the formalism of Caratheodory.

Example 5.1.2. *Let $f(z) = z$ for all $z \in \mathbb{C}$. Consider $z_o \in \mathbb{C}$ and calculate:*

$$f'(z_o) = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o} = \lim_{z \to z_o} \frac{z - z_o}{z - z_o} = \lim_{z \to z_o} 1 = 1 \quad \Rightarrow \quad \boxed{\frac{d}{dz}(z) = 1.}$$

Example 5.1.3. *Let $f(z) = \frac{1}{z}$ and let $z_o \neq 0$. Consider,*

$$f'(z_o) = \lim_{z \to z_o} \frac{\frac{1}{z} - \frac{1}{z_o}}{z - z_o} = \lim_{z \to z_o} \frac{z_o - z}{zz_o(z - z_o)} = \lim_{z \to z_o} \frac{-1}{zz_o} = \frac{-1}{z_o^2} \quad \Rightarrow \quad \boxed{\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{-1}{z^2}.}$$

Example 5.1.4. *Let $f(z) = \bar{z}$ then the difference quotient is $\dfrac{\bar{z} - \bar{a}}{z - a}$. If we consider the path $z = a + t$ where $t \in \mathbb{R}$ then*

$$\frac{\bar{z} - \bar{a}}{z - a} = \frac{\bar{a} + t - \bar{a}}{a + t - a} = 1$$

hence as $t \to 0$ we find the difference quotient tends to 1 along this horizontal path through a. On the other hand, if we consider the path $z = a + it$ then

$$\frac{\bar{z} - \bar{a}}{z - a} = \frac{\bar{a} - it - \bar{a}}{a + it - a} = -1$$

hence as $t \to 0$ we find the difference quotient tends to -1 along this vertical path through a. But, this shows the limit $z \to a$ of the difference quotient does not exist. Moreover, as a was an arbitrary point in $\mathbb{C}$ we have shown that $f(z) = \bar{z}$ is **nowhere complex differentiable** *on $\mathbb{C}$.*

Example 5.1.5. *Let $f(z) = z\bar{z}$ and consider $z_o \neq 0$,*

$$f'(z_o) = \lim_{z \to z_o} \frac{z\bar{z} - z_o\bar{z}_o}{z - z_o}$$

Let $z_o = x_o + iy_o$ and consider the path limit given by $z(t) = x_o + i(t + y_o)$ as $t \to 0$,

$$\frac{z\bar{z} - z_o\bar{z}_o}{z - z_o} = \frac{x_o^2 + (t + y_o)^2 - (x_o^2 + y_o^2)}{x_o + i(t + y_o) - (x_o + iy_o)} = \frac{t^2 + 2ty_o}{it} = -i(t + 2y_o) \to -2iy_o.$$

Yet, if we consider the horizontal path limit approaching $x_o + iy_o$ given by $z(t) = x_o + t + iy_o$ as $t \to 0$,

$$\frac{z\bar{z} - z_o\bar{z}_o}{z - z_o} = \frac{(x_o + t)^2 + y_o^2 - (x_o^2 + y_o^2)}{x_o + t + iy_o - (x_o + iy_o)} = \frac{t^2 + 2tx_o}{t} = (t + 2x_o) \to 2x_o.$$

Since $x_o + iy_o \neq 0$ it is impossible for $-2iy_o = 2x_o$ for this would mean a purely real quantity was nontrivially imaginary. Therefore, $f'(z_o)$ does not exist. This rather nice function $f(x + iy) = x^2 + y^2$ can only hope to be complex differentiable at $z_o = 0$. I'll postpone examining $z_o = 0$ until we have better tools.

Enough suffering. Let us go on to greener pastures.

5.2 Caratheodory definition

The idea we pursue here is that we can prove most things about differentiation through the use of linearizations. To be careful, we'll use the theorem of Caratheodory[2] to make our linearization arguments a bit more rigorous.

The central point is Caratheodory's Theorem which gives us an exact method to implement the linearization. Consider a function f defined near $z = a$, we can write for $z \neq a$

$$f(z) - f(a) = \left[\frac{f(z) - f(a)}{z - a}\right](z - a).$$

If f is differentiable at a then as $z \to a$ the difference quotient $\frac{f(z)-f(a)}{z-a}$ tends to $f'(a)$ and we arrive at the approximation $f(z) - f(a) \approx f'(a)(z - a)$.

Theorem 5.2.1. Caratheodory's Theorem: *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function with $a \in D$ a limit point. Then f is complex differentiable at a iff there exists a function $\phi : D \to \mathbb{C}$ with the following two properties:*

$$(\mathbf{1}.)\ \ \phi \ \text{ is continuous at } a, \qquad (\mathbf{2}.)\ \ f(z) - f(a) = \phi(z)(z - a) \ \text{ for all } z \in D.$$

We say a function ϕ with properties as above is the **difference quotient function** *of f at $z = a$.*

Proof:($\Rightarrow$) Suppose f is differentiable at a. Define $\phi(a) = f'(a)$ and set $\phi(z) = \frac{f(z)-f(a)}{z-a}$ for $z \neq a$. Differentiability of f at a yields:

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a) \quad \Rightarrow \quad \lim_{z \to a} \phi(z) = \phi(a).$$

thus (1.) is true. Finally, note if $z = a$ then $f(z) - f(a) = \phi(z)(z - a)$ as $0 = 0$. If $z \neq a$ then $\phi(z) = \frac{f(z)-f(a)}{z-a}$ multiplied by $(z - a)$ gives $f(z) - f(a) = \phi(z)(z - a)$. Hence (2.) is true.

($\Leftarrow$) Conversely, suppose there exists $\phi : I \to \mathbb{C}$ with properties (1.) and (2.). Note (2.) implies $\phi(z) = \frac{f(z)-f(a)}{z-a}$ for $z \neq a$ hence $\lim_{z \to a} \frac{f(z)-f(a)}{z-a} = \lim_{z \to a} \phi(z)$. However, ϕ is continuous at a thus $\lim_{z \to a} \phi(z) = \phi(a)$. We find f is differentiable at a and $f'(a) = \phi(a)$. $\square$

Here's how we use the theorem: If f is differentiable at a the there exists ϕ such that $f(z) = f(a) + \phi(z)(z - a)$ and $\phi(a) = f'(a)$. Conversely, if we can supply a function fitting the properties of ϕ then it suffices to prove complex differentiability of the given function at the point about which ϕ is based. Let us derive the product rule using this technology.

Suppose f and g are complex differentiable at a and ϕ_f, ϕ_g are the difference quotient functions of f and g respective. Then,

$$f(z) = f(a) + \phi_f(z)(z - a) \qquad \& \qquad g(z) = g(a) + \phi_g(z)(z - a)$$

To derive the linearization of $(fg)(z) = f(z)g(z)$ we need only multiply:

$$\begin{aligned} f(z)g(z) &= [f(a) + \phi_f(z)(z - a)]\,[g(a) + \phi_g(z)(z - a)] \\ &= f(a)g(a) + \underbrace{\left[\phi_f(z)g(a) + f(a)\phi_g(z) + \phi_f(z)\phi_g(z)(z - a)\right]}_{\phi_{fg}(z)}(z - a) \end{aligned}$$

[2] This section was inspired in large part from Bartle and Sherbert's third edition of *Introduction to Real Analysis* and is an adaptation of the corresponding real theorem in my calculus I notes.

Observe that ϕ_{fg} defined above is manifestly continuous as it is the sum and product of continuous functions and by construction $(fg)(z)-(fg)(a) = \phi_{fg}(z)(z-a)$. The product rule is then determined from considering $z \to a$ for the difference quotient function of fg:

$$\lim_{z\to a} \phi_{fg}(z) = \lim_{z\to a}\left[\phi_f(z)g(a) + f(a)\phi_g(z) + \phi_f(z)\phi_g(z)(z-a)\right] = f'(a)g(a) + f(a)g'(a).$$

It is a simple exercise to show $\frac{d}{dz}(c) = 0$ where $c \in \mathbb{C}$ hence as an immediate offshoot of the product rule we find $(cf)'(a) = cf'(a)$.

The quotient rule can also be derived by nearly direct algebraic manipulation of Caratheodory's criteria: suppose f, g are complex differentiable at $z = a$ and $g(a) \neq 0$. Define $h = f/g$ and note $hg = f$ and consider,

$$h(z)\big[g(a) + \phi_g(z)(z-a)\big] = f(a) + \phi_f(z)(z-a).$$

Adding zero,

$$[h(z) - h(a) + h(a)]\big[g(a) + \phi_g(z)(z-a)\big] = f(a) + \phi_f(z)(z-a).$$

We find,

$$[h(z) - h(a)]\big[g(a) + \phi_g(z)(z-a)\big] = f(a) + \phi_f(z)(z-a) - h(a)\big[g(a) + \phi_g(z)(z-a)\big]$$

We may divide by $g(a) + \phi_g(z)(z-a) = g(z)$ as $g(a) \neq 0$ and continuity of g implies $g(z) \neq 0$ for z near a.

$$h(z) - h(a) = \frac{f(a) + \phi_f(z)(z-a) - h(a)\big[g(a) + \phi_g(z)(z-a)\big]}{g(a) + \phi_g(z)(z-a)}$$

Notice $f(a) = h(a)g(a)$ so we obtain the following simplification by multiplying by $g(a)/g(a)$ and factoring out the $z - a$ in the numerator:

$$h(z) - h(a) = \left[\frac{\phi_f(z)g(a) - f(a)\phi_g(z)}{g^2(a) + g(a)\phi_g(z)(z-a)}\right](z-a)$$

By inspection of the expression above it is simple to see we should define:

$$\phi_h(z) = \frac{\phi_f(z)g(a) - f(a)\phi_g(z)}{g^2(a) + g(a)\phi_g(z)(z-a)}$$

which is clearly continuous near $z = a$ and we find:

$$h'(a) = \lim_{z\to a}\phi_h(z) = \lim_{z\to a}\frac{\phi_f(z)g(a) - f(a)\phi_g(z)}{g^2(a) + g(a)\phi_g(z)(z-a)} = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

I leave the chain rule as a homework exercise. That said, have no fear, it's not so bad as I have the proof given for $\mathbb{R}$ in my posted calculus I lecture notes. See Section 4.9 of Calculus !. At this point I think it is worthwhile to compile our work thus far (including the work you will do in homework)

Theorem 5.2.2. *Given functions f, g, w which are complex differentiable (and nonzero for g in the quotient) we have:*

$$\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}, \qquad \frac{d}{dz}(cf) = c\frac{df}{dz}, \qquad \frac{d}{dz}\big(f(w)\big) = \frac{df}{dw}\frac{dw}{dz}$$

where the notation $\frac{df}{dw}$ indicates we take the derivative function of f and evaluate it at the value of the inside function w; that is, $\frac{df}{dw}(z) = f'(w(z))$.

Now I turn to specific functions. We should like to know how to differentiate the functions we introduced in the previous chapter. I will continue to showcase the criteria of Caratheodory. Sometimes it is easier to use Definition 5.1.1 directly and you can compare with a standard real calculus text to see which argument is easier.

Example 5.2.3. *Let $f(z) = c$ where c is a constant. Note $f(z) = c + 0(z-a)$ hence as $\phi(z) = 0$ is continuous and $\lim_{z\to a}\phi(z) = 0$ it follows by Caratheodory's criteria that $\frac{d}{dz}(c) = 0$.*

Example 5.2.4. *Let $f(z) = z$. Note $f(z) = a + 1(z-a)$ hence as $\phi(z) = 1$ is continuous and $\lim_{z\to a}\phi(z) = 1$ it follows by Caratheodory's criteria that $\frac{d}{dz}(z) = 1$.*

Example 5.2.5. *Let $f(z) = z^2$. Note $f(z) = a^2+z^2-a^2 = a^2+(z+a)(z-a)$ hence as $\phi(z) = z+a$ is continuous and $\lim_{z\to a}\phi(z) = 2a$ it follows by Caratheodory's criteria that $\frac{d}{dz}(z^2) = 2z$.*

If you are wondering where the a went. The complete thought of the last example is that $f(z) = z^2$ has $f'(a) = 2a$ hence df/dz is the mapping $a \mapsto 2a$ which we usually denote by $z \mapsto 2z$ hence the claim.

Example 5.2.6. *Let $f(z) = z^4$. Note $f(z) = a^4 + z^4 - a^4 = a^4 + (z^3 + a^2z + az^2 + a^3)(z-a)$ hence as $\phi(z) = z^3 + 3a^2z + 3az^2 + a^3$ is continuous and $\lim_{z\to a}\phi(z) = 4a^3$ it follows by Caratheodory's criteria that $\frac{d}{dz}(z^4) = 4z^3$.*

The factoring in the example above is perhaps mystifying. One way you could find it is to simply divide $z^4 - a^4$ by $z-a$ using polynomial long division. Yes, it still works for complex polynomials. The reader will show $\frac{d}{dz}(z^3) = 3z^2$ in the homework.

Example 5.2.7. *Let $f(z) = 1/z$. Thus $zf(z) = 1$ and we find:*

$$(z-a+a)f(z) = 1 \quad\Rightarrow\quad af(z) = 1 - f(z)(z-a).$$

If $a \neq 0$ then we find by dividing the above by a and noting $f(a) = 1/a$ hence

$$f(z) = f(a) - \frac{f(z)}{a}(z-a).$$

Therefore $\phi(z) = -\frac{f(z)}{a} = \frac{-1}{az}$ is the difference quotient function of f which is clearly continuous for $a \neq 0$ and as $\phi(z) \to -1/a^2$ as $z \to a$ we derive $\frac{d}{dz}\left[\frac{1}{z}\right] = \frac{-1}{z^2}$.

The algebra I show in the example above is merely what first came to mind as I write these notes. You could just as well attack it directly:

$$f(z) - f(a) = \frac{1}{z} - \frac{1}{a} = \frac{a-z}{az} = \frac{-1}{az}(z-a).$$

Perhaps the algebra above is more natural, it also leads to $\phi(z) = \frac{-1}{az}$.

Example 5.2.8. *We can find many additional derivatives from the product or quotient rules. For example,*

$$\frac{d}{dz}\left[\frac{1}{z^2}\right] = -\frac{1}{z^2}\frac{1}{z} - \frac{1}{z^2}\frac{1}{z} = \frac{-2}{z^3}.$$

Or, for $n \in \mathbb{N}$ supposing it is known that $\frac{d}{dz}(z^n) = nz^{n-1}$,

$$\frac{d}{dz}\left[\frac{1}{z^n}\right] = \frac{(0)z^n - 1\cdot nz^{n-1}}{(z^n)^2} = \frac{-n}{z^{2n-(n-1)}} = \frac{-n}{z^{n+1}}.$$

If we prove[3] for $n \in \mathbb{N}$ that $d/dz(z^n) = nz^{n-1}$ then in view of the example above we have shown:

Theorem 5.2.9. Power law for integer powers: *let $n \in \mathbb{Z}$ then $\frac{d}{dz}\big(z^n\big) = nz^{n-1}$.*

Non-integer power functions have phase functions which bring the need for branch cuts. It follows that we ought to discuss derivatives of exponential and log functions before we attempt to extend the power law to other than integer powers. That said, nothing terribly surprising happens. It is in fact the case $\frac{d}{dz}z^n = nz^{n-1}$ for $n \in \mathbb{C}$ however we must focus our attention on just one branch of the function.

Let us attempt to find $\frac{d}{dz}e^z$. We'll begin by showing $f(z) = e^z$ has $f'(0) = 1$. Consider, $f(z) - f(0) = e^z - 1$. Moreover, for $z \neq 0$ we have:

$$f(z) - f(0) = \left[\frac{e^z - 1}{z}\right] z \quad \Rightarrow \quad \phi(z) = \frac{e^z - 1}{z}.$$

To show $f'(0) = 1$ it suffices to demonstrate $\phi(z) \to 1$ as $z \to 0$. If we knew L'Hopital's rule for complex variables then it would be easy, however, we are not in possession of such technology. I will award bonus points to anyone who can prove $\phi(z) \to 1$ as $z \to 0$. I have tried several things to no avail. This example will have to wait for the Cauchy Riemann equation approach.

If a function is complex differentiable over a domain of points it turns out that the complex derivative function **must** be continuous. There is a distinction between complex differentiability at a point and holomorphicity at a point.

Definition 5.2.10. *We say f* **is holomorphic on domain** *D if f is complex differentiable at each point in D. We say f* **is holomorphic at** *z_o if there exists an open disk D centered at z_o on which $f|_D$ is holomorphic.*

Given our calculations thus far we can already see that polynomial functions are holomorphic on $\mathbb{C}$. Furthermore, if $p(z), q(z) \in \mathbb{C}[z]$ then p/q is holomorphic on $\mathbb{C} - \{z \in \mathbb{C} \mid q(z) = 0\}$. We discover many more holomorphic functions via the Cauchy Riemann equations of the next section. It is also good to have some examples which show not all functions on $\mathbb{C}$ are holomorphic.

The following example is taken from [R91] on page 57. I provide proof of the claims made below in the next section as the Cauchy Riemann equations are far easier to calculate that limits.

Example 5.2.11. *Let $f(z) = x^3y^2 + ix^2y^3$ where $z = x + iy$. We can show that f is complex differentiable where $x = 0$ or $y = 0$. In other words, f is complex differentiable on the coordinate axes. It follows this function is* **nowhere holomorphic** *on $\mathbb{C}$ since we cannot find any point about which f is complex differentiable on an whole open disk.*

5.3 complex linearity and the Cauchy Riemann equations

Let me begin by explaining the terminology.

Definition 5.3.1. *Let $f = u + iv$ then $u_x = v_y$ and $u_y = -v_x$ are the* **Cauchy Riemann or (CR)**-*equations for f.*

[3] I invite the reader to prove this by induction

We explain the significance of these equations in this section.

I first learned the approach in this section from [R91]. Here we use the full force of the theory of the real differential calculus and linear algebra on the plane we investigated in the previous chapter. In particular, recall the langauge used in Theorems 4.4.9 and 4.4.10.

Theorem 5.3.2. *If $f = u + iv$ is complex differentiable at z_o then f is real differentiable at z_o in the Frechet sense and $J_f(z_o) = \mathbf{M}(f'(z_o))$ hence $\partial_y f = i\partial_x f$ at z_o. In particular, the partial derivatives of the component functions must satisfy the equations $u_x = v_y$ and $u_y = -v_x$ at z_o. Equivalently, the differential of f at z_o is complex linear.*

Proof: suppose $f = u+iv$ is complex differentiable at $z_o \in \mathbb{C}$. By Caratheodory's criteria we have a continuous complex function ϕ for which $\phi(a) = f'(z_o)$ and

$$f(z) = f(z_o) + \phi(z)(z - z_o) \quad \Rightarrow \quad f(z + h) - f(z_o) = \phi(z_o + h)h$$

I claim $df_{z_o}(h) = f'(z_o)h$. Observe, for $h \neq 0$ we have:

$$\frac{f(z + h) - f(z_o) - f'(z_o)h}{|h|} = \frac{\phi(z_o + h)h - f'(z_o)h}{|h|} = (\phi(z_o + h) - f'(z_o))\frac{h}{|h|}.$$

we wish to show the *Frechet quotient* above tends to 0 as $h \to 0$. Notice we may use a theorem on trivial limits; $|g(z)| \to 0$ as $z \to a$ iff $g(z) \to 0$ as $z \to a$. Therefore, we take the modulus of the Frechet quotient and find

$$\frac{|f(z + h) - f(z_o) - f'(z_o)h|}{|h|} = |\phi(z_o + h) - f'(z_o)|\frac{|h|}{|h|} = |\phi(z_o + h) - f'(z_o)|.$$

Finally, by continuity of ϕ we have $\phi(z_o+h) \to \phi(z_o) = f'(z_o)$ as $h \to 0$ hence the Frechet quotient limits to zero as needed and we have shown that $df_{z_o}(h) = f'(z_o)h$. Thus $df_{z_o} = L_{f'(z_o)}$ in the notation discussed near Theorem 4.4.9. In other words, df_{z_o} is a complex linear map on $\mathbb{R}^2$. Thus $J_f(z_o) = [df_{z_o}] = [f'(z_o)|if'(z_o)]$ by Theorem 4.4.9. On the other hand, recall the form of the Jacobian matrix:

$$J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \left[\frac{\partial f}{\partial x}\middle|\frac{\partial f}{\partial y}\right]$$

Therefore, since $J_f(z_o) = [df_{z_o}] = [f'(z_o)|if'(z_o)]$ we find

$$\frac{\partial f}{\partial x} = f'(z_o) \qquad \& \qquad \frac{\partial f}{\partial y} = if'(z_o)$$

From which we find formulas to calculate the complex derivative via partial derivatives in the case the complex derivative exists:

$$\boxed{f'(z_o) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

Indeed, the equation $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ is simply the vector form of the Cauchy Riemann equations since $f = u + iv$ has $\partial_x f = u_x + iv_x$ and $\partial_y f = u_y + iv_y$ hence

$$u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y \quad \Rightarrow \quad \boxed{u_x = v_y \quad \& \quad v_x = -u_y} \qquad \square.$$

I boxed the equations above since they are computationally central for the remainder of this course. These inform us on how the partial derivatives relate to the complex derivative when it exists. In fact, we learn next the boxed equations suffice to imply complex differentiablity for a continuously differentiable function on the plane. But first, an example of non-complex-differentiability as seen through the CR-equations:

Example 5.3.3. . *At this point we can return to my claim in Example 5.2.11. Let* $f(z) = x^3y^2 + ix^2y^3$ *where* $z = x + iy$ *hence* $u = x^3y^2$ *and* $v = x^2y^3$ *and we calculate:*

$$u_x = 3x^2y^2, \quad u_y = 2x^3y, \quad v_x = 2xy^3, \quad v_y = 3x^2y^2.$$

If f *is holomorphic on some open set disk* D *then it is complex differentiable at each point in* D*. Hence, by our discussion preceding this example it follows* $u_x = v_y$ *and* $v_x = -u_y$*. The only points in* $\mathbb{C}$ *at which the CR-equations hold are where* $x = 0$ *or* $y = 0$*. Therefore, it is impossible for* f *to be complex differentiable on any open disk. Thus our claim made in Example 5.2.11 is true;* f *is nowhere holomorphic.*

Now, let us investigate the converse direction[4]. Let us see that if the CR-equations hold for continuously real differentiable function on a domain then the function is holomorphic on that domain. We assume continuously differentiable on a domain for our expositional convenience. See pages 58-59 of [R91] where he mentions a number of weaker conditions which still are sufficient to guarantee complex differentiability at a given point.

Theorem 5.3.4. *Suppose* $f = u + iv$ *is continuously differentiable on a domain* D*. Then* f *is holomorphic on* D *if and only if the component functions* u, v *satisfy the Cauchy Riemann equations* $u_x = v_y$ *and* $v_x = -u_y$ *throughout* D*. Furthermore, when* f *is holomorphic on* D *the complex derivative may be formulated by* $f'(z) = \partial_x f = -i\partial_y f$ *or in component notation,* $f'(z) = u_x + iv_x = v_y - iu_y$ *on* D*.*

Proof: Assume $f = u + iv$ is continuously differentiable on the domain D. Then then by Theorem 4.4.9 we have f is real differentiable on D. Suppose the partial derivatives satisfy $u_x = v_y$ and $v_x = -u_y$ on D . Notice, the conditions $u_x = v_y$ and $v_x = -u_y$ imply $J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$ thus $J_f = [\partial_x f | i\partial_y f]$ and we find the differential of f is complex linear (using Theorem 4.4.9) for each $z_o \in D$;

$$df_{z_o}(h) = df_{z_o}(1)h = \partial_x f(z_o)h.$$

Let $z_o \in D$ and define $u_x(z_o) = a$ and $v_x(z_o) = b$. We propose $f'(z_o) = a + ib$. We can derive the needed difference quotient by analyzing the Frechet quotient with care. We are given[5]:

$$\lim_{h\to 0} \frac{f(z_o + h) - f(z_o) - (a + ib)h}{h} = 0.$$

Notice, $\displaystyle\lim_{h\to 0} \frac{(a + ib)h}{h} = a + ib$ thus[6]

$$\lim_{h\to 0} \frac{f(z_o + h) - f(z_o)}{h} - \lim_{h\to 0} \frac{(a + ib)h}{h} = 0.$$

Therefore,

$$a + ib = \lim_{h\to 0} \frac{f(z_o + h) - f(z_o)}{h}$$

[4] I'll state the result as a biconditional for our future convenience, of course Theorem 5.3.2 proves the necessity of the CR-equations for a holomorphic function.

[5] I'm cheating, it's a small exercise to show $\lim_{h\to 0} g(h)/|h| = 0$ implies $\lim_{h\to 0} g(h)/h = 0$.

[6] I encourage the reader to verify the little theorem: if $\lim(f - g) = 0$ and $\lim g$ exists then $\lim f = \lim g$.

which verifies our claim $f'(z_o) = a + ib$. But, $z_o \in D$ was arbitrary hence we have shown f is holomorphic on the domain D.

Conversely, apply Theorem 5.3.2 to see that holomorpicity implies the Cauchy Riemann Equations hold throughout D. Let me reiterate the proof once more with a demphasis of the picky details establishing the complex-linearity of the differential. If f is holomorphic on D then f is complex differentiable on D which implies df_{z_o} is complex linear at each $z_o \in D$ and thus $J_f(z_o)$ is in the regular representation of $\mathbb{C}$ which means we have the conditions $u_x = v_y$ and $v_x = -u_y$ since $J_{u+iv} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ must be of the form $\mathbf{M}(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. (by Theorems 4.4.9 and 4.4.10) □

We should reiterate when f is complex differentiable we have the following identities:

$$f'(z) = u_x + iv_x = v_y - iu_y \quad \Rightarrow \quad \frac{df}{dz} = \frac{\partial f}{\partial x} \;\; \& \;\; \frac{df}{dz} = -i\frac{\partial f}{\partial y}$$

where the differential identities hold **only** for holomorphic functions. The corresponding identities for arbitrary functions on $\mathbb{C}$ are discussed on pages 124-126 of Gamelin and in the final section of this chapter on the Wirtinger Derivatives.

As promised, we can show the other elementary functions are holomorphic in the appropriate domain. Let us begin with the complex exponential.

Example 5.3.5. *Let $f(z) = e^z$ then $f(x+iy) = e^x(\cos y + i \sin y)$ hence $u = e^x \cos y$ and $v = e^x \sin y$. Observe u, v clearly have continuous partial derivatives on $\mathbb{C}$ and*

$$u_x = e^x \cos y, \quad v_x = e^x \sin y, \quad u_y = -e^x \sin y, \quad v_y = e^x \cos y.$$

Thus $u_x = v_y$ and $v_x = -u_y$ for each point in $\mathbb{C}$ and we find $f(z) = e^z$ is holomorphic on $\mathbb{C}$ by Theorem 5.3.4. Moreover, as $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$ we find $\frac{d}{dz}e^z = e^z$.

Definition 5.3.6. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$ then f is an* **entire function**. *The set of entire functions on $\mathbb{C}$ is denoted $\mathcal{O}(C)$*

The complex exponential function is entire. Functions constructed from the complex exponential are also entire. In particular, it is a simple exercise to verify $\sin z, \cos z, \sinh z, \cosh z$ are all entire functions. We can either use Theorem 5.3.4 and explicitly calculate real and imaginary parts of these functions, or, we could just use Example 5.3.5 paired with the chain rule. For example:

Example 5.3.7.

$$\begin{aligned}
\frac{d}{dz}\sin z &= \frac{d}{dz}\left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right] \\
&= \frac{1}{2i}\frac{d}{dz}\left[e^{iz}\right] - \frac{1}{2i}\frac{d}{dz}\left[e^{-iz}\right] \\
&= \frac{1}{2i}e^{iz}\frac{d}{dz}\left[iz\right] - \frac{1}{2i}e^{-iz}\frac{d}{dz}\left[-iz\right] \\
&= \frac{1}{2i}e^{iz}i - \frac{1}{2i}e^{-iz}(-i) \\
&= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \cos(z).
\end{aligned}$$

Very similar arguments show the hopefully unsurprising results below:

$$\frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z.$$

You might notice that Theorem 3.1.10 applies to **real-valued** functions on the plane. The theorem below deals with a complex-valued function.

Theorem 5.3.8. *If f is holomorphic on a domain D and $f'(z) = 0$ for all $z \in D$ then f is constant.*

Proof: observe $f'(z) = u_x + iv_x = 0$ thus $u_x = 0$ and $v_x = 0$ thus $v_y = 0$ and $u_y = 0$ by the CR-equations. Thus $\nabla u = 0$ and $\nabla v = 0$ on a connected open set so we may apply Theorem 3.1.10 to see $u(z) = a$ and $v(z) = b$ for all $z \in D$ hence $f(z) = a + ib$ for all $z \in D$. $\square$

There are some striking, but trivial, statements which follow from the Theorem above. For instance:

Theorem 5.3.9. *If f is holomorphic and real-valued on a domain D then f is constant.*

Proof: Suppose $f = u + iv$ is holomorphic on a domain D then $u_x = v_y$ and $v_x = -u_y$ hence $f'(z) = u_x + iv_x = v_y + iv_x$. Yet, $v = 0$ since f is real-valued hence $f'(z) = 0$ and we find f is constant by Theorem 5.3.9. $\square$

You can see the same is true of f which is imaginary and holomorphic. Moreover, these theorems are helpful in proving complex identities. For example:

Example 5.3.10. *Let $f(z) = \sin^2(z) + \cos^2(z)$ for $z \in \mathbb{C}$. Observe f is holomorphic on $\mathbb{C}$ and $f'(z) = 2\sin z \cos z - 2\cos z \sin z = 0$. Furthermore, $f(0) = 1$ since $\sin(0) = 0$ and $\cos(0) = 1$. By Theorem 5.3.9 we find f is constant on $\mathbb{C}$ thus establish the identity $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbb{C}$.*

We could use the idea above to work on proving other identities. That sounds like good homework. We could continue this section to see how to differentiate the reciprocal trigonometric or hyperbolic functions such as $\sec z, \csc z, \text{csch}z, \text{sech}z, \tan z, \tanh z$ however, I will refrain as the arguments are the same as you saw in first semester calculus. It seems likely I ask some homework about these. You may also recall, we needed **implicit differentiation** to find the derivatives of the inverse functions in calculus I. The same is true here and that is the topic of a future section.

The set of holomorphic functions over a domain is an object worthy of study. Notice, if D is a domain in $\mathbb{C}$ then polynomials, rational functions with nonzero denominators in D are all holomorphic. Of course, the functions built from the complex exponential are also holomorphic. A bit later, we'll see any power series is holomorphic in some domain about its center. Each holomorphic function on D is continuous, but, not all continous functions on D are holomorphic. The **antiholomorphic** functions are also continuous. The quintessential antiholomorphic example is $f(z) = \bar{z}$.

Definition 5.3.11. *The set of all holomorphic functions on a domain $D \subseteq \mathbb{C}$ is denoted $\mathcal{O}(D)$.*

On pages 59-60 of [R91] there is a good discussion of the algebraic properties of $\mathcal{O}(D)$. Also, on 61-62 Remmert discusses the notation $\mathcal{O}(D)$ and the origin of the term **holomorphic** which was given in 1875 by Briot and Bouquet. We will eventually uncover the equivalence of the terms holomorphic, analytic , conformal. These terms are in part tied to the approaches of Cauchy, Weierstrauss and Riemann. I'll try to explain this trichotomy in better detail once we know more. It is the theme of Remmert's text [R91].

5.3.1 CR equations in polar coordinates

If we use polar coordinates to rewrite f as follows:

$$f(x(r,\theta), y(r,\theta)) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

we use shorthands $F(r,\theta) = f(x(r,\theta), y(r,\theta))$ and $U(r,\theta) = u(x(r,\theta), y(r,\theta))$ and $V(r,\theta) = v(x(r,\theta), y(r,\theta))$. We derive the CR-equations in polar coordinates via the chain rule from multivariate calculus,

$$U_r = x_r u_x + y_r u_y = \cos(\theta)u_x + \sin(\theta)u_y \quad \text{and} \quad U_\theta = x_\theta u_x + y_\theta u_y = -r\sin(\theta)u_x + r\cos(\theta)u_y$$

Likewise,

$$V_r = x_r v_x + y_r v_y = \cos(\theta)v_x + \sin(\theta)v_y \quad \text{and} \quad V_\theta = x_\theta v_x + y_\theta v_y = -r\sin(\theta)v_x + r\cos(\theta)v_y$$

We can write these in matrix notation as follows:

$$\begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Multiply these by the inverse matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix}$ to find

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)U_r - \frac{1}{r}\sin(\theta)U_\theta \\ \sin(\theta)U_r + \frac{1}{r}\cos(\theta)U_\theta \end{bmatrix}$$

A similar calculation holds for V. To summarize:

$$\boxed{u_x = \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta} \qquad \boxed{v_x = \cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta}$$

$$\boxed{u_y = \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta} \qquad \boxed{v_y = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta}$$

Another way to derive these would be to just apply the chain-rule directly to u_x,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. I leave it to the reader to show you get the same formulas from that approach. The CR-equation $u_x = v_y$ yields:

$$(A.) \quad \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta$$

Likewise the CR-equation $u_y = -v_x$ yields:

$$(B.) \quad \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta = -\cos(\theta)V_r + \tfrac{1}{r}\sin(\theta)V_\theta$$

Multiply (A.) by $r\sin(\theta)$ and $(B.)$ by $r\cos(\theta)$ and subtract (A.) from (B.):

$$U_\theta = -rV_r$$

Likewise multiply (A.) by $r\cos(\theta)$ and $(B.)$ by $r\sin(\theta)$ and add (A.) and (B.):

$$rU_r = V_\theta$$

Finally, recall that $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ hence

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= (\cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta) + i(\cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta) \\ &= (\cos(\theta)U_r + \sin(\theta)V_r) + i(\cos(\theta)V_r - \sin(\theta)U_r) \\ &= (\cos(\theta) - i\sin(\theta))U_r + i(\cos(\theta) - i\sin(\theta))V_r \\ &= e^{-i\theta}(U_r + iV_r) \end{aligned}$$

Theorem 5.3.12. Cauchy Riemann Equations in Polar Form: *If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is a complex function written in polar coordinates r, θ then the Cauchy Riemann equations are written $U_\theta = -rV_r$ and $rU_r = V_\theta$. If $f'(z_o)$ exists then the CR-equations in polar coordinates hold. Likewise, if the CR-equations hold in polar coordinates and all the polar component functions and their partial derivatives with respect to r, θ are continuous on an open disk about z_o then $f'(z_o)$ exists and $f'(z) = e^{-i\theta}(U_r + iV_r)$ which can be written simply as $\dfrac{df}{dz} = e^{-i\theta}\dfrac{\partial f}{\partial r}$.*

Example 5.3.13. *Let $f(z) = z^2$ hence $f'(z) = 2z$ as we have previously derived. That said, lets see how the theorem above works: $f(re^{i\theta}) = r^2e^{2i\theta}$ hence*

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}2re^{2i\theta} = 2re^{i\theta} = 2z.$$

Example 5.3.14. *Let $f(z) = Log(z)$ then for $z \in \mathbb{C}^-$ we find $f(re^{i\theta}) = \ln(r) + i\theta$ for $\theta = Arg(z)$ hence*

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

I mentioned the polar form of Cauchy Riemann equations in these notes since they can be very useful when we work problems on disks. We may not have much occasion to use these, but it's nice to know they exist. I certainly don't expect students to memorize these and I probably will not lecture on this subsection.

5.4 Wirtinger derivative formulation

In physics and engineering you might come across calculations which portray z and $\bar{z}$ as independent variables. This bothered me in that $z = x + iy$ uniquely fixes $\bar{z} = x - iy$. How can z and $\bar{z}$ be independent ? Well, I think the truth of the matter is that when folks talk about z and $\bar{z}$ as independent variables this is actually just a clever way of introducing real variables in the complex setting which can replace x and y and produce formulas which nicely capture the concept of a function being holomorphic on a domain.[7]

Let me share some calculations which elucidate why the definition below is made. Recall $z = x+iy$ and $\bar{z} = x - iy$ thus:

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = i, \quad \frac{\partial \bar{z}}{\partial x} = 1, \quad \frac{\partial \bar{z}}{\partial y} = -i.$$

[7] I've thought a lot about this and I've yet to come to a satisfactory geometric understanding of the conjugate variable scheme, but, perhaps one is already know to those whose life work is several complex variables.

Consider then, by a chain-rule thinking of $f(x,y) = f(z,\bar{z})$[8], (this is **formal nonsense**)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad \& \qquad i\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + i\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = -\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$$

Then add and subtract the equation above to obtain:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] \qquad \& \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]$$

Definition 5.4.1. *The* **Wirtinger Derivatives** *or* **partial complex derivatives** *of a smooth function $f : \mathbb{C} \to \mathbb{C}$ are defined as the following:*

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] \qquad \& \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]$$

Or, more compactly, $\partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f)$ and $\partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$

Sometimes I add a bar to make it easier to see; $\bar{\partial}_{\bar{z}} f = \frac{1}{2}(\partial_x f - i\partial_y f)$. The Theorem below helps bring the slogan that z and $\bar{z}$ are **independent variables** to life:

Theorem 5.4.2. *Observe $\dfrac{\partial \bar{z}}{\partial z} = 0$ and $\dfrac{\partial z}{\partial \bar{z}} = 0$. Moreover, $\dfrac{\partial z}{\partial z} = 1$ and $\dfrac{\partial \bar{z}}{\partial \bar{z}} = 1$.*

Proof: $\dfrac{\partial \bar{z}}{\partial z} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial x}(x - iy) - i\dfrac{\partial}{\partial y}(x - iy)\right] = \dfrac{1}{2}\left[1 + i^2\right] = 0$. The proofs of the remaining identities are similar and might appear in your homework. □

Because the Wirtinger Derivatives are simple linear combinations of real partial derivatives of a complex-valued function we find all the usual rules of calculus apply to the derivatives with respect to z and $\bar{z}$. In abstract algebraic terms, both ∂_z and $\partial_{\bar{z}}$ are **derivations**[9]

Theorem 5.4.3. *If f and g are smooth functions and $c \in \mathbb{C}$ then*

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z} \qquad \& \qquad \frac{\partial}{\partial z}(f+g) = \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \qquad \& \qquad \frac{\partial}{\partial z}(cf) = c\frac{\partial f}{\partial z}$$

and

$$\frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}} \qquad \& \qquad \frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}} \qquad \& \qquad \frac{\partial}{\partial \bar{z}}(cf) = c\frac{\partial f}{\partial \bar{z}}.$$

Proof: the proofs are straightforward, I will just do one to illustrate what is involved,

$$\frac{\partial}{\partial \bar{z}}(cf) = \frac{1}{2}\left[\frac{\partial}{\partial x}(cf) + i\frac{\partial}{\partial y}(cf)\right] = c\frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] = c\frac{\partial f}{\partial \bar{z}}.$$

Here we use Theorem 4.1.6 which states the rules of calculus for the real partial derivatives of complex-valued functions of $x + iy$. The proofs of the other rules are similar. □

Notice the Wirtinger Calculus extend to chain rules and other natural calculations. Forgive me if I skip the justification, but we could prove such calculations as: $\frac{\partial}{\partial z}e^{3z} = 3e^{3z}$ or

$$\frac{\partial}{\partial \bar{z}}\cosh^2(z + 3\bar{z}) = 2\cosh(z + 3\bar{z})\sinh(z + 3\bar{z})\frac{\partial}{\partial \bar{z}}(z + 3\bar{z}) = 6\cosh(z + 3\bar{z})\sinh(z + 3\bar{z}).$$

[8] this is an abuse typical in calculus, we use the same symbol for the map written in different sets of variables.

[9] a derivation on a vector space is a linear transformation which also satifies the product rule, here we consider the vector space of smooth functions on $\mathbb{C}$.

Theorem 5.4.4. *Given functions $f = u + iv$ which is continuously differentiable on a domain D, we find f is holomorphic on D if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ on D. If f is holomorphic then $f'(z) = \frac{\partial f}{\partial z}$.*

Proof: Recall Theorem 5.3.4 stated a continuously differentiable function on $\mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$. Since $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] = 0$ if and only if $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ we find f is holomorphic if and only if $\partial_{\bar{z}} f = 0$. Recall, if f is holomorphic then $f'(z) = \partial_x f$ and $\partial_x f = -i\partial_y f$ hence

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\right] = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] = \frac{\partial f}{\partial z}. \qquad \square.$$

The take-away from the proof above is that $\frac{\partial f}{\partial \bar{z}} = 0$ is simply another notation for the Cauchy Riemann equations.

Example 5.4.5. *Let $f(z) = \bar{z}$ then $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \bar{z}} = 1$. Thus f is not holomorphic on any domain since f does not satisfy the CR-equations on the entire complex plane.*

Example 5.4.6. *Let $f(z) = z\bar{z}$ then $\frac{\partial f}{\partial z} = \bar{z}$ and $\frac{\partial f}{\partial \bar{z}} = z$. Thus f is not holomorphic on any domain since the only place where f satisfies the CR-equations is just $z = 0$.*

Example 5.4.7. *Let $f(z) = \tan(z + z^2 + z^3)$ then*

$$\partial_{\bar{z}} f = \sec^2(z + z^2 + z^3)\partial_{\bar{z}}(z + z^2 + z^3) = 0.$$

Notice, calculating the component functions of $\tan(z+z^2+z^3)$ is a somewhat cumbersome calculation. In contrast, from the perspective of the Wirtinger Calculus, we easily see f is holomorphic at such points as $\cos(z+z^2+z^3) \neq 0$. Ok, I'm starting to feel guilty about not proving the chain-rule for the Wirtinger Calculus. Perhaps that is a good homework problem.

Example 5.4.8. *Let $f(x + iy) = \exp(x^2 + y^2)$ then $f(z) = \exp(z\bar{z})$ hence*

$$\partial_{\bar{z}} f = \exp(z\bar{z})\partial_{\bar{z}}\left[z\bar{z}\right] = z\exp(z\bar{z}).$$

Thus f only satisfies the CR-equations at $z = 0$ which means f is nowhere holomorphic.

5.4.1 chain rule for composite of complex function and path

Notice, we can solve the Wirtinger Derivative equations for the real partial derivatives by adding and subtracting equations:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad \& \qquad i\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \bar{z}} - \frac{\partial f}{\partial z} \quad \Rightarrow \quad \frac{\partial f}{\partial y} = i\left[\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right].$$

Notice, the equations above are not formal nonsense, they're just an algebraic consequence of the notations $\partial_z f$ and $\partial_{\bar{z}} f$. We can write a general chain-rule for a complex function composed with a path in $\mathbb{C}$. From Calculus III, or perhaps Advanced Calculus if you wish, if $\gamma = (x, y) : \mathbb{R} \to \mathbb{C}$ and $f = u + iv : \mathbb{C} \to \mathbb{C}$ then

$$\frac{d}{dt}(u(\gamma(t)) = (\nabla u)(\gamma(t)) \bullet \frac{d\gamma}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}$$

$$\frac{d}{dt}(v(\gamma(t)) = (\nabla v)(\gamma(t)) \bullet \frac{d\gamma}{dt} = v_x \frac{dx}{dt} + v_y \frac{dy}{dt}$$

since $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ by definitions given in the previous chapter, we may add the real chain-rules above and obtain:

$$\frac{d}{dt}(f(\gamma(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

But, we found $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$ and $\frac{\partial f}{\partial y} = i\left[\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right]$ hence

$$\frac{d}{dt}(f(\gamma(t)) = \left[\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right]\frac{dx}{dt} + i\left[\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right]\frac{dy}{dt}$$

where we should understand all the partial derivatives are evaluated along the path $\gamma(t)$. Usually we use the notation $\gamma(t) = z(t)$ so we have $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$ and naturally $\frac{d\bar{z}}{dt} = \frac{dx}{dt} - i\frac{dy}{dt}$. Returning to the chain rule once more we regroup terms to derive:

$$\frac{d}{dt}(f(z(t)) = \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial \bar{z}}\frac{d\bar{z}}{dt}.$$

Theorem 5.4.9. *Given a function $f = u + iv$ which is differentiable near the path $t \mapsto z(t)$ we have*

$$\frac{d}{dt}(f(z(t)) = \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial \bar{z}}\frac{d\bar{z}}{dt}.$$

If $\partial_{\bar{z}} f = 0$ near and on the path then $\frac{d}{dt}(f(z(t)) = \frac{\partial f}{\partial z}\frac{dz}{dt}$. On the other hand, if If $\partial_z f = 0$ near and on the path then $\frac{d}{dt}(f(z(t)) = \frac{\partial f}{\partial \bar{z}}\frac{d\bar{z}}{dt}$.

As we explain in a future chapter, if f is holomorphic on at a point with nonzero derivative then f is **conformal**. The proof of that claim rests on the chain-rule given in the above theorem. I have much less to say about maps with $\partial_z f = 0$, these are known as **anti-holomorphic** maps. Essentially an anti-holomorphic map is a function of $\bar{z}$ alone whereas a holomorphic map is a function of z alone. Again, I feel the tension of the last sentence with the observation $\bar{z}$ is obtained from z. But, there is no danger of confusion provided we follow the rules of the Wirtinger Calculus as defined in this section.

Chapter 6

Holomorphic Functions

In this chapter we examine three major topics involving holomorphic functions.

We begin by examining a somewhat technical result. We see how the inverse function theorem adapts to the context of holomorphic functions. It turns out that the existence of the complex derivative of the local inverse of a function is automatic for a holomorphic function with a nonzero derivatives. That is a nice result which means we are justified in seeking derivatives of inverse functions in the complex domain.

Next we discuss explain why each holomorphic function $f = u + iv$ has a pair of real harmonic functions in the sense that: $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. In other words, each complex differentiable function has component functions which solve Laplace's equation in the plane. Interestingly, given relatively benign topological restrictions on the domain, if we are given a function u for which $u_{xx} + u_{yy} = 0$ then there exists a **harmonic conjugate** v for which $f = u + iv$ is a complex differentiable map. Essentially to find the harmonic congugate we simply integrate the Cauchy Riemann equations.

Last, we begin to contemplate the geometric implications of complex differentiability. We show that any holomorphic map with nonzero derivative will preserve the angle between paths in its domain and their image in its range. This angle-preserving property is called the **conformal** property.

There are other properties of complex differentiable maps which serve to characterize holomorphic maps in a different manner. For the sake of this part of the course I wanted to narrow the focus a bit and just go with what is in this chapter as well as one more HUGE characterization in the next chapter. In particular, we learn in the next chapter that holomorphic maps are necessarily **analytic**. To say a function is *analytic* is to say it has a power series representation at the point. Anyway, more on that in the next chapter.

I left some comments about Gamelin in this Chapter, my apologies since I do not expect you to read Gamelin's text. However, if you are curious, I have a copy in my office if you want to see what I refer to in this chapter.

6.1 Inverse Mappings and the Jacobian

In advanced calculus there are two central theorems of the classical study: the inverse function theorem and the implicit function theorem. In short, the inverse function theorem simply says that if $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable at p and has $\det(F'(p)) \neq 0$ then there exists some neighborhood V of p on which $F|_V$ has a continuously differentiable inverse function. The simplest case of this is calculus I where $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is locally invertible at $p \in U$ if $f'(p) \neq 0$. Note, geometrically this is clear, if the slope were zero then the function will not be $1-1$ near the point so the inverse need not exist. On the other hand, if the derivative is nonzero at a point **and continuous** then the derivative must stay nonzero near the point (by continuity of the derivative function) hence the function is either increasing or decreasing near the point and we can find a local inverse. I remind the reader of these things as they may not have thought through them carefully in their previous course work. That said, I will not attempt a geometric visualization of the complex case. We simply need to calculate the determinant of the derivative matrix and that will allow us to apply the advanced calculus theorem here:

Theorem 6.1.1. *If f is complex differentiable at p then $\det J_f(p) = |f'(p)|^2$.*

Proof: suppose $f = u + iv$ is complex differentiable then the CR equations hold thus:

$$\det \mathrm{J}_f(p) = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = (u_x)^2 + (v_x)^2 = |u_x + iv_x|^2 = |f'(z)|^2. \qquad \square$$

If $f = u + iv$ is holomorphic on a domain D with $(u_x)^2 + (v_x)^2 \neq 0$ on D then f is locally invertible throughout D. The interesting thing about the theorem which follows is we also learn that the inverse function is holomorphic about some small open disk about the point where $f'(p) \neq 0$.

Theorem 6.1.2. *If $f(z)$ is holomorphic on a domain D, $z_o \in D$, and $f'(z_o) \neq 0$. Then there is a (small) disk $U \subseteq D$ containing z_o such that $f|_U$ is $1-1$, the image $V = f(U)$ of U is open, and the inverse function $f^{-1} : V \to U$ is holomorphic and satisfies*

$$(f^{-1})'(f(z)) = 1/f'(z) \qquad \text{for } z \in U.$$

Proof: I will give a proof which springs naturally from advanced calculus. First note that $f'(z_o) \neq 0$ implies $|f'(z_o)|^2 \neq 0$ hence by Theorem 6.1.1 and the inverse function theorem of advanced calculus the exists an open disk U centered about z_o and a function $g : f(U) \to U$ which is the inverse of f restricted to U. Furthermore, we know g is continuously real differentiable. In particular, $g \circ f = Id_U$ and the chain rule in advanced calculus provides $\mathrm{J}_g(f(p))\mathrm{J}_f(p) = I$ for each $p \in U$. Here $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We already learned that the holomorphicity of f implies we can write $\mathrm{J}_f(p) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $u_x(p) = a$ and $v_x(p) = b$. The inverse of such a matrix is given by:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

But, the equation $\mathrm{J}_g(f(p))\mathrm{J}_f(p) = I$ already tells us $(\mathrm{J}_f(p))^{-1} = \mathrm{J}_g(f(p))$ hence we find the Jacobian matrix of $g(f(p))$ is given by:

$$\mathrm{J}_g(f(p)) = \begin{bmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ -b/(a^2 + b^2) & a/(a^2 + b^2) \end{bmatrix}$$

This matrix shows that if $g = m + in$ then $m_x(f(p)) = a/(a^2 + b^2)$ and $n_x = -b/(a^2 + b^2)$. Thus we have $g' = m_x + in_x$ where

$$g'(f(p)) = \frac{1}{a^2 + b^2}(a - ib) = \frac{a - ib}{(a + ib)(a - ib)} = \frac{1}{a + ib} = \frac{1}{f'(p)}. \qquad \square$$

Discussion: I realize some of you have not had advanced calculus so the proof above it not optimal. Thankfully, Gamelin gives an argument on page 52 which is free of matrix arguments. That said, if we understand the form of the Jacobian matrix as it relates the real Jordan form of a matrix then the main result of the conformal mapping section is immediately obvious. In particular, provided $a^2 + b^2 \neq 0$ we can factor as follows

$$\mathrm{J}_f = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2 + b^2} \begin{bmatrix} a/\sqrt{a^2 + b^2} & -b/\sqrt{a^2 + b^2} \\ b/\sqrt{a^2 + b^2} & a/\sqrt{a^2 + b^2} \end{bmatrix}.$$

It follows there exists θ for which

$$\mathrm{J}_f = \pm\sqrt{a^2 + b^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This shows the Jacobian matrix of a complex differentiable mapping has a very special form. Geometrically, we have a scale factor of $\sqrt{a^2 + b^2}$ which either elongates or shrinks vectors. Then the matrix with θ is precisely a rotation by θ. If the $\pm = +$ then in total the Jacobian is just a **dilation** and **rotation**. If the $\pm = -$ then the Jacobian is a **reflection about the origin** followed by a **dilation** and **rotation**. In general, the possible geometric behaviour of 2×2 matrices is much more varied. This decomposition is special to our structure. We discuss the further implications of these observations in Section 6.3.

The application of the inverse function theorem requries less verbosity.

Example 6.1.3. *Note $f(z) = e^z$ has $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$. It follows that there exist local inverses for f about any point in the complex plane. Let $w = Log(z)$ for $z \in \mathbb{C}^-$. Since the inverse function theorem shows us $\frac{dw}{dz}$ exists we may calculate as we did in calculus I. To begin, $w = Log(z)$ hence $e^w = z$ then differentiate to obtain $e^w \frac{dw}{dz} = 1$. But $e^w = z$ thus $\frac{d}{dz} Log(z) = \frac{1}{z}$ for all $z \in \mathbb{C}^-$.*

We should remember, it is not possible to to find a global inverse as we know $e^z = e^{z+2\pi i m}$ for $m \in \mathbb{Z}$. However, given any choice of logarithm $Log_\alpha(z)$ we have $\frac{d}{dz} Log_\alpha(z) = \frac{1}{z}$ for all z in the slit plane which omits the discontinutiy of $Log_\alpha(z)$. In particular, $Log_\alpha(z) \in \mathcal{O}(D)$ for

$$D = \mathbb{C} - \{re^{i\alpha} \mid r \geq 0\}.$$

Example 6.1.4. *Suppose $f(z) = \sqrt{z}$ denotes the principal branch of the square-root function. In particular, we defined $f(z) = e^{\frac{1}{2}Log(z)}$ thus for[1] $z \in \mathbb{C}^-$*

$$\frac{d}{dz}\sqrt{z} = \frac{d}{dz} e^{\frac{1}{2}Log(z)} = e^{\frac{1}{2}Log(z)} \frac{d}{dz}\frac{1}{2}Log(z) = \sqrt{z} \cdot \frac{1}{2z} = \frac{1}{2\sqrt{z}}.$$

[1]we defined $\sqrt{z}$ for all $z \in \mathbb{C}^\times$, however, we cannot find a derivative on all of the punctured plane since if we did that would imply the $\sqrt{z}$ function is continuous on the punctured plane (which is false). In short, the calculation breaks down at the discontinuity of the square root function

Let $\mathcal{L}(z)$ be some branch of the logarithm and define $z^c = e^{c\mathcal{L}(z)}$ we calculate:

$$\frac{d}{dz}z^c = \frac{d}{dz}e^{c\mathcal{L}(z)} = e^{c\mathcal{L}(z)}\frac{d}{dz}c\mathcal{L}(z) = e^{c\mathcal{L}(z)}\frac{c}{z} = cz^{c-1}.$$

To verify the last step, we note:

$$\frac{1}{z} = z^{-1} = e^{-\mathcal{L}(z)} \quad \Rightarrow \quad \frac{1}{z}e^{c\mathcal{L}(z)} = e^{-\mathcal{L}(z)+c\mathcal{L}(z)} = e^{(c-1)\mathcal{L}(z)} = z^{c-1}.$$

Here I used the adding angles property of the complex exponential which we know[2] arises from the corresponding laws for the real exponential and the sine and cosine functions.

6.2 Harmonic Functions

If a function F has second partial derivatives is continuously differentiable then the order of partial derivatives in x and y may be exchanged. In particular,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\big(\ F(x,y)\ \big) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\big(\ F(x,y)\ \big)$$

We will learn as we study the finer points of complex function theory that if a function is complex differentiable at each point in some domain[3] then the complex derivative is **continuous**. In other words, there are no merely complex differentiable functions on a domain, there are only continuously complex differentiable functions on a domain. The word "domain" is crucial to that claim as Example 5.3.3 shows that the complex derivaitve may only exist along stranger sets and yet not exist elsewhere (such a complex derivative function is hardly continuous on $\mathbb{C}$).

In addition to the automatic continuity of the complex derivative on domains[4] we will also learn that the complex derivative function on a domain is itself complex differentiable. In other words, on a domain, if $z \mapsto f'(z)$ exists then $z \mapsto f''(z)$ exists. But, then by the same argument $f^{(3)}(z)$ exists etc. We don't have the theory to develop this claim yet, but, I hope you don't mind me sharing it here. It explains why if $f = u + iv$ is holomorphic on a domain then the second partial derivatives of u, v must exist and be continuous. I suppose it might be better pedagogy to just say we know the second partial derivatives of the component functions of an analytic function are continuous. But, the results I discuss here are a bit subtle and its not bad for us to discuss them multiple times as the course unfolds. We now continue to the proper content of this section.

Laplace's equation is one of the fundamental equations of mathematical physics. The study of the solutions to Laplace's equation is known as **harmonic analysis**. For $\mathbb{R}^n$ the Laplacian is defined:

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

which gives **Laplace's equation** the form $\triangle u = 0$. Again, this is studied on curved spaces and in generality far beyond our scope.

Definition 6.2.1. *Let x, y be Cartesian coordinates on $\mathbb{C}$ then $u_{xx} + u_{yy} = 0$ is* **Laplace's Equation**. *The solutions of Laplace's Equation are called* **harmonic functions**.

[2]perhaps we can give a more fundamental reason based on self-contained arithmetic later in this course!
[3]as we have discussed, a domain is an open and connected set
[4]Gamelin assumes this point as he defines analytic to include this result on page 45

The theorem below gives a very simple way to create new examples of harmonic functions. It also indicates holomorphic functions have very special the component functions.

Theorem 6.2.2. *If $f = u + iv$ is holomorphic on a domain D then u, v are harmonic on D.*

Proof: as discussed at the beginning of this section, we may assume on the basis of later work that u, v have continuous second partial derivatives. Moreover, as f is holomorphic we know u, v solve the CR-equations $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$. Observe

$$\partial_x u = \partial_y v \quad \Rightarrow \quad \partial_x \partial_x u = \partial_x \partial_y v \quad \Rightarrow \quad \partial_x \partial_x u = \partial_y \partial_x v = \partial_y[-\partial_y u]$$

Therefore, $\partial_x \partial_x u + \partial_y \partial_y u = 0$ which shows u is harmonic. The proof for v is similar. $\square$

A fun way to prove the harmonicity of v is to notice that $f = u+iv$ harmonic implies $-if = v - iu$ is harmonic thus $\mathbf{Re}(-if) = v$ and we already showed the real component of f is harmonic thus we may as well apply the result to $-if$.

Example 6.2.3. *Let $f(z) = e^z$ then $e^{x+iy} = e^x \cos y + ie^x \sin y$ hence $u = e^x \cos y$ and $v = e^x \sin y$ are solutions of $\phi_x x + \phi_y y = 0$.*

The functions $u = e^x \cos y$ and $v = e^x \sin y$ have a special relationship. In general:

Definition 6.2.4. *If u is a harmonic function on a domain D and $u + iv$ is holomorphic on D then we say v is a* **harmonic conjugate** *of u on D.*

I chose the word "a" in the definition above rather than the word "the" as the harmonic conjugate is not unique. Observe:

$$\frac{d}{dz}\big(u + i(v + v_o)\big) = \frac{d}{dz}\big(u + iv\big).$$

If v is a harmonic conjugate of u then $v + v_o$ is also a harmonic conjugate of u for any $v_o \in \mathbb{R}$.

A popular introductory exercise is the following:

Given a harmonic function u find a harmonic conjugate v on a given domain.

Gamelin gives a general method to calculate the harmonic conjugate on page 56. This is essentially the same problem we faced in calculus III when we derived potential functions for a given conservative vector field.

Example 6.2.5. *Let $u(x, y) = x^2 - y^2$ then clearly $u_{xx} + u_{yy} = 2 - 2 = 0$. Hence u is harmonic on $\mathbb{C}$. We wish to find v for which $u + iv$ is holomorphic on $\mathbb{C}$. This means we need to solve $u_x = v_y$ and $v_x = -u_y$ which yield $v_y = 2x$ and $v_x = 2y$. Integrating yields:*

$$\frac{\partial v}{\partial y} = 2x \quad \Rightarrow \quad v = 2xy + h_1(x)$$

and

$$\frac{\partial v}{\partial x} = 2y \quad \Rightarrow \quad v = 2xy + h_2(y)$$

from $h_1(x), h_2(y)$ are constant functions and a harmonic conjugate has the form $v(x, y) = 2xy + v_o$. In particular, if we select $v_o = 0$ then

$$u + iv = (x^2 - y^2) + 2ixy = (x + iy)^2$$

The holomorphic function here is just our old friend $f(z) = z^2$.

The shape of the domain was not an issue in the example above, but, in general we need to be careful as certain results have a topological dependence. In Gamelin he proves the theorem below for a rectangle. As he cautions, it is not true for regions with holes like the punctured plane $\mathbb{C}^\times$ or annuli. Perhaps I have assigned problem 7 from page 58 which gives explicit evidence of the failure of the theorem for domains with holes.

Theorem 6.2.6. *Let D be an open disk, or an open rectangle with sides parallel to the axes, and let $u(x, y)$ be a harmonic function on D. Then there is a harmonic function $v(x, y)$ on D such that $u + iv$ is holomorphic on D. The harmonic conjugate v is unique, up to adding a constant.*

6.3 Conformal Mappings

A few nice historical remarks on the importance of the concept discussed in this section is given on page 78 of [R91]. Gauss realized the importance in 1825 and it served as a cornerstone of Riemann's later work. Apparently, Cauchy and Weierstrauss did not make much use of conformality.

Following the proof of the inverse function theorem I argued the 2×2 Jacobian matrix of a holomorphic function was quite special. In particular, we observed it was the product of a reflection, dilation and rotation. That said, at the level of complex notation the same observation is cleanly given in terms of the chain rule and the polar form of complex numbers.

Suppose $f : D \to \mathbb{C}$ is holomorphic on the domain D. Let z_o be a point in D and, for some $\varepsilon > 0$, $\gamma : (-\varepsilon, \varepsilon) \to D$ a path with $\gamma(0) = z_o$. The tangent vector at z_o for γ is simply $\gamma'(0)$. Consider f as the mapping $z \mapsto w = f(z)$; we transport points in the $z = x + iy$-plane to points in the $w = u + iv$-plane. Thus, the curve $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ is naturally a path in the w-plane and we are free to study how the tangent vector of the transformed curve relates to the initial curve in the z-plane. In particular, differentiate and make use of the chain rule for complex differentiable functions[5]:

$$\frac{d}{dt}\big(f(\gamma(t))\big) = \frac{df}{dz}(\gamma(t))\frac{d\gamma}{dt}.$$

Let $\frac{df}{dz}(\gamma(0)) = re^{i\theta}$ and $\gamma'(0) = v$ we find the vector $\gamma'(0) = v$ transforms to $(f \circ \gamma)'(0) = re^{i\theta}v$. Therefore, the tangent vector to the transported curve is stretched by a factor of $r = |(f \circ \gamma)'(0)|$ and rotated by angle $\theta = Arg((f \circ \gamma)'(0))$.

Now, suppose we have c such that $\gamma_1(0) = \gamma_2(0) = z_o$ then $f \circ \gamma_1$ and $f \circ \gamma_2$ are curves through $f(z_o) = w_o$ and we can compare the angle between the curves $f \circ \gamma_1$ at z_o and the angle between the image curves $f \circ \gamma_1$ and $f \circ \gamma_2$ at w_o. Recall the angle between to curves is measured by the angle between their tangent vectors at the point of intersection. In particular, if $\gamma_1'(0) = v_1$ and $\gamma_2'(0) = v_2$ then note $\frac{df}{dz}(\gamma_1(0)) = \frac{df}{dz}(\gamma_1(0)) = re^{i\theta}$ hence both v_1 and v_2 are rotated and stretched in the same fashion. Let us denote $w_1 = re^{i\theta}v_1$ and $w_1 = re^{i\theta}v_1$. Recall the dot-product defines the angle betwen nonzero vectors by $\theta = \dfrac{\vec{A} \bullet \vec{B}}{||\vec{A}||||\vec{B}||}$. Furthermore, we saw shortly after Definition 1.1.3 that the Euclidean dot-product is simply captured by the formula $\langle v, w\rangle = \mathbf{Re}(z\overline{w})$. Hence,

[5] we found this at the end of the last chapter in our discussion of the Wirtinger Calculus

consider:

$$\begin{aligned}\langle w_1, w_2\rangle &= \langle re^{i\theta}v_1,\ re^{i\theta}v_2\rangle \\ &= \mathbf{Re}\left(re^{i\theta}v_1\overline{re^{i\theta}v_2}\right) \\ &= r^2\mathbf{Re}\left(e^{i\theta}v_1\overline{v_2}e^{-i\theta}\right) \\ &= r^2\mathbf{Re}\left(v_1\overline{v_2}\right) \\ &= r^2\langle v_1, v_2\rangle.\end{aligned}$$

Note we have already shown $|w_1| = r|v_1|$ and $|w_2| = r|v_2|$ hence:

$$\frac{\langle v_1, v_2\rangle}{|v_1||v_2|} = \frac{r^2\langle v_1, v_2\rangle}{r|v_1|r|v_2|} = \frac{\langle w_1, w_2\rangle}{|w_1||w_2|}.$$

Therefore, the angle between curves is preserved under holomorphic maps.

Definition 6.3.1. *A smooth complex-valued function $g(z)$ is* **conformal at $\mathbf{z_o}$** *if whenever γ_o, γ_1 are curves terminating at z_o with nonzero tangents, then the curves $g\circ\gamma_o$ and $g\circ\gamma_1$ have nonzero tangents at $g(z_o)$ and the angle between $g\circ\gamma_o$ and $g\circ\gamma_1$ at $g(z_o)$ is the same as the angle between γ_o and γ_1 at z_o.*

Therefore, we have the following result from the calculation of the previous page:

Theorem 6.3.2. *If $f(z)$ is holomorphic at z_o and $f'(z_o) \neq 0$ then $f(z)$ is conformal at z_o.*

This theorem gives beautiful geometric significance to holomorphic functions. The converse of the theorem requires we impose an additional condition. The function $f(z) = \bar{z} = x - iy$ has $\mathrm{J}_f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\det(\mathrm{J}_f) = -1 < 0$. This means that the function does not maintain the **orientation** of vectors. On page 74 of [R91] the equivalence of real differentiable, angle-preserving, orientation-preserving maps and nonzero f' holomorphic maps is asserted. The proof is already contained in the calculations we have considered.

We all should recognize $x = x_o$ and $y = y_o$ as the equations of vertical and horizontal lines respective. At the point (x_o, y_o) these lines intersect at right angles. It follows that the image of the coordinate grid in the $z = x + iy$ plane gives a family of orthogonal curves in the w-plane. In particular, the lines which intersect at (x_o, y_o) give orthogonal curves which intersect at $f(x_o + iy_o)$. In particular $x \mapsto w = f(x + iy_o)$ and $y \mapsto w = f(x_o + iy)$ are paths in the w-plane which intersect orthogonally at $w_o = f(x_o + iy_o)$.

Example 6.3.3. *Consider $f(z) = z^2$. We have $f(x + iy) = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$. Thus,*

$$t \mapsto t^2 - y_o^2 + 2iy_o t \qquad \& \qquad t \mapsto x_o^2 - t^2 + 2ix_o t$$

Let u, v be coordinates on the w-plane. The image of $y = y_o$ has

$$u = t^2 - y_o^2 \qquad \& \qquad v = 2y_o t$$

If $y_o \neq 0$ then $t = v/2y_o$ which gives $u = \frac{1}{4y_o^2}v^2 - y_o^2$. This is a parabola which opens horizontally to the right in the w-plane. The image of $x = x_o$ has

$$u = x_o^2 - t^2 \qquad \& \qquad v = 2x_o t$$

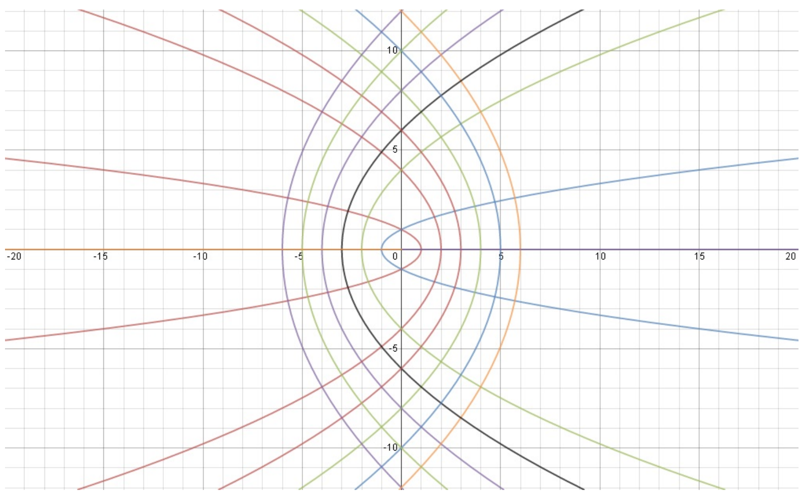
If $x_o \neq 0$ then $t = v/2x_o$ which gives $u = x_o^2 - \frac{1}{4x_o^2}v^2$. This is a parabola which opens horizontally to the left in the w-plane. As level-curves in the w-plane the right-opening parabola is $F(u,v) = u - \frac{1}{4y_o^2}v^2 + y_o^2 = 0$ whereas the left-opening parabola is given by $G(u,v) = u - x_o^2 + \frac{1}{4x_o^2}v^2$. We know the gradients of F and G are normals to the curves. Calculate,

$$\nabla F = \langle 1, -\frac{v}{2y_o^2}\rangle \qquad \& \qquad \nabla G = \langle 1, \frac{v}{2x_o^2}\rangle \qquad \Rightarrow \qquad \nabla F \bullet \nabla G = 1 - \frac{v^2}{4x_o^2y_o^2}$$

At a point of intersection we have $x_o^2 - \frac{1}{4x_o^2}v^2 = \frac{1}{4y_o^2}v^2 - y_o^2$ from which we find $x_o^2 + y_o^2 = v^2(\frac{1}{4x_o^2} + \frac{1}{4y_o^2})$. Multiply by $x_o^2y_o^2$ to obtain $x_o^2y_o^2(x_o^2+y_o^2) = \frac{v^2}{4}(y_o^2+x_o^2)$. But, this gives $1 = \frac{v^2}{4x_o^2y_o^2}$. Therefore, at the point of intersection we find $\nabla F \bullet \nabla G = 0$. It follows the sideways parabolas intersect orthogonally.

If $x_o = 0$ then $t \mapsto -t^2$ is a parametrization of the image of the y-axis which is the negative real axis in the w-plane. If $y_o = 0$ then $t \mapsto t^2$ is a parametrization of the image of the x-axis which is the positive real axis in the w-plane. The point at which these exceptional curves intersect is $w = 0$ which is the image of $z = 0$. That point, is the only point at which $f'(0) \neq 0$.

I plot several of the curves in the w-plane. You can see how the intersections make right angles at each point except the origin.



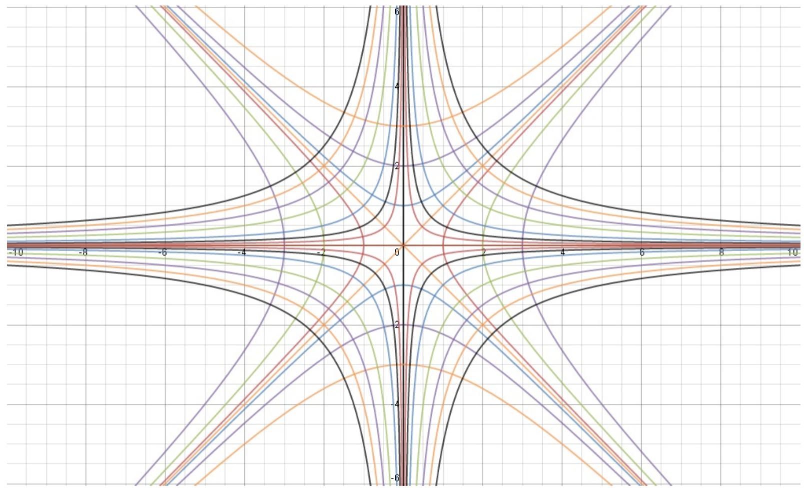
The plot above was produced using www.desmos.com which I whole-heartedly endorse for simple graphing tasks.

We can also study the inverse image of the cartesian coordinate lines $u = u_o$ and $v = v_o$ in the z-plane. In particular,

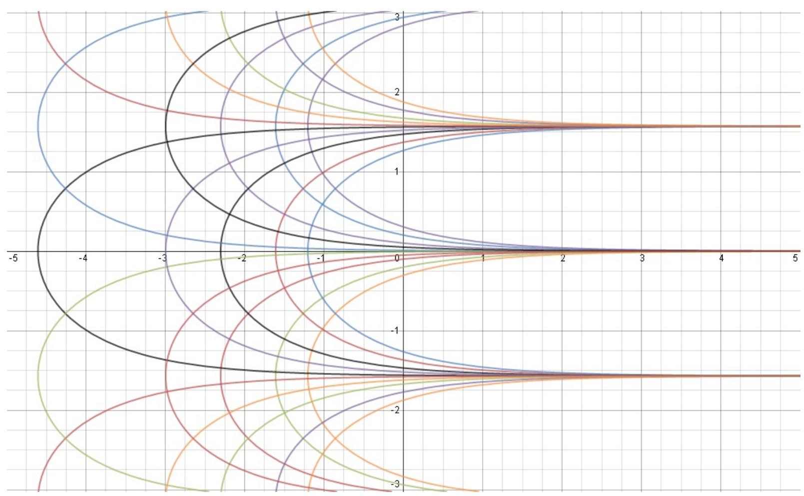
$$u(x,y) = u_o \qquad \& \qquad v(x,y) = v_o$$

give curves in $z = x + iy$-plane which intersect at z_o orthogonally provided $f'(z_o) \neq 0$.

Example 6.3.4. *We return to Example 6.3.3 and as the reverse question: what is the inverse image of $u = u_o$ or $v = v_o$ for $f(z) = z^2$ where $z = x + iy$ and $u = x^2 - y^2$ and $v = 2xy$. The curve $x^2 - y^2 = u_o$ is a hyperbola with asymptotes $y = \pm x$ whereas $2xy = v_o$ is also a hyperbola, but, it's asymptotes are the x, y axes. Note that $u_o = 0$ gives $y = \pm x$ whereas $v_o = 0$ gives the x, y-axes. These meet at the origin which is the one point where $f'(z) \neq 0$.*



Example 6.3.5. *Consider $f(z) = e^z$ then $f(x+iy) = e^x \cos y + ie^x \sin y$. We observe $u = e^x \cos y$ and $v = e^x \sin y$. The curves $u_o = e^x \cos y$ and $v_o = e^x \sin y$ map to the vertical and horizontal lines in the w-plane. I doubt these are familar curves in the xy-plane. Here is a plot of the z-plane with the inverse images of a few select u, v-coordinate lines:*



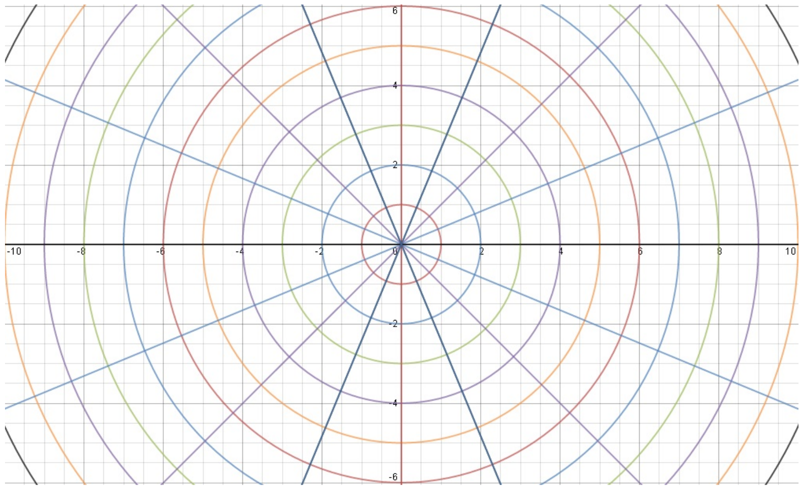
On the other hand, we can study how $z \mapsto w = e^z$ distorts the x, y-coordinate grid. The horizontal line through $x_o + iy_o$ is parametrized by $x = x_o + t$ and $y = y_o$ has image

$$t \mapsto f(x_o + t + iy_o) = e^{x_o + t} e^{iy_o}$$

as t varies we trace out the ray from the origin to ∞ in the w-plane at angle y_o. The vertical line through $x_o + iy_o$ is parametrized by $x = x_o$ and $y = y_o + t$ has image

$$t \mapsto f(x_o + t + iy_o) = e^{x_o} e^{i(y_o + t)}$$

as t varies we trace out a circle of radius e^{x_o} centered at the origin of the w-plane. Therefore, the image of the x, y-coordinate lines in the w-plane is a family of circles and rays eminating from the origin. Notice, the origin itself is not covered as $e^z \neq 0$.



There is another simple calculation to see the orthogonality of constant u or v curves. Calculate $\nabla u = \langle u_x, u_y \rangle$ and $\nabla v = \langle v_x, v_y \rangle$. But, if $f = u + iv$ is holomorphic then $u_x = v_y$ and $v_x = -u_y$. By CR-equations,

$$\nabla u = \langle u_x, u_y \rangle = \langle v_y, -v_x \rangle$$

but, $\nabla v = \langle v_x, v_y \rangle$ hence $\nabla u \bullet \nabla v = 0$. Of course, this is just a special case of our general result on conformality of holomorphic maps.

Chapter 7

Integration

In this chapter we discover many surprising theorems which connect a holomorphic function and its integrals and derivatives. In part, the results here are merely a continuation of the complex-valued multivariate analysis studied in the previous chapter. However, the Theorem of Goursat and Cauchy's integral formula lead to striking results which are not analogus to the real theory. In particular, if a function is complex differentiable on a domain then Goursat's Theorem provides that $z \mapsto f'(z)$ is automatically a continuous mapping. There is no distinction between complex differentiable and continuously complex differentiable in the function theory on a complex domain. Moreover, if a function is once complex differentiable then it is twice complex differentiable. Continuing this thought, there is no distinction between the complex smooth functions and the complex once-differentiable functions on a complex domain. These distinctions are made in the real case and the distinctions are certainly aspects of the more subtle side of real analysis. These truths and more we discover in this chapter.

Before going into the future, let us pause to enjoy a quote by Gauss from 1811 to a letter to Bessel:

> What should we make of $\int \phi x \cdot dx$ for $x = a + bi$? Obviously, if we're to proceed from clear concepts, we have to assume that x passes, via infinitely small increments (each of the form $\alpha + i\beta$), from that value at which the integral is supposed to be 0, to $x = a + bi$ and that then all the $\phi x \cdot dx$ are summed up. In this way the meaning is made precise. But the progression of x values can take place in infinitely many ways: Just as we think of the realm of all real magnitudes as an infinite straight line, so we can envision the realm of all magnitudes, real and imaginary, as an infinite plane wherein every point which is determined by an abscissa a and ordinate b represents as well the magnitude $a + bi$. The continuous passage from one value of x to another $a + bi$ accordingly occurs along a curve and is consequently possible in infinitely many ways. But I maintain that the integral $\int \phi x \cdot dx$ computed via two different such passages always gets the same value as long as $\phi x = \infty$ never occurs in the region of the plane enclosed by the curves describing these two passages. This is a very beautiful theorem, whose not-so-difficult proof I will give when an appropriate occassion comes up. It is closedly related to other beautiful truths having to do with developing functions in series. The passage from point to point can always be carried out without touching one where $\phi x = \infty$. However, I demand that those points be avoided lest the original basic conception $\int \phi x \cdot dx$ lose its clarity and lead to contradictions. Moreover, it is also clear how a function generated by $\int \phi x \cdot dx$ could have several values for the same values of x depending on whether a

> point where $\phi x = \infty$ is gone around not at all, once, or several times. If, for example, we define $\log x$ having gone around $x = 0$ one of more times or not at all, every circuit adds the constant $2\pi i$ or $-2\pi i$; thus the fact that every number has multiple logarithms becomes quite clear" (*Werke* 8, 90-92 according to [R91] page 167-168)

This quote shows Gauss knew complex function theory before Cauchy published the original monumental works on the subject in 1814 and 1825. Apparently, Poisson also published an early work on complex integration in 1813. See [R91] page 175.

7.1 Contour Integral

The definition of the complex integral is naturally analogus to the usual Riemann sum in $\mathbb{R}$. In the real integral one considers a partition of $x_o, x_1, \dots, x_n$ which divides $[a, b]$ into n-subintervals. In the complex integral, to integrate along a path γ we consider points $z_o, z_1, \dots, z_n$ along the path. In both cases, as $n \to \infty$ we obtain the integral.

Definition 7.1.1. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $f(z)$ a complex-valued function which is continuous on and near γ. Let $z_o, z_1, \dots, z_n \in trace(\gamma)$ where $a \leq t_o < t_1 < \cdots < t_n \leq b$ and $\gamma(t_j) = z_j$ for $j = 0, 1, \dots, n$. We define:*

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

Equivalently, as a complex-valued integral over the real parameter of the path:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt.$$

Or, as a complex combination of real line-integrals:

$$\int_\gamma f(z)\,dz = \int_\gamma udx - vdy + i\int_\gamma udy + vdx.$$

The initial definition above is not our typical method of calculation! In fact, the boxed formulas we find in the next page or so are equivalent to the initial, Riemann sum definition given above. I thought I should start with this so you better appreciate the boxed-definitions which we uncover below. Consider,

$$z_j - z_{j-1} = \gamma(t_j) - \gamma(t_{j-1}) = \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}(t_j - t_{j-1})$$

Applying the mean value theorem[1] we select $t_j^* \in [t_{j-1}, t_j]$ for which $\gamma'(t_j^*) = \frac{\gamma(t_j)-\gamma(t_{j-1})}{t_j - t_{j-1}}$. Returning to the integral, and using $\triangle t_j = t_j - t_{j-1}$ we obtain

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\gamma(t_j))\frac{d\gamma}{dt}(t_j^*)\triangle t_j = \boxed{\int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt.}$$

I sometimes use the boxed formula above as the definition of the complex integral. Moreover, in practice, we set $z = \gamma(t)$ as to symbolically replace dz with $\frac{dz}{dt}dt$. See Example 7.1.3 for an example

[1] this will be justified in homework

of this notational convenience. That said, the expression above can be expressed as a complex-linear combination of two real integrals. If we denote $\gamma = x+iy$ and $f = u+iv$ then (I omit some t-dependence to make it fit in second line)

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \lim_{n\to\infty}\sum_{j=1}^{n}(u(\gamma(t_j))+iv(\gamma(t_j)))\left(\frac{dx}{dt}(t_j^*)+i\frac{dy}{dt}(t_j^*)\right)\triangle t_j \\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\left(u\circ\gamma\frac{dx}{dt}-v\circ\gamma)\frac{dy}{dt}\right)\triangle t_j + i\lim_{n\to\infty}\sum_{j=1}^{n}\left(u\circ\gamma\frac{dy}{dt}+v\circ\gamma\frac{dx}{dt}\right)\triangle t_j \\
&= \int_a^b\left(u(\gamma(t))\frac{dx}{dt}-v(\gamma(t))\frac{dy}{dt}\right)dt + i\int_a^b\left(u(\gamma(t))\frac{dy}{dt}+v(\gamma(t))\frac{dx}{dt}\right)dt \\
&= \boxed{\int_\gamma udx-vdy+i\int_\gamma udy+vdx.}
\end{aligned}$$

Definition 7.1.2. *Introduce notation for integration of vector fields with complex components:*

$$\int_\gamma (P_1+iP_2)dx+(Q_1+iQ_2)dy = \int_\gamma P_1dx+Q_1dy+i\int_\gamma P_2dx+Q_2dy.$$

Then the contour integral of $f = u+iv$ can be rewritten as follows:

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma udx-vdy+i\int_\gamma udy+vdx \\
&= \int_\gamma (u+iv)dx+(-v+iu)dy.
\end{aligned}$$

If we let $dz = dx+idy$ then the formula above can be factored since $(-v+iu)dy = (u+iv)idy$,

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)dx+(-v+iu)dy = \int_\gamma (u+iv)(dx+idy).$$

If we could imagine ourselves in the place of a 19-th century mathematician where dx and dy we intuitively little increments of real variables then supposing $dz = dx+idy$ is just the thought that a little increment of complex values decomposes into a complex combination of real increments dx and dy. In any event, the differential notation is surprisingly consistent so we have many ways to approach the contour integral. If the reader knows how to calculate line-integrals in the plane then he is well prepared to calculate contour integrals. That said, a reasonable method to calculate $\int_C f(z)dz$ is as follows:

- **(i.)** find a parametrization for C, say $z = g(t)$ for $t_1 \leq t \leq t_2$
- **(ii.)** calculate $f(g(t)$ by setting $z = g(t)$ in the formula for $f(z)$ and calculate $dz = \frac{dg}{dt}dt$
- **(iii.)** integrate $f(z)dz = f(g(t))\frac{dg}{dt}dt$ over $[t_1,t_2]$ (simplify before integrating !)

Calculational Comment: For your convenience, let us pause to note some basic properties of an integral of a complex-valued function of a real variable. In particular, suppose $f(t), g(t)$ are continuous complex-valued functions of $t \in \mathbb{R}$ and $c \in \mathbb{C}$ and $a, b \in \mathbb{R}$ then

$$\int (f(t)+g(t))\,dt = \int f(t)dt + \int g(t)dt \qquad \& \qquad \int cf(t)dt = c\int f(t)dt$$

More importantly, the FTC naturally extends; if $\frac{dF}{dt} = f$ then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Notice, this is not quite the same as first semester calculus. Theorem 5.4.9 is immensely useful in what follows from here on out. Often, as we calculate dz by $\frac{d\gamma}{dt}dt$ we have $\gamma(t)$ written as the composition of a holomorphic function of z and some simple function of t. I already used this in Examples 7.1.3 and 7.1.4. Did you notice?

Example 7.1.3. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the unit-circle $\gamma(t) = e^{it}$. Calculate $\int_\gamma \frac{dz}{z}$. Note, if $z = e^{it}$ then $dz = ie^{it}dt$ hence:*

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i.$$

Example 7.1.4. *Let C be the line-segment from p to q parametrized by $t \in [0, 1]$; $z = p + t(q - p)$ hence $dz = (q - p)dt$. We calculate, for $n \in \mathbb{Z}$ with $n \neq -1$,*

$$\int_C z^n dz = \int_0^1 (p + t(q - p))^n (q - p)dt = \frac{(p + t(q - p))^{n+1}}{n + 1}\bigg|_0^1 = \frac{q^{n+1}}{n + 1} - \frac{p^{n+1}}{n + 1}.$$

Example 7.1.5. *Let $\gamma = [p, q]$ and let $c \in \mathbb{C}$ with $c \neq -1$. Recall $f(z) = z^c$ is generally a multiply-valued function whose set of values is given by $z^c = exp(c\log(z))$. Suppose p, q fall in a subset of $\mathbb{C}$ on which a single-value of z^c is defined and let z^c denote that* **function** *of z. Let $\gamma(t) = p + tv$ where $v = q - p$ for $0 \leq t \leq 1$ thus $dz = vdt$ and:*

$$\int_\gamma z^c dz = \int_0^1 (p + tv)^c v dt$$

notice $\dfrac{d}{dt}\left[\dfrac{(p + tv)^{c+1}}{c + 1}\right] = (p+tv)^c v$ *as we know $f(z) = z^{c+1}$ has $f'(z) = (c+1)z^c$ and $\frac{d}{dt}(p+tv) = v$. The chain rule (Theorem 5.4.9) completes the thought. Consequently, by FTC for complex-valued integrals of a real variable,*

$$\int_\gamma z^c dz = \frac{(p + tv)^{c+1}}{c + 1}\bigg|_0^1 = \frac{p^{c+1}}{c + 1} - \frac{q^{c+1}}{c + 1}.$$

Notice, the $n = 0$ case of this example yields:

$$\int_{[p,q]} dz = q - p.$$

An **arc** is a curve which is formed from joining finitely many smooth paths. We should extend our definition of integration to an arc:

Definition 7.1.6. *In particular, if γ is a curve formed by joining the smooth paths $\gamma_1, \gamma_2, \dots, \gamma_n$. In terms of the trace denoted trace$(\gamma) = [\gamma]$ we have $[\gamma] = [\gamma_1] \cup [\gamma_2] \cup \cdots \cup [\gamma_n]$. Let $f(z)$ be complex valued and continuous near the trace of γ. Define:*

$$\int_\gamma f(z)\,dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz.$$

Example 7.1.7. *Let $\gamma = [z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$. Calculate, using the $n = 0$ case from Example 7.1.5 on each line-segment:*

$$\begin{aligned}
\int_\gamma dz &= \sum_{i=1}^{n} \int_{[z_{i-1}, z_i]} dz \\
&= \sum_{i=1}^{n} (z_i - z_{i-1}) \\
&= z_1 - z_0 + z_2 - z_1 + \cdots + z_{n-1} - z_{n-2} + z_{n-1} - z_n \\
&= z_n - z_0.
\end{aligned}$$

So, if $z_0 = z_n$ then the integral reduces to zero.

Example 7.1.8. *Let $f(z) = z^2$ and let $C = C_R^+ \cup [-R, R]$ be the CCW-oriented boundary of the upper half-disk of the closed R-disk centered at the origin. Calculate $\int_C z^2 dz$. Notice,*

$$\int_C z^2 dz = \int_{C_R^+} z^2 dz + \int_{[-R,R]} z^2 dz$$

For $z \in C_R^+$ we write $z = Re^{it}$ with $dz = iRe^{it} dt$ for $0 \leq t \leq \pi$ and for $z \in [-R, R]$ we write $z = x$ with $dz = dx$ for $-R \leq x \leq R$. Hence calculate:

$$\begin{aligned}
\int_C z^2 dz &= \int_0^\pi \left(Re^{it}\right)^2 iRe^{it} dt + \int_{-R}^{R} x^2 dx \\
&= \int_0^\pi iR^3 e^{3it} dt + \int_{-R}^{R} x^2 dx \\
&= \frac{iR^3 e^{3it}}{3i}\bigg|_0^\pi + \frac{x^3}{3}\bigg|_{-R}^{R} \\
&= \frac{R^3 e^{3i\pi}}{3} - \frac{R^3}{3} + \frac{R^3}{3} - \frac{(-R)^3}{3} \\
&= 0. \qquad (7.1)
\end{aligned}$$

Notice, in $\mathbb{C}^\times$, any loop not containing the origin can be smoothly deformed to a point in and thus it is true that $\int_\gamma \frac{dz}{z} = 0$ if 0 is not within the interior of the loop.

Example 7.1.9. *Let $R > 0$ and z_o a fixed point in the complex plane. Assume the integration is taken over a positively oriented parametrization of the pointset indicated: for $m \in \mathbb{Z}$,*

$$\int_{|z-z_o|=R} (z - z_o)^m \, dz = \begin{cases} 2\pi i & \text{for } m = -1 \\ 0 & \text{for } m \neq -1. \end{cases}$$

Let $z = z_o + Re^{it}$ for $0 \leq t \leq 2\pi$ parametrize $|z - z_o| = R$. Note $dz = iRe^{it} dt$ hence

$$\begin{aligned}
\int_{|z-z_o|=R} (z - z_o)^m \, dz &= \int_0^{2\pi} (Re^{it})^m iRe^{it} dt \\
&= iR^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt \\
&= iR^{m+1} \int_0^{2\pi} \big(\cos[(m+1)t] + i \sin[(m+1)t]\big) dt.
\end{aligned}$$

The integral of any integer multiple of periods of trigonometric functions is trivial. However, in the case $m = -1$ the calculation reduces to $\int_{|z-z_o|=R}(z-z_o)^{-1}\,dz = i\int_0^{2\pi}\cos(0)dt = 2\pi i$. I encourage the reader to extend this calculation to arbitrary loops which encircle z_o by applying Corollary 7.2.10 part (iii.).

Let γ be a loop containing z_o in its interior. An interesting aspect of the example above is the contrast of $\int_\gamma \frac{dz}{z-z_o} = 2\pi i$ and $\int_\gamma \frac{dz}{(z-z_o)^2} = 0$. One might be tempted to think that divergence at a point necessitates a non-trivial loop integral after seeing the $m = -1$ result. However, it is not the case. At least, not at this naive level of investigation. Later we will see the quadratic divergence generates nontrivial integrals for $f'(z)$. Cauchy's Integral formula studied later in this chapter will make this clear. Next, we consider less exact methods. Often, what follows it the only way to calculate something. In contrast to the usual presentation of real-valued calculus, the inequality theorem below is a weapon we will wield to conquer formiddable enemies later in this course. So, sharpen your blade now as to prepare for war.

Following Gamelin, denote the infinitesimal arclength $ds = |dz|$ and define the integral with respect to arclength of a complex-valued function by:

Definition 7.1.10. *Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path and $f(z)$ a complex-valued function which is continuous on and near γ. Let $z_o, z_1, \dots, z_n \in trace(\gamma)$ where $a \leq t_o < t_1 < \cdots < t_n \leq b$ and $\gamma(t_j) = z_j$ for $j = 0, 1, \dots, n$. We define:*

$$\int_\gamma f(z)\,|dz| = \lim_{n\to\infty}\sum_{j=1}^{n} f(z_j)|z_j - z_{j-1}|.$$

Equivalently, as a complex-valued integral over the real parameter of the path:

$$\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t))\left|\frac{d\gamma}{dt}\right| dt.$$

We could express this as a complex-linear combination of the standard real-arclength integrals of multivariate calculus, but, I will abstain. It is customary in Gamelin to denote the length of the path γ by L. We may calculate L by integration of $|dz|$ along $\gamma = x + iy : [a,b] \to \mathbb{C}$:

$$L = \int_\gamma |dz| = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt.$$

Of course, this is just the usual formula for arclength of a parametrized curve in the plane. The Theorem below is often called the **ML-estimate** or **ML-theorem** throughout the remainder of this course.

Theorem 7.1.11. *Let $h(z)$ be a continuous near a smooth path γ with length L. Then*

1. $\left|\int_\gamma h(z)\,dz\right| \leq \int_\gamma |h(z)|\,|dz|$.

2. *If $|h(z)| \leq M$ for all $z \in [\gamma]$ then* $\left|\int_\gamma h(z)\,dz\right| \leq ML$.

Proof: in terms of the Riemann sum formulation of the complex integral and arclength integral the identities above are merely consequences of the triangle inequality applied to a particular approximating sum. Note:

$$\left|\sum_{j=1}^{n} h(z_j)(z_j - z_{j-1})\right| \leq \sum_{j=1}^{n} |h(z_j)(z_j - z_{j-1})| = \sum_{j=1}^{n} |h(z_j)||z_j - z_{j-1}|$$

where we used **multiplicativity of the norm**[2] in the last equality and the triangle inequality in the first inequality. Now, as $n \to \infty$ we obtain (1.). The proof of (2.) is one more step:

$$\left|\sum_{j=1}^{n} h(z_j)(z_j - z_{j-1})\right| \leq= \sum_{j=1}^{n} |h(z_j)||z_j - z_{j-1}| \leq \sum_{j=1}^{n} M|z_j - z_{j-1}| = M\sum_{j=1}^{n} |z_j - z_{j-1}| = ML. \quad \square$$

I should mention, last time I taught this course I tried to prove this on the fly directly from the definition written as $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt$. It went badly. There are proofs which are not at the level of the Riemann sum and it's probably worthwhile to share a second proof. I saw this proof in my complex analysis course given by my advisor Dr. R.O. Fulp in 2005 at NCSU.

Alternate Proof: we begin by developing a theorem for complex-valued functions of a real-variable. We claim **Lemma:** $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|\,dt$. Notice that $w(t)$ denotes the modulus of the complex value $w(t)$. If $w(t) = 0$ on $[a, b]$ then the claim is true. Hence, suppose $w(t)$ is continuous and hence the integral of $w(t)$ exists and we set $R > 0$ and $\theta \in \mathbb{R}$ such that $\int_a^b w(t)dt = Re^{i\theta}$. Let's get real: in particular $R = e^{-i\theta}\int_a^b w(t)dt = \int_a^b e^{-i\theta}w(t)dt$ hence:

$$\begin{aligned}
R &= \int_a^b e^{-i\theta}w(t)dt \\
&= \mathbf{Re}\left(\int_a^b e^{-i\theta}w(t)dt\right) \\
&= \int_a^b \mathbf{Re}(e^{-i\theta}w(t))\,dt \\
&\leq \int_a^b \left|e^{-i\theta}w(t)\right|dt \qquad \text{due to a property of modulus; } \mathbf{Re}(z) \leq |z| \\
&= \int_a^b |w(t)|\,dt
\end{aligned}$$

Thus, the Lemma follows as: $|\int_a^b w(t)\,dt| = |Re^{i\theta}| \leq \int_a^b |w(t)|\,dt$. Now, suppose $h(z)$ is complex-valued and continuous near $\gamma : [a, b] \to \mathbb{C}$. We calculate, using the Lemma, then multiplicative property of the modulus:

$$\left|\int_\gamma h(z)\,dz\right| = \left|\int_a^b h(\gamma(t))\frac{d\gamma}{dt}\,dt\right| \leq \int_a^b \left|h(\gamma(t))\frac{d\gamma}{dt}\right|\,dt = \int_a^b |h(\gamma(t))|\left|\frac{d\gamma}{dt}\right|\,dt = \int_\gamma |h(z)|\,|dz|.$$

This proves (1.) and the proof of (2.) is essentially the same as we discussed in the first proof. $\square$

[2]in $\mathcal{A}$-Calculus we get a modified ML-theorem accordinng to the size of the structure constants. Note, the alternate proof would not go well in $\mathcal{A}$ since we do not have a polar representation of an arbtrary $\mathcal{A}$-number.

Example 7.1.12. *Consider $h(z) = 1/z$ on the unit-circle γ. Clearly, $|z| = 1$ for $z \in [\gamma]$ hence $|h(z)| = 1$ which means this estimate is* **sharp***, it cannot be improved. Furthermore, $L = 2\pi$ and the ML-estimate shows $\left|\int_\gamma \frac{dz}{z}\right| \leq 2\pi$. Indeed, in Example 7.1.3 $\int_\gamma \frac{dz}{z} = 2\pi i$ so the estimate is not too shabby.*

Typically, the slightly cumbersome part of applying the ML-estimate is fiinding M. Helpful techniques include: using the polar form of a number, $\mathbf{Re}(z) \leq |z|$ and $\mathbf{Im}(z) \leq |z|$ and naturally $|z + w| \leq |z| + |w|$ as well as $|z - w| \geq ||z| - |w||$ which is useful for manipulating denomiinators.

Example 7.1.13. *Let γ_R be the half-circle of radius R going from R to $-R$ on the real-axis. Find an bound on the modulus of $\int_{\gamma_R} \frac{dz}{z^2+6}$. Notice, on the circle we have $|z| = R$. Furthermore,*

$$\frac{1}{|z^2+6|} \leq \frac{1}{||z^2| - |6||} = \frac{1}{||z|^2 - 6|} = \frac{1}{|R^2 - 6|}$$

If $R > \sqrt{6}$ then we have bound $M = \frac{1}{R^2-6}$ for which $|h(z)| \leq M$ for all $z \in \mathbb{C}$ with $|z| = R$. Note, $L = \pi R$ for the half-circle and the ML-estimate gives:

$$\left|\int_{\gamma_R} \frac{dz}{z^2+6}\right| \leq \frac{\pi R}{R^2-6}.$$

Notice, if we consider $R \to \infty$ then we find from the estimate above and the squeeze theorem that $\left|\int_{\gamma_R} \frac{dz}{z^2+6}\right| \to 0$. It follows that the integral of $\frac{dz}{z^2+6}$ over an infinite half-circle is zero.

A similar calculation shows any rational function $f(z) = p(z)/q(z)$ with $deg(p(z)) + 2 \leq deg(q(z))$ has an integral which vanishes over sections of a cricle which has an infinite radius. There is a general proof of this assertion which I owe you for homework.

7.2 FTC for Complex Integral and Cauchy's Theorem

The term **primitive** means antiderivative. In particular:

Definition 7.2.1. *We say $F(z)$ is a* **primitive** *of $f(z)$ on D iff $F'(z) = f(z)$ for each $z \in D$.*

The fundamental theorem of calculus part II has a natural analog in our context.

Theorem 7.2.2. Complex FTC II: *Let $f(z)$ be continuous with primitive $F(z)$ on domain D then if γ is a path from A to B in D then $\displaystyle\int_\gamma f(z)\,dz = F(b) - F(a)$.*

Proof: Suppose $\gamma : [t_1, t_2] \to \mathbb{C}$ is a path from A to B in a domain D. recall the complex derivative can be cast as a partial derivative with respect to x or y in the following sense: $\frac{dF}{dz} = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$.

Thus:

$$\begin{aligned}
\int_\gamma f(z)\,dz = \int_\gamma \frac{dF}{dz}\,dz &= \int_\gamma \frac{dF}{dz}\,dx + i\int_\gamma \frac{dF}{dz}\,dy \\
&= \int_\gamma \frac{\partial F}{\partial x}\,dx + i\int_\gamma -i\frac{\partial F}{\partial y}\,dy \\
&= \int_\gamma \left(\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right) \\
&= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}\right)\frac{d\gamma}{dt}dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left[F(\gamma(t))\right]dt \\
&= F(\gamma(t_2)) - F(\gamma(t_1)) \\
&= F(B) - F(A).
\end{aligned}$$

where we used Theorem 5.4.9 to rewrite the integrand as a total derivative. Not surprisingly, the usual FTC II from basic calculus is crucial to the proof above. □

In the context of the above theorem we sometimes use the notation $\int_\gamma f(z)\,dz = \int_A^B f(z)\,dz$. This notation should be used with care.

Example 7.2.3.

$$\int_0^{1+3i} z^3\,dz = \frac{1}{4}z^4\Big|_0^{1+3i} = \frac{(1+3i)^4}{4}.$$

Corollary 7.2.4. *Suppose C is a closed curve and the complex function f has antiderivative F at all points of C then $\int_C f(z)dz = 0$.*

Proof: pick a point A in C and note that C is a path from A to A. Apply Theorem 7.2.2 to find $\int_C f(z)dz = F(A) - F(A) = 0$. □

The example below was inspired from page 108 of Gamelin.

Example 7.2.5. *The function $f(z) = 1/z$ has primitive $Log(z) = \ln|z| + iArg(z)$ on $\mathbb{C}^-$. We can capture the integral around the unit-circle by a limitiing process. Consider the unit-circle, positively oriented, with an $\pm\varepsilon$-sector deleted about negative x-axis; $\gamma_\varepsilon : [-\pi+\epsilon, \pi-\varepsilon] \to \mathbb{C}$ with $\gamma(t) = e^{it}$. The path has starting point $\gamma(-\pi+\varepsilon) = e^{i(-\pi+\varepsilon)}$ and ending point $\gamma(\pi-\epsilon) = e^{i(\pi-\epsilon)}$. Note $[\gamma_\varepsilon] \subset \mathbb{C}^-$ hence for each $\epsilon > 0$ we are free to apply the complex FTC:*

$$\int_{\gamma_\varepsilon} \frac{dz}{z} = Log(e^{i(\pi-\varepsilon)}) - Log(e^{i(-\pi+\varepsilon)}) = i(\pi-\varepsilon) - i(-\pi+\varepsilon) = 2\pi i - 2i\varepsilon.$$

Thus, as $\varepsilon \to 0$ we find $2\pi i - 2i\varepsilon \to 2\pi i$ and $\gamma_\varepsilon \to \gamma_0$ where γ_0 denotes the positively oriented unit-circle. Therefore, we find: $\int_{\gamma_0} \frac{dz}{z} = 2\pi i$.

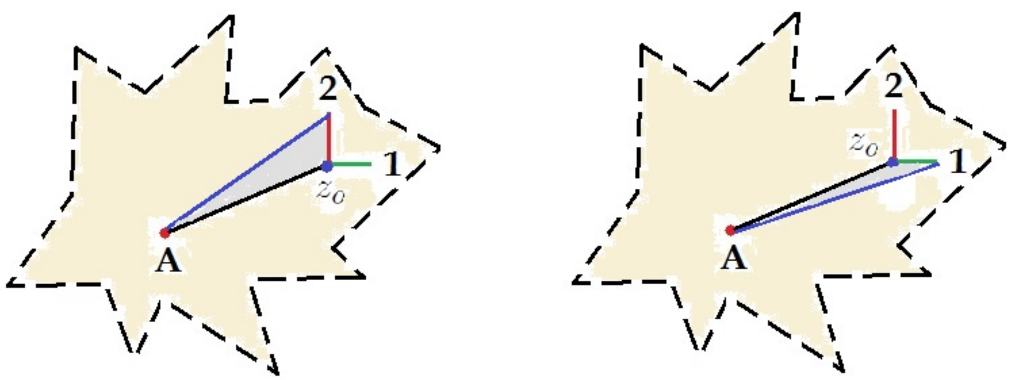
The example above gives us another manner to understand Example 7.1.3. It all goes back to the $2\pi\mathbb{Z}$ degeneracy of the standard angle. Notice the other examples in the previous section can also be calculated directly by application of FTC II since the primitives of the functions we considered in the last section were all easily antidifferentiated. Next we consider the analog of FTC I for complex integration. Remember, the existential import of FTC I in basic calculus is that an antiderivative for a continuous function exists in that the area function of the given continuous function serves as its antiderivative. In the complex context, a holomorphic function on a star-shaped domain[3] has an antiderivative given by a path integral from the star-center.

Theorem 7.2.6. Complex FTC I: *let D be star-shaped and let $f(z)$ be holomorphic on D. Then $f(z)$ has a primitive on D and the primitive is unique up to an additive constant. A primitive for $f(z)$ is given by*[4]

$$F(z) = \int_A^z f(\zeta)\, d\zeta$$

where A is a star-center of D and the integral is taken along some path in D from A to z.

Proof: assume D is a star-shaped domain with star-center A and suppose $f \in \mathcal{O}(D)$. If $f = u+iv$ then we have $u_x = v_y$ and $u_y = -v_x$ on D. Define $h(z) = \int_{[A,z]}(u+iv)(dx+idy) = \int_{[A,z]} f(\zeta)d\zeta$. Expanding $d\zeta = dx + idy$ we expand $fd\zeta = f(dx+idy) = fdx + ifdy$.



Fix a point z_o in D and note $[A, z_o]$ is in D. Furthermore, γ_1 is given by $x = t$ for $x_o \leq t \leq x$ and $y = y_o$. Likewise, γ_2 is the line-segment $[z_o, x_o + iy]$ where $x = x_o$ and $y = t$ for $y_o \leq t \leq y$. Note $[A, x_o + iy]$ and $[A, x + iy_o]$ are in D. Apply Green's Theorem[5] on the triangle T_2 with vertices $A, z_o, x_o + iy$:

$$\int_{\partial T_2} f(\zeta)d\zeta = \int_{\partial T_2} fdx + ifdy = \iint_{T_2} \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_{T_2} (0)dA = 0.$$

Thus, notice the reversal of $[x_o + iy, A]$ is $[A, x_o + iy]$ and so we place a minus on the third integral:

$$\int_{[A,z_o]} f(\zeta)d\zeta + \int_{[z_o,x_o+iy]} f(\zeta)d\zeta - \int_{[A,x_o+iy]} f(\zeta)d\zeta = 0$$

but, we defined $h(z) = \int_{[A,z]} f(\zeta)d\zeta$ thus:

$$h(x_o, y) - h(x_o, y_o) = i\int_{y_o}^{y} f(x_o, t)dt \quad \Rightarrow \quad \frac{\partial h}{\partial y}(x_o, y_o) = if(x_o, y_o).$$

[3] this is for convenience of exposition, we could replace star-shaped with simply connected and let the base point of the integration be any point in the region of holomorphicity.

[4] the symbol ζ is used here since z has another set meaning, this is the Greek letter "zeta"

[5] we require a modest generalization of the usual Green's Theorem. I'll let you work it out in a homework. Basically, we just apply Green's Theorem to the real and imaginary parts separately and thus derive the theorem used here

Examine triangle T_1 formed by $A, z_o, x_o + iy$ and apply Green's Theorem:

$$\int_{\partial T_1} f(\zeta)d\zeta = \int_{\partial T_1} f dx + i f dy = \iint_{T_1} \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_{T_1} (0) dA = 0.$$

Thus, notice the reversal of $[x_o + iy, A]$ is $[A, x_o + iy]$ and so we place a minus on the third integral:

$$\int_{[A,z_o]} f(\zeta)d\zeta + \int_{[z_o,x+iy_o]} f(\zeta)d\zeta - \int_{[A,x+iy_o]} f(\zeta)d\zeta = 0$$

Therefore,

$$h(x, y_o) - h(x_o, y_o) = \int_{x_o}^{x} f(t, y_o)dt \quad \Rightarrow \quad \frac{\partial h}{\partial x}(x_o, y_o) = f(x_o, y_o).$$

We've shown $f(x_o, y_o) = \frac{\partial h}{\partial x}(x_o, y_o) = \frac{1}{i}\frac{\partial h}{\partial y}(x_o, y_o)$ hence $\frac{\partial h}{\partial y}(x_o, y_o) = i\frac{\partial h}{\partial x}(x_o, y_o)$ which shows h is complex differentiable at $z_o = x_o + iy_o$. However, the point z_o was arbitrary hence we've shown $\frac{dh}{dz} = h_x = f$ on the given star-shaped domain. Finally, set $F = h$. $\square$

Example 7.2.7. *Since $f(z) = \frac{1}{z}$ is holomorphic on star-shaped $\mathbb{C}^-$ we have path-indendence of the integral of $f(z)$ on $\mathbb{C}^-$ and I claim we could define Log by the integral below:*

$$Log(z) = \int_1^z \frac{d\zeta}{\zeta}$$

Notice by FTC I we have $\frac{d}{dz}Log(z) = \frac{1}{z}$ for $z \in \mathbb{C}^-$ in this approach. However, proving Log serves as an inverse of the exponential suitably restricted would require some calculation. It is interesting to actually carry out the integration. Let C denote the path which goes from 1 to $z_o \in \mathbb{C}^-$. In particular, form C by the union of the path $C_1 = [1, |z_o|]$ followed by the arc C_2 from $|z_o|$ to z_o. For specificity, assume $z_o = x_o + iy_o$ where $y_o > 0$ and $x_o > 1$ then $z_o = |z_o|e^{i\theta_o}$ for $\theta_o \in (0, \pi)$. We calculate, for C_1, $y = 0$ hence $dz = dx$ for $1 \leq x \leq |z_o|$ whereas for C_2, $z = |z_o|e^{it}$ for $0 \leq t \leq \theta_o$ hence $dz = i|z_o|e^{it}dt$

$$\begin{aligned}
\int_C \frac{dz}{z} &= \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} \\
&= \int_1^{|z_o|} \frac{dx}{x} + \int_0^{\theta_o} \frac{i|z_o|e^{it}dt}{|z_o|e^{it}} \\
&= \int_1^{|z_o|} \frac{dx}{x} + i\int_0^{\theta_o} dt \\
&= \ln|z_o| + i\theta_o.
\end{aligned}$$

Similar calculations show $Log(z_o) = \ln|z_o| + i\theta_o$ in the case $0 < |z| < 1$ and also in the case $y_o < 0$ where $-\pi < \theta_o < 0$. In summary, the integral definition of the principle logarithm agrees with our previous definition.

Example 7.2.8. *If $\mathbb{C}^\alpha$ denotes the slit-plane where $te^{i\alpha}$ for $t \geq 0$ is deleted then we may define the branch of the log with domain $\mathbb{C}^\alpha$ via:*

$$Log_\alpha(z) = i(\alpha + \pi) + \int_{-e^{i\alpha}}^{z} \frac{d\zeta}{\zeta}$$

Notice the previous example examines $\alpha = -\pi$ so $-e^{i\alpha} = -e^{i\pi} = 1$ and the shift of $i(\alpha + \pi) = 0$ is absent. I invite the reader to affirm the formula given here is in agreement with our previous definition of $Log_\alpha(z) = \ln|z| + iArg_\alpha(z)$ where $\alpha < Arg_\alpha(z) < \alpha + 2\pi$ for $z \in \mathbb{C}^\alpha$.

Theorem 7.2.9. Cauchy's Theorem: *If $f(z)$ is holomorphic and continuously differentiable on D and extends continuously to ∂D then $\int_{\partial D} f(z)\,dz = 0$. Here ∂D is the oriented boundary of D where inner boundaries are oriented CW whereas outer boundaries are oriented CCW*[6]

Proof: If $f \in \mathcal{O}(U)$ then the Cauchy Riemann equations give $\partial_y f = i\partial_x f$. Apply Green's Theorem,

$$\int_{\partial D} f(z)dz = \int_{\partial D} f dx + i f dy = \int_D (\partial_x(if) - \partial_y f)dA = \int_D (i\partial_x f - \partial_y f)dA = 0.\square$$

Technically, the assumption in both proofs above of the continuity of $f'(z)$ throughout D is needed in order that Green's Theorem apply. That said, we shall soon study Goursat's Theorem and gain an appreciation for why this detail is superfluous.

The complex integral of a holomorphic function is path independent. In addition, if the function is holomorphic within a given loop then the integral around the loop is zero. I leave the proof of the Corollary to Cauchy's Theorem to the reader, possibly in homework.

Corollary 7.2.10. *(Path Independence, Trivial holonomy, Deformation Theorem)*

- **(i.)** *If $f(z)$ is holomorphic and continuously differentiable on D and C_1 and C_2 are two coterminal paths in D then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.*
- **(ii.)** *If C is a simple closed curve whose interior is within D then $\int_C f(z)dz = 0$.*
- **(iii.)** *If C_{in} and C_{out} are two CCW oriented loops which bound an annulus where f is continuously differentiable and holomorphic then $\int_{C_{in}} f(z)dz = \int_{C_{out}} f(z)dz$.*

I refer to part (iii.) of the Corollary above as *the deformation theorem.* It allows us to trade simple calculation as in Example 7.1.3 in place of a potentially very challenging integral we would face if we tried to perform the integration directly for deformed loop with lots of wiggles.

Example 7.2.11. *Let γ be a, postively oriented, simple, closed, curve containg the unit circle in its interior. Notice $f(z) = \frac{1}{z}$ is holomorphic between γ and the unit circle. Hence, by Example 7.1.3 and the deformation theorem we find*

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

Definition 7.2.12. *A complex one-form has the form $\omega = Pdx + Qdy$ where P, Q are complex-valued smooth functions. If $Q_x = P_y$ on a domain D then we say $\omega = Pdx + Qdy$ is* **closed** *on D. On the other hand, if there exists h such that $dh = Pdx + Qdy$ then we say $\omega = Pdx + Qdy$ is* **exact** *on D.*

We can prove that every exact form is closed. However, depending on the shape of the domain, there may exist closed forms which are not exact. In particular, the assumption of star-shaped (or simply connected to be a bit more general) is needed since there are closed forms on domains with holes which are not exact. The standard example is $\mathbb{C}^\times$ where $\frac{dz}{z}$ is closed, but $\int_{|z|=1} \frac{dz}{z} = 2\pi i$

[6] CW means clockwise and CCW means counterclockwise. If you imagine yourself a tiny person walking the boundary then if you walk in the positively oriented direction then the interior of the space will be on your left. Or you could imagine D as being huge and yourself as a giant, still, the interior is on the left if you walk the positively oriented boundary.

shows we cannot hope for a primitive to exist on all of $\mathbb{C}^{\times}$. If such a primitive did exist then the integral around $|z| = 1$ would necessarily be zero which contradicts the always important Example 7.1.3. This discussion is the beginning of the study of DeRahm Cohomology. In DeRahm Cohomology we use the calculus of differential forms on manifolds to capture and characterize holes in manifolds. It's part of a larger discussion about algebraic topology in which abstract algebra is used to calculate topological invariants. We learn a little more about this in the complex context when we study winding number later in this course.

Example 7.2.13. *The function $f(z) = \frac{2}{1+z^2}$ has natural domain of $\mathbb{C} - \{i, -i\}$. Moreover, partial fractions decomposition provides the following identity:*

$$f(x) = \frac{1}{z+i} + \frac{1}{z-i}$$

If $\epsilon < 1$ and $\gamma_\epsilon(p)$ denotes the circle centered at p with positive orientation and radius ϵ then I invite the student to verify that:

$$\int_{\gamma_\epsilon(-i)} \frac{dz}{z+i} = 2\pi i \qquad \& \qquad \int_{\gamma_\epsilon(-i)} \frac{dz}{z-i} = 0$$

whereas

$$\int_{\gamma_\epsilon(i)} \frac{dz}{z+i} = 0 \qquad \& \qquad \int_{\gamma_\epsilon(i)} \frac{dz}{z-i} = 2\pi i.$$

Suppose D is a domain which includes $\pm i$. Let $S = D - \text{interior}(\gamma_\epsilon(\pm i))$. That is, S is the domain D with the points inside the circles $\gamma_\epsilon(-i)$ and $\gamma_\epsilon(i)$ deleted. Furthermore, we suppose ϵ is small enough so that the circles are interior to D. This is possible as we assumed D is an open connected set when we said D is a domain. All of this said: note $\frac{d}{dz}\left[\frac{2}{z^2+1}\right] = \frac{-4z}{(z^2+1)^2}$ hence $f(z)$ is holmorphic on D and we may apply Cauchy's Theorem on S:

$$0 = \int_{\partial S} \frac{2dz}{z^2+1} = \int_{\partial D} \frac{2dz}{z^2+1} - \int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) - \int_{\gamma_\epsilon(i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right)$$

But, we know the integrals around the circles and it follows:

$$\int_{\partial D} \frac{2dz}{z^2+1} = 4\pi i.$$

Notice the nontriviality of the integral above is due to the singular points $\pm i$ in the domain.

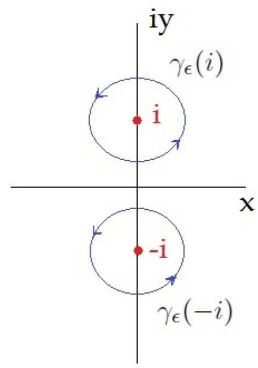
Look back at Example 7.1.9 if you are rusty on how to calculate the integrals around the circles. It is fun to think about the calculation above in terms of what we can and can't do with logarithms:

$$\int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + d[\log(z-i)]\right) = \int_{\gamma_\epsilon(-i)} \frac{dz}{z+i} = 2\pi i.$$

where the $\log(z-i)$ is taken to be a branch of the logarithm which is holomorphic on the given circle; for example, $\log(z-i) = \text{Log}_{\pi/2}(z-i)$ would be a reasonable choice since the circle is centered at $z = -i$ which falls on $\theta = -\pi/2$. The jump in the $\text{Log}_{\pi/2}(z-i)$ occurs away from where the integration is taken and so long as $\epsilon < 1$ we have that $dz/(z-i)$ is exact with potential $\text{Log}_{\pi/2}(z-i)$. That said, we prefer the notation $\log(z-i)$ when the details are not important to the overall calculation. Notice, see for $dz/(z+i)$ as the differential of a logarithm because the circle of integration necessarily contains the singularity which forbids the existence of the logarithm on the whole punctured plane $\mathbb{C} - \{-i\}$. Similarly,

$$\int_{\gamma_\epsilon(i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(i)} \left(d[\log(z+i)] + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(i)} \frac{dz}{z-i} = 2\pi i$$

is a slick notation to indicate the use of an appropriate branch of $\log(z+i)$. In particular, $\text{Log}_{-\pi/2}(z+i)$ is appropriate for $\epsilon < 1$.



7.3 The Cauchy Integral Formula

Once again, when we assume holomorphic on a domain we also add the assumption of continuity of $f'(z)$ on the domain. Gamelin assumes continuity of $f'(z)$ when he says $f(z)$ is *analytic* on D. As I have mentioned a few times now, we show in Section 7.5 that $f(z)$ holomorphic on a domain automatically implies that $f'(z)$ is continuous. This means we can safely delete the assumption of continuity of $f'(z)$ once we understand Goursat's Theorem.

The theorem below is rather surprising in my opinion.

Theorem 7.3.1. Cauchy's Integral Formula ($m = 0$)**:** *let D be a bounded domain with piecewise smooth boundary ∂D. If $f(z)$ is holomorphic with continuous $f'(z)$ on D and $f(z), f'(z)$ extend continuously to ∂D then for each $z \in D$,*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw$$

Proof: Assume the preconditions of the theorem. Fix a point $z \in D$. Note D is open hence z is interior thus we are free to choose $\epsilon > 0$ for which $\{w \in \mathbb{C} \mid |w - z| < \epsilon\} \subseteq D$. Define:

$$D_\epsilon = D - \{w \in \mathbb{C} \mid |w - z| \leq \epsilon\}$$

Observe the boundary of D_ϵ consists of the outer boundary ∂D and the circle γ_ϵ^- which is $|w-z| = \epsilon$ given CW-orientation; $\partial D_\epsilon = \partial D \cup \gamma_\epsilon^-$. Further, observe $g(w) = \frac{f(w)}{w-z}$ is holomorphic as

$$g'(w) = \frac{f'(w)}{w-z} - \frac{f(w)}{(w-z)^2}$$

and $g'(w)$ continuous on D_ϵ and $g(w), g'(w)$ both extend continuously to ∂D_ϵ as we have assumed from the outset that $f(w), f'(w)$ extend likewise. We obtain from Cauchy's Theorem 7.2.9 that:

$$\int_{\partial D_\epsilon} \frac{f(w)}{w-z}dw = 0 \quad \Rightarrow \quad \int_{\partial D} \frac{f(w)}{w-z}dw + \int_{\gamma_\epsilon^-} \frac{f(w)}{w-z}dw = 0.$$

However, if γ_ϵ^+ denotes the CCW-oriented circle, we have $\int_{\gamma_\epsilon^-} \frac{f(w)}{w-z}dw = -\int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw$ hence:

$$\int_{\partial D} \frac{f(w)}{w-z}dw = \int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw$$

The circle γ_ϵ^+ has $w = z + \epsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ thus $dz = i\epsilon e^{i\theta} d\theta$ and we calculate:

$$\int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw = \int_0^{2\pi} \frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = 2\pi i \int_0^{2\pi} f(z+\epsilon e^{i\theta})\frac{d\theta}{2\pi} = 2\pi i f(z).$$

In the last step we used the Mean Value Property which was proved in homework. Finally, solve for $f(z)$ to obtain the desired result. □

We can formally derive the higher-order formulae by differentiation:

$$f'(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_{\partial D} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial D} \frac{d}{dz}\left[\frac{f(w)}{w-z}\right]dw = \frac{1!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^2}\,dw$$

Differentiate once more,

$$f''(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_{\partial D} \frac{f(w)}{(w-z)^2}\,dw = \frac{1}{2\pi i}\int_{\partial D} \frac{d}{dz}\left[\frac{f(w)}{(w-z)^2}\right]dw = \frac{2!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^3}\,dw$$

continuing, we would arrive at:

$$\boxed{f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dw}$$

which is known as **Cauchy's generalized integral formula**. Note that $0! = 1$ and $f^{(0)}(z) = f(z)$ hence Theorem 7.3.1 naturally fits into the formula above.

It is probably worthwhile to examine a proof of the formulas above which is not based on differentiating under the integral. The arguments below show that our formal derivation above were valid. In the case $m = 1$ the needed algebra is simple enough:

$$\frac{1}{w-(z+\triangle z)} - \frac{1}{w-z} = \frac{\triangle z}{(w-(z+\triangle z))(w-z)}.$$

Then, appealing to the $m = 0$ case to write the functions as integrals:

$$\begin{aligned}\frac{f(z+\triangle z)-f(z)}{\triangle z} &= \frac{1}{2\pi i\triangle z}\int_{\partial D}\frac{1}{w-(z+\triangle z)}dw + \frac{1}{2\pi i\triangle z}\int_{\partial D}\frac{1}{w-z}dw \\ &= \frac{1}{2\pi i\triangle z}\int_{\partial D}\left[\frac{1}{w-(z+\triangle z)} - \frac{1}{w-z}\right]f(w)\,dw \\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-(z+\triangle z))(w-z)}dw.\end{aligned}$$

Finally, as $\triangle z \to 0$ we find $f'(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^2}\,dw$. We assume that the limiting process $\triangle z \to 0$ can be interchanged with the integration process. Gamelin comments this is acceptable due to the uniform continuity of the integrand.

We now turn to the general case, assume Cauchy's generalized integral formula holds for $m-1$. We need to make use of the binomial theorem:

$$((w-z)+\triangle z)^m = (w-z)^m - m(w-z)^{m-1}\triangle z + \frac{m(m-1)}{2}(w-z)^{m-2}(\triangle z)^2 + \cdots + (\triangle z)^m$$

Clearly, we have $((w-z)+\triangle z)^m = (w-z)^m - m(w-z)^{m-1}\triangle z + g(z,w)(\triangle z)^2$ It follows that:

$$\begin{aligned}\frac{1}{(w-(z+\triangle z))^m} - \frac{1}{(w-z)^m} &= \frac{m(w-z)^{m-1}\triangle z + g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m} \\ &= \frac{m\triangle z}{(w-(z+\triangle z))(w-z)^m}. + \frac{g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m}\end{aligned}$$

Apply the induction hypothesis to obtain the integrals below: $\dfrac{f^{(m-1)}(z+\triangle z) - f^{(m-1)}(z)}{\triangle z} =$

$$\begin{aligned}&= \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\frac{f(w)}{(w-(z+\triangle z))^m}dw + \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\frac{f(w)}{(w-z)^m}dw \\ &= \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\left[\frac{m\triangle z}{(w-(z+\triangle z))(w-z)^m}. + \frac{g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m}\right]f(w)\,dw \\ &= \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)dw}{(w-(z+\triangle z))(w-z)^m}. + \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{g(z,w)\triangle z f(w)dw}{(w-(z+\triangle z))^m(w-z)^m}.\end{aligned}$$

As $\triangle z \to 0$ we see the right integral vanishes and the left integral has a denominator which tends to $(w-z)^{m+1}$ hence, by the definition of the m-th derivative,

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)dw}{(w-z)^{m+1}}$$

The arguments just given provide proof of the following theorem:

Theorem 7.3.2. Cauchy's Generalized Integral Formula ($m \in \mathbb{N}\cup\{0\}$)**:** *let D be a bounded domain with piecewise smooth boundary ∂D. If $f(z)$ is holomorphic with continuous $f'(z)$ on D and $f(z), f'(z)$ extend continuously to ∂D then for each $z \in D$,*

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^{m+1}}\,dw$$

Often we need to use the theorem above with the role of z as the integration variable. For example:

$$f^{(m)}(z_o) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(z)}{(z-z_o)^{m+1}}\,dz$$

from which we obtain the useful identity:

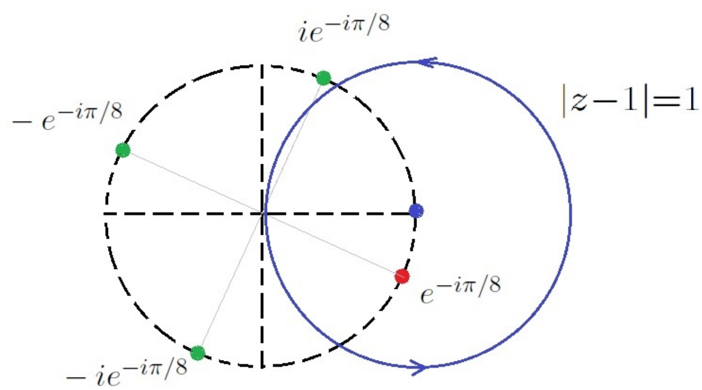
$$\boxed{\int_{\partial D}\frac{f(z)}{(z-z_o)^{m+1}}\,dz = \frac{2\pi i f^{(m)}(z_o)}{m!}}$$

This formula allows us to calculate many difficult integrals by simple evaluation of an approrpriate derivative. That said, we do improve on this result when we uncover the technique of residues later in the course. Think of this as an intermediate step in our calculational maturation.

Example 7.3.3. *Let the integral below be taken over the CCW-oriented curve $|z| = 1$:*

$$\oint_{|z|=2} \frac{\sin(2z)}{(z-i)^6}\,dz = \frac{2\pi i}{5!}\frac{d^5}{dz^5}\bigg|_{z=i} \sin(2z) = \frac{2\pi i}{5\cdot 4\cdot 3\cdot 2}(-32\cos(2i)) = \frac{-8\pi i\cosh(2)}{15}.$$

Example 7.3.4. *Notice that $z^4 + i = 0$ for $z \in (-i)^{1/4} = \left(e^{-i\pi/2}\right)^{1/4} = e^{-i\pi/8}\{1, i, -1, -i\}$ hence $z^4 + i = (z - e^{-i\pi/8})(z - ie^{-i\pi/8})(z + e^{-i\pi/8})(z + ie^{-i\pi/8})$. Consider the circle $|z - 1| = 1$ (blue). The dotted circle is the unit-circle and the intersection near $ie^{-i\pi/8}$ is at $\theta = \pi/3$ which is roughly as illustrated.*



The circle of integration below encloses the principal root (red), but not the other three non-principal fourth roots of $-i$ (green). Consequently, we apply Cauchy's integral formula based on the divergence of the principal root:

$$\begin{aligned}
\oint_{|z-1|=1} \frac{dz}{z^4+i} &= \oint_{|z-1|=1} \frac{dz}{(z - e^{-i\pi/8})(z - ie^{-i\pi/8})(z + e^{-i\pi/8})(z + ie^{-i\pi/8})} \\
&= \frac{2\pi i}{(z - ie^{-i\pi/8})(z + e^{-i\pi/8})(z + ie^{-i\pi/8})}\bigg|_{z=e^{-i\pi/8}} \\
&= \frac{2\pi i}{(e^{-i\pi/8} - ie^{-i\pi/8})(e^{-i\pi/8} + e^{-i\pi/8})(e^{-i\pi/8} + ie^{-i\pi/8})} \\
&= \frac{2\pi i}{e^{-3i\pi/8}(1-i)(1+1)(1+i)} \\
&= \frac{\pi i}{2}e^{3i\pi/8}.
\end{aligned}$$

Of course, you could simplify the answer further and present it in Cartesian form.

Finally, one last point:

Corollary 7.3.5. *If $f(z)$ is holomorphic with continuous derivative $f'(z)$ on a domain D then $f(z)$ is infinitely complex differentiable. That is, $f', f'', \dots$ all exist and are continuous on D.*

The proof of this is that Cauchy's integral formula gives us an explicit expression (which exists) for any possible derivative of f. There are no just once or twice continuously complex differentiable functions. You get one continuous derivative on a domain, you get infinitely many. Pretty good deal. Moreover, the continuity of the derivative is not even needed as we discover soon.

7.4 Liouville's Theorem

It is our convention to say $f(z)$ is holomorphic on a closed set D iff there exists an open set $\tilde{D}$ containing D on which $f(z) \in \mathcal{O}(\tilde{D})$. Consider a function $f(z)$ for which $f'(z)$ exists and is continuous for $z \in \mathbb{C}$ such that $|z - z_o| \leq \rho$. In such a case Cauchy's integral formula applies:

$$f^{(m)}(z_o) = \frac{m!}{2\pi i} \int_{|z-z_o|\leq\rho} \frac{f(z)}{(z-z_o)^{m+1}}\, dz$$

We parametrize the circle by $z = z_o + \rho e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ where $dz = i\rho e^{i\theta} d\theta$. Therefore,

$$f^{(m)}(z_o) = \frac{m!}{2\pi i} \int_0^{2\pi} \frac{f(z_o + \rho e^{i\theta})}{(\rho e^{i\theta})^{m+1}}\, i\rho e^{i\theta} d\theta = \frac{m!}{2\pi \rho^m} \int_0^{2\pi} f(z_o + \rho e^{i\theta}) e^{-im\theta} d\theta$$

If we have $|f(z_o + \rho e^{i\theta})| \leq M$ for $0 \leq \theta \leq 2\pi$ then the we find

$$\begin{aligned} \left| \int_0^{2\pi} f(z_o + \rho e^{i\theta}) e^{-im\theta} d\theta \right| &\leq \int_0^{2\pi} \left| f(z_o + \rho e^{i\theta}) e^{-im\theta} \right| d\theta \\ &= \int_0^{2\pi} \left| f(z_o + \rho e^{i\theta}) \right| d\theta \\ &\leq 2\pi M. \end{aligned}$$

The discussion above serves to justify the bound given below:

Theorem 7.4.1. Cauchy's Estimate: *suppose $f(z)$ is holomorphic with continuous derivative on a domain D then for any closed disk $\{z \in \mathbb{C} \mid |z - z_o| \leq \epsilon\} \subset D$ on which $|f(z)| \leq M$ for all $z \in \mathbb{C}$ with $|z - z_o| = \rho$ we find*

$$\left| f^{(m)}(z_o) \right| \leq \frac{Mm!}{\rho^m}$$

Many interesting results flow from the estimate above. For example:

Theorem 7.4.2. Liouville's Theorem: *Suppose $f(z)$ is holomorphic with continuous derivative on $\mathbb{C}$. If $|f(z)| \leq M$ for all $z \in \mathbb{C}$ then $f(z)$ is constant.*

Proof: Assume $f(z), f'(z)$ are continuous on $\mathbb{C}$ and $|f(z)| \leq M$ for all $\mathbb{C}$. Let us consider the disk of radius R centered at z_o. From Cauchy's Estimate with $m = 1$ we obtain:

$$|f'(z_o)| \leq \frac{M}{R}.$$

Observe, as $R \to \infty$ we find $|f'(z_o)| \to 0$ hence $f'(z_o) = 0$. But, z_o was an arbitrary point in $\mathbb{C}$ hence $f'(z) = 0$ for all $z \in \mathbb{C}$ and as $\mathbb{C}$ is connected we find $f(z) = c$ for all $z \in \mathbb{C}$. $\square$

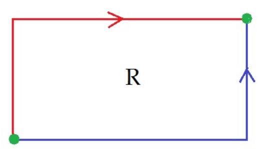
We saw in the homework that this theorem allows a relatively easy proof of the Fundamental Theorem of Algebra. In addition, we were able to show that an entire function whose range misses a disk of values must be constant. As I mentioned in class, the take-away message here is simply this: *every bounded entire function is constant.*

7.5 Theorems of Morera and Goursat

In this section we prove that every holomorphic function on a domain is necessarily a continuously differentiable function. This means the theorems in previous sections can be fine-tuned to delete the precondition of continuous differentiability since holomorphic implies the desired continuity. It is important to note that continuous differentiability of $f(z)$ is **not** assumed as a precondition of the theorem.

Theorem 7.5.1. Morera's Theorem: *Let $f(z)$ be a continuous function on a domain U. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in U with sides parallel to the coordinate axes then $f(z)$ is holomorphic with continuous $f'(z)$ in U.*

Proof: the vanishing of the rectangular integral allows us to exchange the lower path between two vertices of a rectangle for the upper path:



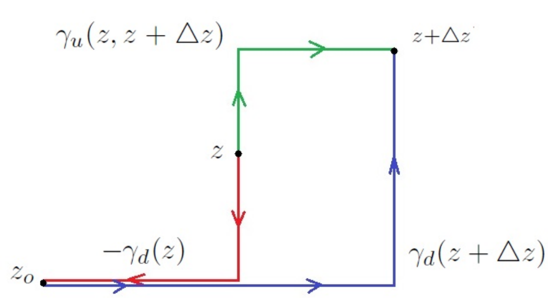
It suffices to prove the theorem for a disk D with center z_o where $D \subseteq U$[7]. Define:

$$F(z) = \int_{\gamma_d(z)} f(w)dw$$

where $\gamma_d(z) = [x_o + iy_o,\ iy_o + x] \cup [iy_o + x,\ x + iy]$ where $z_o = x_o + iy_o$ and $z = x + iy$. To show $F'(z)$ exists we consider the difference: here $\triangle z$ is a small enough displacement as to keep $z + \triangle z \in D$, the calculation below is supported by the diagram which follows after:

$$\begin{aligned} F(z + \triangle z) - F(z) &= \int_{\gamma_d(z+\triangle z)} f(w)dw - \int_{\gamma_d(z)} f(w)dw \\ &= \int_{\gamma_d(z+\triangle z)} f(w)dw + \int_{-\gamma_d(z)} f(w)dw \\ &= \int_{\gamma_u(z, z+\triangle z)} f(w)dw \qquad \star . \end{aligned}$$

Where $-\gamma_d(z)$ denotes the reversal of $\gamma_d(z)$. I plotted it as the red path below. The blue path is $\gamma_d(z + \triangle z)$. By the assumption of the theorem we are able to replace the sum of the blue and red paths by the green path $\gamma_u(z, z + \triangle z)$.



[7]do you understand why this is true and no loss of generality here?

Notice, $f(z)$ is just a constant in the integral below hence:

$$\int_{\gamma_u(z,z+\triangle z)} f(z)dw = f(z)\int_z^{z+\triangle z} dw = f(z)w\Big|_z^{z+\triangle z} = f(z)\triangle z.$$

Return once more to $\star$ and add $f(z)-f(z)$ to the integrand:

$$\begin{aligned} F(z \ + \ \triangle z) - F(z) &= \int_{\gamma_u(z,z+\triangle z)} [f(z)+f(w)-f(z)]dw \\ &= f(z)\triangle z + \int_{\gamma_u(z,z+\triangle z)} \big(f(w)-f(z)\big)dw \quad \star\star \end{aligned}$$

Note $L(\gamma_u(z,z+\triangle z)) < 2|\triangle z|$ and if we set $M = sup\{|f(w)-f(z)| \mid z \in \gamma_u(z,z+\triangle z)\}$ then the ML-estimate provides

$$\left|\int_{\gamma_u(z,z+\triangle z)} \big(f(w)-f(z)\big)dw\right| \leq ML < 2M|\triangle z|$$

Rearranging $\star\star$ we find:

$$\left|\frac{F(z \ + \ \triangle z) - F(z)}{\triangle z} - f(z)\right| \leq 2M.$$

Notice that as $\triangle z \to 0$ we have $2M \to 0$ hence $F'(z) = f(z)$ be the inequality above. Furthermore, we assumed $f(z)$ continuous hence $F'(z)$ is continuous. Consequently $F(z)$ is both holomorphic and possesses continuous derivative $F'(z)$ on D. Apply the Corollary 7.3.5 to Cauchy's Generalized Integral Formula to see that $F''(z) = f'(z)$ exists and is continuous. □

Theorem 7.5.2. Goursat's Theorem: *If $f(z)$ is a complex-valued function on a domain D such that*

$$f'(z_o) = \lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o}$$

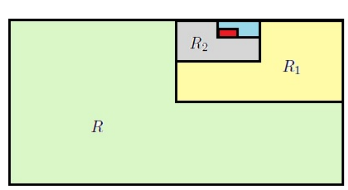
exists at each point z_o of D then $f'(z)$ is continuous on D.

In other words: *If a function is holomorphic on a domain D then $z \to f'(z)$ is continuous.*

Proof: let R be a closed rectangle in D with sides parallel to the coordinate axes. Divide R into four identical sub-rectangles and let R_1 be the sub-rectangle for which $\left|\int_{\partial R_1} f(z)dz\right|$ is largest (among the 4 sub-rectangles). Observe that $\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)dz\right|$ or, equivalently, $\left|\int_{\partial R} f(z)dz\right| \leq 4\left|\int_{\partial R_1} f(z)dz\right|$. Then, we subdivide R_1 into 4 sub-rectangles and the rectangle with largest integral R_2. Continuing in this fashion we obtain a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$. It is a simple exercise to verify:

$$\left|\int_{\partial R_n} f(z)dz\right| \leq 4\left|\int_{\partial R_{n+1}} f(z)dz\right| \quad \Rightarrow \quad \left|\int_{\partial R} f(z)dz\right| \leq 4^n\left|\int_{\partial R_n} f(z)dz\right| \quad \star.$$

The subdivision process is illustrated below:



As $n \to \infty$ it is clear that the sequence of nested rectangles converges to a point $z_o \in R$. Furthermore, if L is the length of the perimeter of R then $L/2^n$ is the length of ∂R_n. As $f(z)$ is complex-differentiable at z_o we know for each $z \in R_n$ there must exist an ϵ_n such that

$$\left| \frac{f(z) - f(z_o)}{z - z_o} - f'(z_o) \right| \leq \epsilon_n$$

hence

$$\big| f(z) - f(z_o) - f'(z_o)(z - z_o) \big| \leq \epsilon_n |z - z_o| \leq 2\epsilon_n L/2^n \quad \star\star.$$

The last inequality is very generous since $z_o, z \in R_n$ surely implies they are closer than the perimeter $L/2^n$ apart. Notice, the function $g(z) = f(z_o) + f'(z_o)(z - z_o)$ has primitive $G(z) = f(z_o)z + f'(z_o)(z^2/2 - zz_o)$ on R_n hence[8] $\int_{\partial R_n} g(z)dz = 0$. Subtracting this zero is crucial:

$$\left| \int_{\partial R_n} f(z)dz \right| = \left| \int_{\partial R_n} \big[f(z) - f(z_o) - f'(z_o)(z - z_o) \big] dz \right| \leq (2\epsilon_n L/2^n)(L/2^n) = \frac{2L^2\epsilon_n}{4^n}.$$

where we applied the ML-estimate by $\star\star$ and $L(\partial R_n) = L/2^n$. Returning to $\star$,

$$\left| \int_{\partial R} f(z)dz \right| \leq 4^n \left| \int_{\partial R_n} f(z)dz \right| \leq 4^n \cdot \frac{2L^2\epsilon_n}{4^n} = 2L^2\epsilon_n.$$

Finally, as $n \to 0$ we have $\epsilon_n \to 0$ thus it follows $\int_{\partial R} f(z)dz = 0$. But, this shows the integral around an arbitrary rectangle in D is zero hence by Morera's Theorem 7.5.1 we find $f(z)$ is holomorphic with continuous $f'(z)$ on D. $\square$

A good part of the reason I keep this section is that I think it is important for Math majors to see the nesting argument given in the proof above.

7.6 Complex Notation and Pompeiu's Formula

My apologies, it seems I have failed to write much here. I have many things to say, some of them I said in class. Recently, we learned how to generalize the idea of this section to nearly arbitrary associative algebras. More on that somewhere else.

[8] this application of Cauchy's Theorem does not beg the question by assuming continuity of $g'(z)$

Chapter VIII

Power Series

A power series is simply a polynomial without end. But, this begs questions. What does "without end" mean? How can we add, subtract, multiply and divide things which have no end? In this chapter we give a careful account of things which go on without end.

History provides examples of the need for caution[1]. For example, even Cauchy wrongly asserted in 1821 that an infinite series of continuous functions was once more continuous. In 1826 Abel[2] provided a counter-example and in the years to follow the concept of uniform convergence was invented to avoid such blunders. Abel had the following to say about the state of the theory as he saw it: from page 114 of [R91]

> If one examines more closely the reasoning which is usually employed in the treatment of infinite series, he will find that by and large it is unsatisfactory and that the number of propositions about infinite series which can be regarded as rigorously confirmed is small indeed

The concept of uniform convergence is apparently due to the teacher of Weierstrauss. Christoph Gudermann wrote in 1838: "it is a fact worth noting that... the series just found have *all the same convergence rate*". Weierstrauss used the concept of uniform convergence throughout his work. Apparently, Seidel and Stokes independently in 1848 and 1847 also used something akin to uniform convergence of a series, but the emminent British mathematician G.H Hardy gives credit to Weierstrauss:

> Weierstrauss's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis

It is fun to note Cauchy's own view of his 1821 oversight. In 1853 in the midst of a work which used and made significant contributions to the theory of uniformly convergent series, he wrote that *it is easy to see how one should modify the statement of the theorem.* See page 102 of [R91] for more details as to be fair to Cauchy.

In this chapter, we study convergence of sequence and series. Ultimately, we find how power series work in the complex domain. The results are surprisingly simple as we shall soon discover. Most importantly, we introduce the term *analytic* and see in what sense it is equivalent to our term *holomorphic*. Obviously, we differ from Gamelin on this point of emphasis.

[1] the facts which follow here are taken from [R91] pages 96-98 primarily

[2] did work on early group theory, we name commutative groups **Abelian** groups in his honor

8.1 Infinite Series

We discussed and defined complex sequences in Chapter 2. See Definition 3.2.1. We now discuss series of complex numbers. In short, a complex series is formed by adding the terms in some sequence of complex numbers:

$$\sum_{n=0}^{\infty} z_n = z_o + z_1 + z_2 + \cdots$$

If this sum exists as a complex number then the series is convergent whereas if the sum above does not converge then the series is said to be divergent. The convergence (or divergence) of the series is described precisely by the convergence (or divergence) of the sequence of partial sums:

Definition 8.1.1. *Let $a_n \in \mathbb{C}$ for each $n \in \mathbb{N} \cup \{0\}$ then we define*

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

If $\lim_{n\to\infty} \sum_{j=0}^{n} a_j = S \in \mathbb{C}$ then the series $a_o + a_1 + \cdots$ is said to **converge** *to S.*

The linearity theorems for sequences induce similar theorems for series. In particular, Theorem 3.2.3 leads us to:

Theorem 8.1.2. *Let $c \in \mathbb{C}$, $\sum a_j = A$ and $\sum b_j = B$ then $\sum(a_j + b_j) = A + B$ and $\sum ca_j = cA$;*

$$\sum(a_j + b_j) = \sum a_j + \sum b_j \qquad \textit{additivity of convergent sums}$$

$$\sum cb_j = c\sum b_j \qquad \textit{homogeneity of convergent sums}$$

Proof: let $S_n = \sum_{j=0}^{n} a_j$ and $T_n = \sum_{j=0}^{n} b_j$. We are given, from the definition of convergent series, that these partial sums converge; $S_n \to A$ and $T_n \to B$ as $n \to \infty$. Consider then,

$$\sum_{j=0}^{n}(a_j + cb_j) = \sum_{j=0}^{n} a_j + c\sum_{j=0}^{n} b_j$$

Thus, the sequence of partial sums for $\sum_{j=0}^{\infty}(a_j + cb_j)$ is found to be $S_n + cT_n$. Apply Theorem 3.2.3 and conclude $S_n + cT_n \to A + cB$ as $n \to \infty$. Therefore,

$$\sum_{j=0}^{\infty}(a_j + cb_j) = \sum_{j=0}^{\infty} a_j + c\sum_{j=0}^{\infty} b_j.$$

If we set $c = 1$ we obtain additivity, if we set $A = 0$ we obtain homogeneity. □

I offered a proof for series which start at $j = 0$, but, it ought to be clear the same holds for series which start at any particular $j \in \mathbb{Z}$.

Let me add a theorem which is a simple consequence of Theorem 3.2.10 applied to partial sums:

Theorem 8.1.3. *Let $x_k, y_k \in \mathbb{R}$ then $\sum x_k + iy_k$ converges iff $\sum x_k$ and $\sum y_k$ converge. Moreover, in the convergent case, $\sum x_k + iy_k = \sum x_k + i\sum y_k$.*

Series of real numbers enjoy a number of results which stem from the ordering of the real numbers. The theory of series with non-negative terms is particularly intuitive. Suppose $a_o, a_1, \dots > 0$ then $\{a_o, a_o + a_1, a_o + a_1 + a_2, \dots\}$ is a monotonically increasing sequence. Recall Theorem 3.2.5 which said that a monotonic sequence converged iff it was **bounded**.

Theorem 8.1.4. *If $0 \leq a_k \leq r_k$, and if $\sum r_k$ converges, then $\sum a_k$ converges, and $\sum a_k \leq \sum r_k$.*

Proof: obviously $a_k, r_k \in \mathbb{R}$ by the condition $0 \leq a_k \leq r_k$. Observe $\sum_{k=0}^{n+1} r_k = r_{n+1} + \sum_{k=0}^{n} r_k$ hence $\sum_{k=0}^{n+1} r_k \geq \sum_{k=0}^{n} r_k$. Thus the sequence of partial sums of $\sum r_k$ is increasing. Since $\sum r_k$ converges it follows that the convergent sequence of partial sums is bounded. That is, there exists $M \geq 0$ such that $\sum_{k=0}^{n} r_k \leq M$ for all $n \in \mathbb{N} \cup \{0\}$. Notice $a_k \leq r_k$ implies $\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} r_k$. Therefore, $\sum_{k=0}^{n} a_k \leq M$. Observe $a_k \geq 0$ implies $\sum_{k=0}^{n} a_k$ is increasing by the argument we already offered for $\sum_{k=0}^{n} r_k$. We find $\sum_{k=0}^{n} a_k$ is a bounded, increasing sequence of non-negative real numbers thus $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = A \in \mathbb{R}$ by Theorem 3.2.5. Finally, we appeal to part of the sandwhich theorem for real sequences, if $c_n \leq d_n$ for all n and both c_n and d_n converge then $\lim_{n\to\infty} c_n \leq \lim_{n\to\infty} d_n$. Think of $c_n = \sum_{k=0}^{n} a_k$ and $d_n = \sum_{k=0}^{n} r_k$. Note $\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} r_k$ implies $\lim_{n\to\infty} \sum_{k=0}^{n} a_k \leq \lim_{n\to\infty} \sum_{k=0}^{n} r_k$. The theorem follows. $\square$

Can you appreciate the beauty of how Gamelin discusses convergence and proofs ? Compare the proof I give here to his paragraph on page 130-131. His prose captures the essential details of what I wrote above without burying you in details which obscure. In any event, I will continue to add uglified versions of Gamelin's prose in this chapter. I hope that by seeing both your understanding is fortified.

We return to the study of complex series once more. Suppose $a_j \in \mathbb{C}$ in what follows. The definition of a finite sum is made recursively by $\sum_{j=0}^{0} a_j = a_o$ and for $n \geq 1$:

$$\sum_{j=0}^{n} a_j = a_n + \sum_{j=0}^{n-1} a_j.$$

Notice this yields:

$$a_n = \sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j.$$

Suppose $\sum_{j=0}^{\infty} a_j = S \in \mathbb{C}$. Observe, as $n \to \infty$ we see that $\sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j \to S - S = 0$. Therefore, the condition $a_n \to 0$ as $n \to \infty$ is a **necessary** condition for convergence of $a_o + a_1 + \cdots$.

Theorem 8.1.5. *If $\sum_{j=0}^{\infty} a_j$ converges then $a_j \to 0$ as $j \to \infty$.*

Of course, you should recall from calculus that the criteria above is not **sufficient** for convergence of the series. For example, $1 + 1/2 + 1/3 + \cdots$ diverges despite the fact $1/n \to 0$ as $n \to \infty$.

I decided to elevate Gamelin's example on page 131 to a proposition.

Proposition 8.1.6. *Let $z_j \in \mathbb{C}$ for $j \in \mathbb{N} \cup \{0\}$.*
If $|z| < 1$ then $\displaystyle\sum_{j=0}^{\infty} z^n = \frac{1}{1-z}$. If $|z| \geq 1$ then $\displaystyle\sum_{j=0}^{\infty} z^n$ diverges.

Proof: if $|z| \geq 1$ then the n-th term test shows the series diverges. Suppose $|z| < 1$. Consider,

$$S_n = 1 + z + z^2 \cdots + z^n \quad \Rightarrow \quad zS_n = z + z^2 + \cdots + z^n + z^{n+1}$$

and we find $S_n - zS_n = 1 - z^{n+1}$ thus $(1-z)S_n = 1 - z^{n+1}$ and derive:

$$S_n = \frac{1 - z^{n+1}}{1 - z}$$

This is a rare and wonderful event that we were able to explicitly calculate the n-th partial sum with such small effort. Note $|z| < 1$ implies $|z|^{n+1} \to 0$ as $n \to \infty$. Therefore,

$$\sum_{j=0}^{\infty} z^n = \lim_{n \to \infty} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}. \qquad \square$$

Definition 8.1.7. *A complex series $\sum a_k$ is said to* **converge absolutely** *if $\sum |a_k|$ converges.*

Notice that $|a_k|$ denotes the modulus of a_k. In the case $a_k \in \mathbb{R}$ this reduces to the usual[3] definition of absolute convergence since the modulus is merely the absolute value function in that case. If you'd like to see a proof of absolute convergence in the real case, I recommend page 82 of [J02]. The proof there is based on parsing the real series into non-negative and negative terms. We have no such dichotomy to work with here so something else must be argued.

Theorem 8.1.8. *If $\sum a_k$ is absolutely convergent then $\sum a_k$ converges and $\left|\sum a_k\right| \leq \sum |a_k|$.*

Proof: assume $\sum |a_k|$ converges. Let $a_k = x_k + iy_k$ where $x_k, y_k \in \mathbb{R}$. Observe:

$$|a_k| = \sqrt{x_k^2 + y_k^2} \geq \sqrt{x_k^2} = |x_k| \qquad \& \qquad |a_k| \geq |y_k|.$$

Thus, $|x_k| \leq |a_k|$ hence by comparison test the series $\sum |x_k|$ converges with $\sum |x_k| \leq \sum |a_k|$. Likewise, $|y_k| \leq |a_k|$ hence by comparison test the series $\sum |y_k|$ converges with $\sum |y_k| \leq \sum |a_k|$. Recall that absolute convergence of a real series implies convergence hence $\sum x_k$ and $\sum y_k$ exist. Theorem 8.1.3 allows us to conclude $\sum x_k + iy_k = \sum a_k$ converges. $\square$

Given that I have used the absolute convergence theorem for real series I think it is appropriate to offer the proof of that theorem since many of you may either have never seen it, or at a minimum, have forgotten it. Following page 82 of [J02] consider a real series $\sum_{n=0}^{\infty} x_n$. We define:

$$p_n = \begin{cases} x_n & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0 \end{cases} \qquad \& \qquad q_n = \begin{cases} 0 & \text{if } x_n \geq 0 \\ -x_n & \text{if } x_n < 0 \end{cases}$$

Notice $x_n = p_n - q_n$. Furthermore, notice p_n, q_n are non-negative terms. Observe

$$p_0 + p_1 + \cdots + p_n \leq |x_o| + |x_1| + \cdots + |x_n|$$

Hence $\sum |x_n|$ converging implies $\sum p_n$ converges by Comparison Theorem 8.1.4 and $\sum p_n \leq \sum |x_n|$. Likewise,

$$q_0 + q_1 + \cdots + q_n \leq |x_o| + |x_1| + \cdots + |x_n|$$

[3] in the sense of second semester calculus where you probably first studied series

Hence $\sum |x_n|$ converging implies $\sum q_n$ converges by Comparison Theorem 8.1.4 and $\sum q_n \leq \sum |x_n|$. But, then $\sum x_n = \sum (p_n - q_n) = \sum p_n - \sum q_n$ by Theorem 8.1.2. Finally, notice

$$x_0 + x_1 + \cdots + x_n \leq |x_0| + |x_1| + \cdots + |x_n|$$

thus as $n \to \infty$ we obtain $\sum x_n \leq \sum |x_n|$. This completes the proof that absolute convergence implies convergence for series with real terms.

I challenge you to see that my proof here is really not that different from what Gamelin wrote[4].

Example 8.1.9. *Consider $|z| < 1$. Proposition 8.1.6 applies to show $\sum z_j$ is absolutely convergent by direct calculation and:*

$$\left| \frac{1}{1-z} \right| = \left| \sum_{j=0}^{\infty} z^j \right| \leq \sum_{j=0}^{\infty} |z|^j = \frac{1}{1-|z|}.$$

Following Gamelin,

$$\frac{1}{1-z} - \sum_{k=0}^{n} z^k = \sum_{k=0}^{\infty} z^k - \sum_{k=0}^{n} z^k = \sum_{k=n+1}^{\infty} z^k = z^{n+1} \sum_{k=0}^{\infty} z^k = \frac{z^{n+1}}{1-z}.$$

Therefore,

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{|z|^{n+1}}{1-|z|}.$$

The inequality above gives us a bound on the error for the n-th partial sum of the geometric series.

If you are interested in the history of absolute convergence, you might look at pages 29-30 of [R91] where he describes briefly the influence of Cauchy, Dirichlet and Riemann on the topic. It was Riemann who proved that a series which converges but, does not converge absolutely, could be rearranged to converge to any value in $\mathbb{R}$.

8.2 Sequences and Series of Functions

A sequence of functions on $E \subseteq \mathbb{C}$ is an assignment of a function on E for each $n \in \mathbb{N} \cup \{0\}$. Typically, we denote the sequence by $\{f_n\}$ or simply by f_n. In addition, although we are ultimately interested in the theory of sequences of complex functions, I will give a number of real examples to illustrate the subtle issues which arise in general.

Definition 8.2.1. *A sequence of functions f_n on E is said to* **pointwise converge** *to f if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in E$.*

You might be tempted to suppose that if each function of the sequence is continuous and the limit exists then surely the limit function is continuous. Well, you'd be wrong:

Example 8.2.2. *Let $n \in \mathbb{N} \cup \{0\}$ and define $f_n(x) = x^n$ for $x \in [0,1]$. We can calculate the limit function:*

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

[4] Bailu, notice the proof I give here easily extends to an associative algebra

Notice, f_n is continuous for each $n \in \mathbb{N}$, but, the limit function f is **not** *continuous. In particular, you can see we cannot switch the order of the limits below:*

$$0 = \lim_{x \to 1^-} \left(\lim_{n \to \infty} x^n \right) \neq \lim_{n \to \infty} \left(\lim_{x \to 1^-} x^n \right) = 1$$

To guarantee the continuity of the limit function we need a stronger mode of convergence. Following Gamelin (and a host of other analysis texts) consider:

Example 8.2.3. *We define a sequence for which each function g_n makes a triangular tent of slope $\pm n^2$ from $x = 0$ to $x = 2/n$. In particular, for $n \in \mathbb{N}$ define:*

$$g_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

Notice,

$$\int_0^{1/n} n^2 x dx = n^2 \frac{(1/n)^2}{2} = \frac{1}{2}$$

and

$$\int_{1/n}^{2/n} (2n - n^2 x) dx = 2n(2/n - 1/n) - \frac{n^2}{2}[(2/n)^2 - (1/n)^2] = 2 - \frac{3}{2} = \frac{1}{2}.$$

Therefore, $\int_0^1 g_n(x)\,dx = 1$ for each $n \in \mathbb{N}$. However, as $n \to \infty$ we find $g_n(x) \to 0$ for each $x \in [0,1]$. Observe:

$$1 = \lim_{n \to \infty} \int_0^1 g_n(x)\,dx \neq \int_0^1 \lim_{n \to \infty} g_n(x)\,dx = 0.$$

To guarantee the integral of the limit function is the limit of the integrals of the sequence we need a stronger mode of convergence. Here I break from Gamelin and add one more example.

Example 8.2.4. *For each $n \in \mathbb{N}$ define $f_n(x) = x^n/n$ for $0 \leq x \leq 1$. Notice that $\lim_{n \to \infty} x^n/n = 0$ for each $x \in [0,1]$. Furthermore, $\lim_{x \to a} x^n/n = a^n/n$ for each $a \in [0,1]$ where we use one-sided limits at $a = 0^+, 1^-$. It follows that:*

$$\lim_{n \to \infty} \lim_{x \to a} \frac{x^n}{n} = \lim_{n \to \infty} \frac{a^n}{n} = 0$$

likewise,

$$\lim_{x \to a} \lim_{n \to \infty} \frac{x^n}{n} = \lim_{x \to a} 0 = 0$$

Thus, the limit $n \to \infty$ and $x \to a$ commute for this sequence of functions.

The example above shows us there is hope for the limit of a sequence of continuous function to be continuous. Perhaps we preserve derivatives under the limit ? Consider:

Example 8.2.5. *Once more study $f_n(x) = x^n/n$ for $0 \leq x \leq 1$. Notice $\frac{df_n}{dx} = x^{n-1}$. However, this is just the sequence we studied in Example 8.2.2,*

$$\lim_{n \to \infty} \frac{df_n}{dx} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \Rightarrow \quad \lim_{x \to 1^-} \lim_{n \to \infty} \frac{df_n}{dx} = \lim_{x \to 1^-} (0) = 0.$$

On the other hand,

$$\lim_{x\to 1^-}\frac{df_n}{dx} = \lim_{x\to 1^-} x^{n-1} = 1 \quad \Rightarrow \quad \lim_{n\to\infty}\lim_{x\to 1^-}\frac{df_n}{dx} = \lim_{n\to\infty}(1) = 1.$$

Therefore, the limit of the sequence of derivatives is not the derivative of the limit function.

The examples above lead us to define a stronger type of convergence which preserves continuity and integrals to the limit. However, in the real case, differentiation is still subtle.

The standard definition of uniform convergence is given below:[5]

Definition 8.2.6. *Let $\{f_n\}$ be a sequence of functions on E. Let f be a function on E. We say $\{f_n\}$* **converges uniformly to** *f if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$.*

This is not quite Gamelin's presentation. Instead, from page 134, Gamelin says:

> We say a sequence of functions $\{f_j\}$ **converges uniformly** to f on E if $|f_j(x) - f(x)| \leq \epsilon_j$ for all $x \in E$ where $\epsilon_j \to 0$ as $j \to \infty$. We call ϵ_j the worst-case estimator of the difference $f_j(x) - f(x)$ and usually take ϵ_j to be the supremum (maximum) of $|f_j(x) - f(x)|$ over $x \in E$,
> $$\epsilon_j = \sup_{x\in E} |f_j(x) - f(x)|.$$

Very well, are these definitions of uniform convergence equivalent? For a moment, let us define the uniform convergence of Gamelin as G-uniform convergence whereas that given in the Definition 8.2.6 defines S-uniform convergence. The question becomes:

> Can we show a sequence of functions $\{f_n\}$ on E is S-uniformly convergent to f on E iff the sequence of functions is G-uniformly convergent to f on E ?

This seems like an excellent homework question, so, I will merely assert it's verity for us here:

Theorem 8.2.7. *Let $\{f_n\}$ be a sequence of functions on E. Then $\{f_n\}$ is S-uniformly convergent to f on E if and only if $\{f_n\}$ is G-uniformly convergent to f on E.*

Proof: by trust in Gamelin, or as is my preference, your homework. □

The beautiful feature of Gamelin's definition is that it gives us a method to calculate the worst-case estimator. We merely need to find the maximum difference between the n-th function in the sequence and the limit function over the given domain of interest (E).

If you think about it, the supremum gives you the *best* worst-case estimator. Let me explain, if ϵ_j has $|f_j(z) - f(z)| \leq \epsilon_j$ for all $z \in E$ then ϵ_j is an upper bound on $|f_j(z) - f(z)|$. But, the supremum is the **least upper bound** hence $|f_j(w) - f(w)| \leq \sup_{z\in E} |f_j(z) - f(z)| \leq \epsilon_j$ for all $w \in E$. This simple reasoning shows us that when the supremum exists and we may use it as a worst-case estimator **provided** we also know $\sup_{z\in E} |f_j(z) - f(z)| \to 0$ as $j \to \infty$. On the other hand, if no supremum exists or if the supremum does not go to zero as $j \to \infty$ then we have no hope of finding a worst case estimator.

[5] for instance, see page 246 of [J02].

The paragraph above outlines the logic used in the paragraphs to follow.

In Example 8.2.2 we had $f_n(x) = x^n$ for $x \in [0,1]$ pointwise converged to $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ from which we may calculate[6] $\sup_{x\in[0,1]} |x^n - f(x)| = 1$. Therefore, it is not possible to find $\epsilon_n \to 0$. In Gamelin's terminology, *the worst-case estimator is* 1 hence this sequence is not uniformly convergent to $f(x)$ on $[0,1]$.

In Example 8.2.3 we had

$$g_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

which is point-wise convergent to $g(x) = 0$ for $x \in [0,1]$. The largest value attained by $g_n(x)$ is found at $x = 1/n$ where

$$g_n(1/n) = n^2(1/n) = n$$

Therefore,

$$\sup_{x\in[0,1]} |g_n(x) - g(x)| = n.$$

Therefore, the convergence of $\{g_n\}$ to g is not uniform on $[0,1]$.

Next, consider Example 8.2.4 where we noted that $f_n(x) = x^n/n$ converges pointwise to $f(x) = 0$ on $[0,1]$. In this case it is clear that $f_n(1) = 1/n$ is the largest value attained by $f_n(x)$ on $[0,1]$ hence:

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1/n = \epsilon_n \to 0 \quad \text{as } n \to \infty.$$

Hence $\{x^n/n\}$ converges uniformly to $f(x) = 0$ on $[0,1]$. Apparently, continuity is preserved under uniform convergence. On the other hand, Example 8.2.5 shows us that, for real functions, derivatives need not be preserved in a uniformly convergent limit.

We now present the two major theorems about uniformly convergent sequences of functions.

Theorem 8.2.8. *Let $\{f_j\}$ be a sequence of complex-valued functions on $E \subseteq \mathbb{C}$. If each f_j is continuous on E and if $\{f_j\}$ converges uniformly to f on E then f is continuous on E.*

Proof: let $\epsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ for which

$$|f_N(z) - f(z)| < \frac{\epsilon}{3} \qquad \star$$

for all $z \in E$. However, by continuity of f_N at $z = a$ there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies

$$|f_N(z) - f_N(a)| < \frac{\epsilon}{3} \qquad \star\star.$$

[6] sometimes the supremum is also known as the **least upper bound**, it is the smallest possible upper bound on the set in question. In this case, 1 is not attained in the set, but numbers arbitrary close to 1 are attained. Technically, this set has **no maximum** which is why the parenthetical comment in Gamelin suggesting supremum and maximum are synonyms is sometimes not helpful.

We claim $f(z)$ is continuous at $z = a$ by the same choice of δ. Consider, for $0 < |z - a| < \delta$,

$$\begin{aligned}|f(z) - f(a)| &= |f(z) - f_N(z) + f_N(z) - f_N(a) + f_N(a) - f(a)| \\ &\leq |f_N(z) - f(z)| + |f_N(z) - f_N(a)| + |f_N(a) - f(a)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

where I have used $\star\star$ for the middle term and $\star$ for the left and rightmost terms. Thus $\lim_{z\to a} f(z) = f(a)$ and as $a \in E$ was arbitrary we have shown f continuous on E. $\square$

I followed the lead of [J02] page 246 where they offer the same proof for an arbitary *metric space*.

Theorem 8.2.9. *Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of continuous complex-valued functions on γ, and if $\{f_j\}$ converges uniformly to f on γ then $\int_\gamma f_j(z)dz$ converges to $\int_\gamma f(z)dz$.*

Proof: let ϵ_j be the worst-case estimator for $f_j - f$ on γ then $|f_j(z) - f(z)| \leq \epsilon_j$ for all $z \in [\gamma]$. Let γ have length L and apply the ML-estimate:

$$\left| \int_\gamma (f_j(z) - f(z))dz \right| \leq \epsilon_j L.$$

Thus, as $j \to \infty$ we find $\left| \int_\gamma f_j(z)dz - \int_\gamma f(z)dz \right| \to 0$. $\square$

This theorem is also true in the real case as you may read on page 249 of [J02]. However, that proof requires we understand the real analysis of integrals which is addressed by our real analysis course. The ML-theorem is the hero here. Furthermore, in the same section of [J02] you'll find what additional conditions are needed to preserve differentiability past the limiting process.

The definitions given for series below are quite natural. As a guiding concept, we say X is a feature of a series if X is a feature of the sequence of partial sums.

Definition 8.2.10. *Let $\sum_{j=0}^{\infty} f_j$ be a sequence of complex-valued functions on E. The partial sums are functions defined by $S_n(z) = \sum_{j=0}^{n} f_j(z) = f_0(z) + f_1(z) + \cdots + f_n(z)$ for each $z \in E$. The series $\sum_{j=0}^{\infty} f_j$ converges pointwise on E iff $\{S_n(z)\}$ converges pointwise on E. The series $\sum_{j=0}^{\infty} f_j$ converges uniformly on E iff $\{S_n(z)\}$ converges uniformly on E.*

The theorem below gives us an analog of the comparison test for series of complex functions.

Theorem 8.2.11. Weierstrauss M-Test: *suppose $M_k \geq 0$ and $\sum M_k$ converges. If g_k are complex-valued functions on a set E such that $|g_k(z)| \leq M_k$ for all $z \in E$ then $\sum g_k$ converges uniformly on E.*

Proof: let $z \in E$ and note that $|g_k(z)| \leq M_k$ implies that $\sum |g_k(z)|$ is convergent by the comparison test Theorem 8.1.4. Moreover, as absolute convergence implies convergence we have $\sum_{k=0}^{\infty} g_k(z) = g(z) \in \mathbb{C}$ with $|g(z)| \leq \sum |g_k(z)| \leq \sum M_k$ by Theorem 8.1.8. The difference between the series and the partial sum is bounded by the tail of the **majorant series**

$$\left| g(z) - \sum_{k=0}^{n} g_k(z) \right| = \left| \sum_{k=n+1}^{\infty} g_k(z) \right| \leq \sum_{k=n+1}^{\infty} M_k.$$

However, this shows a worst-case estimator for $S_n(z) - g(z)$ is given by $\epsilon_n = \sum_{k=n+1}^{\infty} M_k$. We argue $\epsilon_n = \sum_{k=n+1}^{\infty} M_k \to 0$ as $n \to \infty$ for each $z \in E$ hence $\sum g_k$ converges uniformly on E. $\square$
For future reference:

Definition 8.2.12. *A given series of functions $\sum f_j$ on E is dominated by M_j if $|f_j(z)| \leq M_j$. When $\sum M_j$ converges we call M_j a* **majorant** *for $\sum f_j$.*

Just to reiterate: if we can find a majorant for a given series of functions then it serves to show the series is uniformly convergent by Weierstrauss' M-Test. Incidentally, as a historical aside, Weierstrauss gave this M-test as a footnote on page 202 of his 1880 work *Zur Functionenlehre* see [R91] page 103.

Example 8.2.13. *The geometric series $\displaystyle\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ converges for each $z \in \mathbb{C}$ with $|z| < 1$. Consider that in Example 8.1.9 we derived:*

$$\left| \sum_{k=0}^{\infty} z^k - \sum_{k=0}^{n} z^k \right| = \frac{|z|^{n+1}}{|1-z|}.$$

Notice $\sup_{|z|<1} \left(\frac{|z|^{n+1}}{1-|z|} \right)$ is unbounded hence $\sum_{k=0}^{\infty} z^k$ does not converge uniformly on $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. However, if $0 < R < 1$ we consider a disk $D_R = \{z \in \mathbb{C} \mid |z| < R\}$. We can find a majorant for the geometric series $\sum_{k=0}^{\infty} z^k$ as follows: let $M_k = R^k$ for each $z \in D_R$ note $|z^k| = |z|^k \leq R^k$ and $\sum_{k=0}^{\infty} R^k = \frac{1}{1-R}$. Therefore, $\sum_{k=0}^{\infty} z^k$ is uniformly convergent on D_R by Weierstrauss' M-Test.

The example above explains why $\sum_{k=0}^{\infty} z^k$ is pointwise convergent, but not uniformly convergent, on the entire open unit-disk $\mathbb{E}$. On the other hand, we have uniform convergence on any closed disk inside $\mathbb{E}$.

Example 8.2.14. *Consider $\sum_{k=1}^{\infty} \frac{z^k}{k^3}$. If we consider $|z| < 1$ notice we have the inequality $\left| \frac{z^k}{k^3} \right| = \frac{|z|^k}{k^3} \leq \frac{1}{k^3}$. Recall from calculus II that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is the $p = 3$ series which converges. Therefore, by the Weierstrauss M-test, we find $\sum_{k=1}^{\infty} \frac{z^k}{k^3}$ converges uniformly on $|z| < 1$.*

We now turn to complex analysis. In particular, we work to describe how holomorphicity filters through sequential limits. The theorem below is somewhat shocking given what we saw in the real case in Example 8.2.5.

Theorem 8.2.15. *If $\{f_j\}$ is a sequence of holomorphic functions on a domain D that converge uniformly to f on D then f is holomorphic on D.*

Proof: We follow Gamelin and use Morera's Theorem. To begin, We need continuity to apply Morera's Theorem. Notice f_j holomorphic implies f_j converges to f which is continuous on D by the supposed uniform covergence and Theorem 8.2.8.

let R be a rectangle in D with sides parallel to the coordinate axes. Uniform convergence of the sequence and Theorem 8.2.9 shows:

$$\lim_{j\to\infty} \int_{\partial R} f_j(z)dz = \int_{\partial R} \lim_{j\to\infty} (f_j(z))\, dz = \int_{\partial R} f(z)dz.$$

Consider that $f_j \in \mathcal{O}(D)$ allows us to apply Morera's Theorem to deduce $\int_{\partial R} f_j(z)dz = 0$ for each j. Therefore, $\int_{\partial R} f(z)dz = \lim_{j\to\infty}(0) = 0$. However, as R was arbitrary, we have by Morera's Theorem that f is holomorphic on D. $\square$

I suspect the discussion of continuity above is a vestige of our unwillingness to embrace Goursat's result in Gamelin.

Theorem 8.2.16. *Suppose that $\{f_j\}$ is holomorphic for $|z-z_o| \leq R$, and suppose that the sequence $\{f_j\}$ converges uniformly to f for $|z-z_o| \leq R$. Then for each $r < R$ and for each $m \geq 1$, the sequence of m-th derivatives $\{f_j^{(m)}\}$ converges uniformly to $f^{(m)}$ for $|z-z_o| \leq r$.*

Proof: as the convergence of $\{f_j\}$ is uniform we may select ϵ_j such that $|f_j(z) - f(z)| \leq \epsilon_j$ for $|z-z_o| < R$ where $\epsilon_j \to 0$ as $j \to \infty$. Fix s such that $r < s < R$. Apply the Cauchy Integral Formula for the m-th derivative of $f_j(z) - f(z)$ on the disk $|z-z_o| \leq s$:

$$f_j^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i}\oint_{|z-z_o|=s} \frac{f_j(w)-f(w)}{(w-z)^{m+1}}\, dw$$

for $|z-z_o| \leq r$. Consider, if $|w-z_o| = s$ and $|z-z_o| \leq r$ then

$$|w-z| = |w-z_o+z_o-z| \geq ||w-z_o| - |z-z_o|| = |s-|z-z_o|| \geq |s-r|.$$

Thus $|w-z| \geq s-r$ and it follows that

$$\left|\frac{f_j(w)-f(w)}{(w-z)^{m+1}}\right| \leq \frac{\epsilon_j}{(s-r)^{m+1}}$$

Therefore, as $L = 2\pi s$ for $|z-z_o| = s$ the ML-estimate provides:

$$|f_j^{(m)}(z) - f^{(m)}(z)| \leq \frac{m!}{2\pi i} \cdot \frac{\epsilon_j}{(s-r)^{m+1}} \cdot 2\pi s = \rho_j \qquad \text{(this defines } \rho_j\text{)}$$

for $|z-z_o| \leq r$. Notice, m is fixed thus $\rho_j \to 0$ as $j \to \infty$. In other words, ρ_j serves as the worst-case estimator for the m-th derivative and we have established the uniform convergence of $\{f_j^{(m)}\}$ for $|z-z_o| \leq r$. $\square$

I believe there are a couple small typos in Gamelin's proof on 136-137. They are corrected in what is given above.

Definition 8.2.17. *A sequence $\{f_j\}$ of holomorphic functions on a domain D* **converges normally** *to an analytic function f on D if it converges uniformly to f on each closed disk contained in D.*

Gamelin points out this leads immediately to our final theorem for this section: (this is really just Theorem 8.2.16 rephrased with our new normal convergence terminology)

Theorem 8.2.18. *Suppose that $\{f_j\}$ is a sequence of holomorphic functions on a domain D that converges normally on D to the holomorphic function f. Then for each $m \geq 1$, the sequence of m-th derivatives $\{f_j^{(m)}\}$ converges normally to $f^{(m)}$ on D.*

We already saw this behaviour with the geometric series. Notice that Example 8.2.13 shows $\sum_{j=0}^{\infty} z^j$ converges normally to $\frac{1}{1-z}$ on $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. Furthermore, we ought to note that the Weierstrauss M-test provides normal convergence. See [R91] page 92-93 for a nuanced discussion of the applicability and purpose of each mode of convergence. In summary, local uniform convergence is a natural mode for sequences of holomorphic functions whereas, normal convergence is the prefered mode of convergence for series of holomorphic functions. If the series are not normally convergent then we face the rearrangement ambiguity just as we did in the real case. Finally, a historical note which is a bit amusing. The term *normally convergent* is due to Baire of the famed Baire Catagory Theorem. From page 107 of [R91]

> Although in my opinion the introduction of new terms must only be made with extreme prudence, it appeared indispensable to me to characterize by a brief phrase the simplest and by far the most prevalent case of uniformly convergent series, that of series whose terms are smaller in modulus than positive numbers forming a convergent series (what one sometimes calls the Weierstrauss criterion). I call these series *normally* convergent, and I hope that people will be willing to excuse this innovation. A great number of demonstrations, be they in theory of series or somewhat further along in the theory of infinite products, are considerably simplified when one advances this notion, which is much more manageable than that of uniform convergence. (1908)

8.3 Power Series

In this section we study series of power functions.

Definition 8.3.1. *A* **power series centered at** z_o *is a series of the form* $\displaystyle\sum_{k=0}^{\infty} a_k(z-z_o)^k$ *where* $a_k, z_o \in \mathbb{C}$ *for all* $k \in \mathbb{N} \cup \{0\}$. *We say* a_k *are the* **coefficients of the series**.

Example 8.3.2. $\sum_{k=0}^{\infty} \frac{2^k}{k!}(z-3i)^k$ *is a power series centered at* $z_o = 3i$ *with coefficient* $a_k = \frac{2^k}{k!}$.

I will diverge from Gamelin slightly here and add some structure from [R91] page 110-111.

Lemma 8.3.3. Abel's Convergence Lemma: *Suppose for the power series* $\sum a_k z^k$ *there are positive real numbers* s *and* M *such that* $|a_k|s^k \leq M$ *for all* k. *Then this power series is normally convergent in* $\{z \in \mathbb{C} \mid |z| < s\}$.

Proof: consider r with $0 < r < s$ and let $q = r/s$. Observe, for $z \in \{z \in \mathbb{C} \mid |z| < r\}$,

$$|a_k z^k| < |a_k| r^k = |a_k| s^k \left(\frac{r}{s}\right)^k \leq M q^k$$

The series $\sum M q^k$ is geometric with $q = r/s < 1$ hence $\sum M q^k = \frac{M}{1-q}$. Therefore, by Weierstrauss' criterion we find $\sum a_k z^k$ is normally convergent on $\{z \in \mathbb{C} \mid |z| < s\}$. $\square$

This leads to the insightful result below:

Corollary 8.3.4. *If the series* $\sum a_k z^k$ *converges at* $z_o \neq 0$, *then it converges normally in the open disk* $\{z \in \mathbb{C} \mid |z| < |z_o|\}$.

Proof: as $\sum a_k z_o^k$ converges we have $a_k z_o^k \to 0$ as $k \to \infty$. Thus, $|a_k||z_o^k| \to 0$ as $k \to \infty$. Consequently, the sequence $\{|a_k||z_o^k|\}$ of positive terms is convergent and hence bounded. That is, there exists $M > 0$ for which $a_k||z_o^k| \leq M$ for all k. $\square$

The result above is a guiding principle as we search for possible domains of a given power series. If we find even one point at a certain distance from the center of the expansion then the whole disk is included in the domain. On the other hand, if we found the series diverged at a particular point then we can be sure no larger disk is included in the domain of power series. However, there might be points closer to the center which are also divergent. To find the domain of convergence we need to find the closest singularity to the center of the expansion (the center was $z = 0$ in Lemma and Corollary above, but, clearly these results translate naturally to series of the form $\sum a_k(z - z_o)^k$). Indeed, we should make a definition in view of our findings:

Definition 8.3.5. *A power series* $\sum_{k=0}^{\infty} a_k(z - z_o)^k$ *has* **radius of convergence** R *if the series converges for* $|z - z_o| < R$ *but diverges for* $|z - z_o| > R$*. In the case the series converges everywhere we say* $R = \infty$ *and in the case the series only converges at* $z = z_o$ *we say* $R = 0$*.*

It turns out the concept above is meaningful for **all** power series:

Theorem 8.3.6. *Let* $\sum a_k(z - z_o)^k$ *be a power series. Then there is* R*,* $0 \leq R \leq \infty$ *such that* $\sum a_k(z - z_o)^k$ *converges normally on* $\{z \in \mathbb{C} \mid |z - z_o| < R\}$*, and* $\sum a_k(z - z_o)^k$ *does not converge if* $|z - z_o| > R$*.*

Proof: Let us define (this can be a non-negative real number or ∞)

$$R = \sup\{t \in [0, \infty) \mid |a_k|t^k \text{ is a bounded sequence}\}$$

If $R = 0$ then the series converges only at $z = z_o$. Suppose $R > 0$ and let s be such that $0 < s < R$. By construction of R, the sequence $|a_k|s^k$ is bounded and by Abel's convergence lemma $\sum a_k(z-z_o)^k$ is normally convergent in $\{z \in \mathbb{C} \mid |z - z_o| < s\}$. However, $\{z \in \mathbb{C} \mid |z - z_o| < R\}$ is formed by a union of the open s-disks and thus we find normal convergence on the open R-disk centered at z_o. $\square$

The proof above is from page 111 of [R91]. Note the union argument is similar to $V.2\#10$ of page 138 in Gamelin where you were asked to show uniform convergence extends to finite unions.

Example 8.3.7. *The series* $\sum_{k=0}^{\infty} z^k$ *is the geometric series. We have shown it converges iff* $|z| < 1$ *which shows* $R = 1$*.*

Example 8.3.8. *The series* $\sum_{k=1}^{\infty} \frac{z^k}{k^4}$ *has majorant* $M_k = 1/k^4$ *for* $|z| < 1$*. Recall, by the p-series test, with* $p = 4 > 1$ *the series* $\sum_{k=1}^{\infty} \frac{1}{k^4}$ *converges. Thus, the given series in* z *is normally convergent on* $|z| < 1$*.*

Example 8.3.9. *Consider* $\sum_{j=0}^{\infty} \frac{(-1)^j}{4^j}(z - i)^{2j}$*. Notice this is geometric, simply let* $w = -(z - i)^2/4$ *and note:*

$$w^j = \left(\frac{-(z - i)^2}{4}\right)^j = \frac{(-1)^j (z - i)^{2j}}{4^j} \;\; \Rightarrow \;\; \sum_{j=0}^{\infty} \frac{(-1)^j}{4^j}(z - i)^{2j} = \sum_{j=0}^{\infty} w^j = \frac{1}{1 - w} = \frac{1}{1 + (z - i)^2/4}.$$

The convergence above is only given if we have $|w| < 1$ which means $|-(z-i)^2/4| < 1$ which yields $|z-i| < 2$. The given series represents the function $f(z) = \frac{1}{1+(z-i)^2/4}$ on the open disk $|z-i| < 2$. The power series $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j}{4^j}(z-i)^{2j}$ is centered at $z_o = i$ and has $R = 2$.

It is customary to begin series where the formula is reasonable when the start of the sum is not indicated.

Example 8.3.10. *The series $\sum k^k z^k$ has $R = 0$. Notice this series diverges by the n-th term test whenever $z \neq 0$.*

Example 8.3.11. *The series $\sum k^{-k} z^k$ has $R = \infty$. To see this, apply of Theorem 8.3.17 .*

At times I refer to what follows as Taylor's Theorem. This is probably not a good practice since Taylor's work was in the real domain and we make no mention of an estimate on the remainder term. That said, Cauchy has enough already so I continue this abuse of attribution.

Theorem 8.3.12. *Let $\sum a_k(z-z_o)^k$ be a power series with radius of convergence $R > 0$. Then, the function*

$$f(z) = \sum a_k(z-z_o)^k, \qquad |z-z_o| < R,$$

is holomorphic. The derivatives of $f(z)$ are obtained by term-by-term differentiation ,

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z-z_o)^{k-1}, \qquad f''(z) = \sum_{k=2}^{\infty} k(k-1)a_k(z-z_o)^{k-2},$$

and similarly for higher-order derivatives. The coefficients are given by:

$$a_k = \frac{1}{k!} f^{(k)}(z_o), \qquad k \geq 0.$$

Proof: by Theorem 8.3.6 the given series is normally convergent on $D_R(z_o)$; recall, $D_R(z_o) = \{z \in \mathbb{C} \mid |z-z_o| < R\}$. Notice that, for each $k \in \{0\} \cup \mathbb{N}$, $f_k(z) = a_k(z-z_o)^k$ is holomorphic on $D_R(z_o)$ hence by Theorem 8.2.15 we find $f(z)$ is holomorphic on $D_R(z_o)$. Furthermore, by Theorem 8.2.16, f' and f'' are holomorphic on $D_R(z_o)$ and are formed by the series of derivatives and second derivatives of $f_k(z) = a_k(z-z_o)^k$. We can calculate,

$$\frac{df_k}{dz} = k a_k (z-z_o)^{k-1} \qquad \& \qquad \frac{d^2 f_k}{dz^2} = k(k-1)a_k(z-z_o)^{k-2}.$$

Finally, the k-th coefficients of the series may be selected by evaluation at z_o of the k-th derivative of f. For $k = 0$ notice

$$f(z_o) = a_o + a_1(z_o - z_o) + a_2(z_o - z_o)^2 + \cdots = a_o$$

thus, as $f^{(0)}(z) = f(z)$ we have $f^{(0)}(z_o) = a_o$. Consider $f^{(k)}(z)$, apply the earlier result of this theorem for the k-th derivative,

$$f^{(k)}(z) = \sum_{j=k}^{\infty} j(j-1)(j-2)\cdots(j-k+1)a_j(z-z_o)^{j-k}$$

evaluate the above at $z = z_o$, only $j - k = 0$ gives nonzero term:

$$f^{(k)}(z_o) = k(k-1)(k-2)\cdots(k-k+1)a_k = k!a_k \qquad \Rightarrow \qquad a_k = \frac{f^{(k)}(z_o)}{k!}. \qquad \square$$

The next few examples illustrate an important calculational technique in this course. Basically, the idea is to twist geometric series via the term-by-term calculus to obtain near-geometric series. This allows us a wealth of examples with a minimum of calculation. I begin with a basic algebra trick before moving to the calculus-based slight of hand.

Example 8.3.13.

$$\sum_{k=0}^{\infty} z^{3k+4} = \sum_{k=0}^{\infty} z^4 z^{3k} = z^4 \sum_{k=0}^{\infty} (z^3)^k = \frac{z^4}{1-z^{3k}}.$$

The series above normally converges to $f(z) = \frac{z^4}{1-z^{3k}}$ *for* $|z^3| < 1$ *which is simply* $|z| < 1$.

Example 8.3.14.

$$\sum_{k=0}^{\infty} \left(z^{2k} + (z-1)^{2k}\right) = \sum_{k=0}^{\infty} z^{2k} + \sum_{k=0}^{\infty} (z-1)^{2k} = \frac{1}{1-z^2} + \frac{1}{1-(z-1)^2}$$

where the geometric series both converge only if we have a simultaneous solution of $|z| < 1$ *and* $|z-1| < 1$. *The open region on which the series above converges is not a disk. Why does this not contradict Theorem 8.3.6 ?*

Ok, getting back to the calculus tricks I mentioned previous to the above pair of examples,

Example 8.3.15. *Notice* $f(z) = \frac{1}{1-z^2}$ *has* $\frac{df}{dz} = \frac{2z}{(1-z^2)^2}$. *However, for* $|z^2| < 1$ *which is more naturally presented as* $|z| < 1$ *we have:*

$$f(z) = \frac{1}{1-z^2} = \sum_{k=0}^{\infty} z^{2k} \quad \Rightarrow \quad \frac{df}{dz} = \sum_{k=1}^{\infty} 2kz^{2k-1}.$$

Therefore, we discover, for $|z| < 1$ *the function* $g(z) = \frac{2z}{(1-z^2)^2}$ *has the following power series representation centered at* $z_o = 0$,

$$\frac{2z}{(1-z^2)^2} = \sum_{k=1}^{\infty} 2kz^{2k-1} = 2z + 4z^3 + 6z^5 + \cdots .$$

Example 8.3.16. *The singularity of* $f(z) = Log(1-z)$ *is found at* $z = 1$ *hence we have hope to look for power series representations for this function away from* $z_o = 1$. *Differentiate* $f(z)$ *to obtain (note, the* -1 *is from the chain rule):*

$$\frac{df}{dz} = \frac{-1}{1-z} = -\sum_{k=0}^{\infty} z^k.$$

Integrate both sides of the above to see that there must exist a constant C *for which*

$$Log(1-z) = C - \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}$$

But, we have Log$(1-0) = 0 = C$ hence,

$$-Log(1-z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots .$$

The calculation above holds for $|z| < 1$ according to the theorems we have developed about the geometric series and term-by-term calculus. However, in this case, we may also observe $z = -1$ produces the negative of alternating harmonic series which converges. Thus, there is at least one point on which the series for $-Log(1-z)$ converges where the differentiated series did not converge. This is illustrative of a general principle which is worth noticing: differentiation may remove points from the boundary of the disk of convergence whereas integration tends to add points of convergence on the boundary.

Theorem 8.3.17. *If $|a_k/a_{k+1}|$ has a limit as $k \to \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k(z-z_o)^k$*

Proof: Let $L = \lim_{k\to\infty} |a_k/a_{k+1}|$. If $r < L$ then there must exist $N \in \mathbb{N}$ such that $|a_k/a_{k+1}| > r$ for all $k > N$. Observe $|a_k| > r|a_{k+1}|$ for $k > N$. It follows,

$$|a_N|r^N \geq |a_{N+1}|r^{N+1} \geq |a_{N+2}|r^{N+2} \geq \cdots$$

Let $M = \max\{|a_o|, |a_1|r, \dots, |a_{N-1}|r^{N-1}, |a_N|r^N\}$ and note $|a_k|r^k \leq M$ for all k hence by Abel's Convergence Lemma, the power series $\sum a_k(z-z_o)^k$ is normally convergent for $|z| < r$. Thus, $r \leq R$ as R defines the maximal disk on which $\sum a_k(z-z_o)^k$ is normally convergent. Let $\{r_n\}$ be a sequence of such that $r_n < L$ for each n and $r_n \to L$ as $n \to \infty$. For $r_n < L$ we've shown $r_n \leq R$ hence $\lim_{n\to\infty} r_n \leq \lim_{n\to\infty} R$ by the sandwhich theorem. Thus $L \leq R$.

Suppose $s > L$. We again begin with an observation that there exists an $N \in \mathbb{N}$ such that $|a_k/a_{k+1}| < s$ for $k > N$. It follows,

$$|a_N|s^N \leq |a_{N+1}|s^{N+1} \leq |a_{N+2}|s^{N+2} \leq \cdots$$

and clearly $\sum a_k(z-z_o)^k$ fails the n-th term test for $z \in \mathbb{C}$ with $|z-z_o| > s$. We find the series diverges for $|z-z_o| > s$ and thus we find $s \geq R$. Let $\{s_n\}$ be a sequence of values with $s_n > L$ for each n and $\lim_{n\to\infty} s_n = L$. The argument we gave for s equally well applies to each s_n hence $s_n \geq R$ for all n. Once again, take $n \to \infty$ and apply the sandwhich lemma to obtain $\lim_{n\to\infty} s_n = L \leq R$.

Thus $L \leq R$ and $L \geq R$ and we conclude $L = R$ as desired. $\square$

Theorem 8.3.18. *If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite or $+\infty$, then the radius of convergence R of $\sum a_k(z-z_o)^k$ is given by:*

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}.$$

Proof: see page 142. Again, you can see Abel's Convergence Lemma at work. $\square$

One serious short-coming of the ratio and root tests is their failure to apply to series with infinitely many terms which are zero. The **Cauchy Hadamard** formula gives a refinement which allows us to capture such examples. In short, the limit superior replaces the limit in Theorem 8.3.18. If you would like to read more, I recommend page 112 of [R91].

8.4 Power Series Expansion of an Analytic Function

In the previous section we studied some of the basic properties of complex power series. Our main result was that a function defined by a power series is holomorphic on the open disk of convergence. We discover a converse in this section: holomorphic functions on a disk admit power series representation on the disk. We finally introduce the term *analytic*

Definition 8.4.1. *A function $f(z)$ is* **analytic** *on $D_R(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < R\}$ if there exist coefficients $a_k \in \mathbb{C}$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$ for all $z \in D_R(z_o)$.*

Of course, by Theorem 8.3.12 we immediately know $f(z)$ analytic on some disk about z_o forces the coefficients to follow Taylor's Theorem $a_k = f^{(k)}(z_o)/k!$. Thus, another way of characterizing an analytic function is that an analytic function is one which is generated by its Taylor series[7].

Theorem 8.4.2. *Suppose $f(z)$ is holomorphic for $|z - z_o| < \rho$. Then $f(z)$ is represented by the power series*

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k, \qquad |z - z_o| < \rho,$$

where

$$a_k = \frac{f^{(k)}(z_o)}{k!}, \qquad k \geq 0,$$

and where the power series has radius of convergence[8] $R \geq \rho$. For any fixed r, $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \oint_{|w - z_o| = r} \frac{f(w)}{(w - z_o)^{k+1}}\, dw, \qquad k \geq 0.$$

Further, if $|f(z)| \leq M$ for $|z - z_o| = r$, then

$$|a_k| \leq \frac{M}{r^k}, \qquad k \geq 0.$$

Proof: assume $f(z)$ is as stated in the theorem. Let $z \in \mathbb{C}$ such that $|z| < r < \rho$. Suppose $|w| = r$ then by the geometric series Proposition 8.1.6

$$\frac{f(w)}{w - z} = \frac{f(w)}{w} \frac{1}{1 - z/w} = \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k = \sum_{k=0}^{\infty} f(w) \frac{z^k}{w^{k+1}}.$$

Moreover, we are given the convergence of the above series is uniform for $|w| = r$. This allows us to expand Cauchy's Integral formula into the integral of a series of holomorphic functions which converges uniformly. It follows we are free to apply Theorem 8.2.9 to exchange the order of the integration and the infinite summation in what follows:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z}\, dw \\ &= \frac{1}{2\pi i} \int_{|w|=r} \left(\sum_{k=0}^{\infty} f(w) \frac{z^k}{w^{k+1}} \right) dw \\ &= \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w^{k+1}}\, dw \right)}_{a_k} z^k. \end{aligned}$$

[7] again, I feel obligated to mention Taylor's work was in the real domain, so this term is primarily to allow the reader to connect with their experience with real power series

[8] we should remember Theorem 8.3.6 provides the series is normally convergent

This suffices to prove the theorem in the case $z_o = 0$. Notice the result holds whenever $|z| < r$ and as $r < \rho$ is arbitrary, we must have the radius of convergence[9] $R \geq \rho$. Continuing, I reiterate the argument for $z_o \neq 0$ as I think it is healthy to see the argument twice and as the algebra I use in this proof **is relevant** to future work on a multitude of examples.

Suppose $z \in \mathbb{C}$ such that $|z - z_o| < r < \rho$. Suppose $|w - z_o| = r$ hence $|z - z_o|/|w - z_o| < 1$ thus:

$$\begin{aligned} \frac{f(w)}{w-z} &= \frac{f(w)}{w - z_o - (z - z_o)} \\ &= \frac{f(w)}{w - z_o} \cdot \frac{1}{1 - \left(\frac{z-z_o}{w-z_o}\right)} \\ &= \frac{f(w)}{w - z_o} \sum_{k=0}^{\infty} \left(\frac{z - z_o}{w - z_o}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{f(w)(z - z_o)^k}{(w - z_o)^{k+1}} \end{aligned}$$

Thus, following the same logic as in the $z_o = 0$ case, but now for $|w - z_o| = r$, we obtain:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \int_{|w-z_o|=r} \left(\sum_{k=0}^{\infty} \frac{f(w)(z - z_o)^k}{(w - z_o)^{k+1}} \right) dw \\ &= \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{(w - z_o)^{k+1}}\, dw \right)}_{a_k} (z - z_o)^k. \end{aligned}$$

Once again we can argue that as $|z - z_o| < r < \rho$ gives $f(z)$ presented as the power series centered at z_o above for arbitrary r it must be that the radius of convergence $R \geq \rho$.

The derivative identity $a_k = \frac{f^{(k)}(z_o)}{k!}$ is given by Theorem 8.3.12 and certain applies here as we have shown the power series representation of $f(z)$ exists. Finally, if $|f(z)| \leq M$ for $|z - z_o| < r$ then apply Cauchy's Estimate 7.4.1

$$|a_k| = \left| \frac{f^{(k)}(z_o)}{k!} \right| \leq \frac{1}{k!} \frac{Mk!}{r^k} = \frac{M}{r^k} \qquad \square$$

Consider the argument of the theorem above. If you were a carefree early nineteenth century mathematician you might have tried the same calculations. If you look at was derived for a_k and compare the differential to the integral result then you would have **derived** the Generalized Cauchy Integral Formula:

$$a_k = \frac{f^{(k)}(z_o)}{k!} = \frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{(w - z_o)^{k+1}}\, dw.$$

You can contrast our viewpoint now with that which we proved the Generalized Cauchy Integral Formula back in Theorem 7.3.2. The technique of expanding $\frac{1}{w-z}$ into a power series for which

[9]this can be made rigorous with a sequential argument as I offered twice in the proof of Theorem 8.3.17

integration and differentiation term-by-term was to be utilized was known and practiced by Cauchy at least as early as 1831 see page 210 of [R91]. In retrospect, it is easy to see how once one of these theorems was discovered, the discovery of the rest was inevitable to the curious.

What follows is a corollary to Theorem 8.4.2.

Corollary 8.4.3. *Suppose $f(z)$ and $g(z)$ are holomorphic for $|z - z_o| < r$. If $f^{(k)}(z_o) = g^{(k)}(z_o)$ for $k \geq 0$ then $f(z) = g(z)$ for $|z - z_o| < r$.*

Proof: if f, g are holomorphic on $|z - z_o| < r$ then Theorem 8.4.2 said they are also analytic on $|z - z_o| < r$ with coefficients fixed by the values of the function and their derivatives at z_o. Consequently, both functions share identical power series on $|z - z_o| < r$ hence their values match at each point in the disk. □

Theorem 8.3.6 told us that the domain of a power series included an open disk of some maximal radius R. Now, we learn that if $f(z)$ is holomorphic on an open disk centered at z_o then it has a power series representation on the disk. It follows that the function cannot be holomorphic beyond the radius of convergence given to us by Theorem 8.3.6 for if it did then we would find the power series centered at z_o converged beyond the radius of convergence.

Corollary 8.4.4. *Suppose $f(z)$ is analytic at z_o, with power series expansion centered at z_o; $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$. The radius of convergence of the power series is the largest number R such that $f(z)$ extends to be holomorphic on the disk $\{z \in \mathbb{C} \mid |z - z_o| < R\}$*

Notice that power series converge normally on the disk of their convergence. It seems that Gamelin is unwilling to use the term *normally convergent* except to introduce it. Of course, this is not a big deal, we can either use the term or state it's equivalent in terms of uniform convergence on closed subsets.

Example 8.4.5. *Let $f(z) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} z^k = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$. We can show $f(z)f(w) = f(z+w)$ by direct calculation of the Cauchy product. Once that is known and we observe $f(0) = 0$ then it is simple to see $f(z)f(-z) = f(z-z) = f(0) = 1$ hence $\frac{1}{f(z)} = f(-z)$. Furthermore, we can easily show $\frac{df}{dz} = f$. All of these facts are derived from the arithmetic of power series alone. That said, perhaps you recognize these properties as those of the exponential function. There are two viewpoints to take here:*

1. *define the complex exponential function by the power series here and derive the basic properties by the calculus of series*

2. *define the complex exponential function by $e^{x+iy} = e^x(\cos y + i \sin y)$ and verify the given series represents the complex exponential on $\mathbb{C}$.*

Whichever viewpoint you prefer, we all agree:

$$\boxed{e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots}$$

Notice $a_k = 1/k!$ hence $a_k/a_{k+1} = (k+1)!/k! = k+1$ hence $R = \infty$ by ratio test for series.

Example 8.4.6. *Consider* $f(z) = \cosh z$ *notice* $f'(z) = \sinh z$ *and* $f''(z) = \cosh z$ *and in general* $f^{(2k)}(z) = \cosh z$ *and* $f^{(2k+1)}(z) = \sinh z$*. We calculate* $f^{(2k)}(0) = \cosh 0 = 1$ *and* $f^{(2k+1)}(0) = \sinh 0 = 0$*. Thus,*

$$\boxed{\cosh z = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k} = 1 + \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \cdots}$$

Example 8.4.7. *Following from Definition 2.5.2 we find* $e^z = \cosh z + \sinh z$*. Thus,* $\sinh z = e^z - \cosh z$*. Therefore,*

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n - \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k}.$$

However, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1}$ *hence the even terms cancel and we find the odd series below for hyperbolic sine:*

$$\boxed{\sinh z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1} = 1 + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots}$$

Example 8.4.8. *To derive the power series for* $\sin z$ *and* $\cos z$ *we use the relations* $\cosh(iz) = \cos(z)$ *and* $\sinh(iz) = i \sin z$ *hence*

$$\cos z = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (iz)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

since $i^{2k} = (i^2)^k = (-1)^k$*. Likewise, as* $i^{2k+1} = i(-1)^k$

$$i \sin z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (iz)^{2k+1} = i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

Therefore,

$$\boxed{\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \cdots}$$

and

$$\boxed{\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots.}$$

Once again, I should comment, we could use the boxed formulas above to **define** cosine and sine. It is then straightforward to derive all the usual properties of sine and cosine. A very nice presentation of this is found on pages 274-278 of [J02]. You might be interested to know that π can be carefully defined as twice the smallest positive zero of $\cos z$. Since the series definition of cosine does not implicitly use the definition of π, this gives us a careful, non-geometric, definition of π.

8.5 Power Series Expansion at Infinity

The technique used in this section could have been utilized in earlier discussions of ∞. To study the behaviour of $f(z)$ at $z = \infty$ we simple study the corresponding function $g(w) = f(1/w)$ at $w = 0$.

Example 8.5.1. *Notice* $\lim_{z\to\infty} f(z) = \lim_{w\to 0} f(1/w)$ *allows us to calculate:*

$$\lim_{z\to\infty} \frac{z}{z+1} = \lim_{w\to 0} \frac{1/w}{1/w+1} = \lim_{w\to 0} \frac{1}{1+w} = \frac{1}{1+0} = 1.$$

Definition 8.5.2. *A function $f(z)$ is* **analytic** *at $z = \infty$ if $g(w) = f(1/w)$ is analytic at $w = 0$.*

In particular, we mean that there exist coefficients $b_o, b_1, \dots$ and $\rho > 0$ such that $g(w) = b_o + b_1 w + b_2 w^2 + \cdots$ for all $w \in \mathbb{C}$ such that $0 < |w| < \rho$. Recall, by Theorem 8.4.2 we have $\sum_{k=0} b_k w^k$ converging normally to $g(w)$ on the open disk of convergence. If $|z| > 1/\rho$ then $1/|z| < \rho$ hence

$$f(z) = g(1/z) = b_o + b_1/z + b_2/z^2 + \cdots .$$

The series $b_o + b_1/z + b_2/z^2 + \cdots$ coverges normally to $f(z)$ on the **exterior domain** $\{z \in \mathbb{C} \mid |z| > R\}$ where $R = 1/\rho$. Recall that normal convergence previous meant we had uniform convergence on all closed subdisks, in this context, it means we have uniform convergence for any $S > R$. In particular, for each $S > R$, the series $b_o + b_1/z + b_2/z^2 + \cdots$ converges uniformly to $f(z)$ for $\{z \in \mathbb{C} \mid |z| > S\}$.

Example 8.5.3. *Let $P(z) \in \mathbb{C}[z]$ be a polynomial of order N. Then $P(z) = a_o + a_1 z + \cdots + a_N z^N$ is not analytic at $z = \infty$ as the function $g(w) = a_o + a_1/w + \cdots + a_n/z^N$ is not analytic at $w = 0$.*

Example 8.5.4. *Let $f(z) = \frac{1}{z^2} + \frac{1}{z^{42}}$ is analytic at $z = \infty$ since $g(w) = f(1/w) = w^2 + w^{42}$ is analytic at $w = 0$. In fact, g is entire which goes to show $f(z) = \frac{1}{z^2} + \frac{1}{z^{42}}$ on $\mathbb{C}^\times$. Refering to the terminology just after 8.5.2 we have $\rho = \infty$ hence $R = 0$.*

The example above is a rather silly example of a **Laurent Series**. It is much like being asked to find the Taylor polynomial for $f(z) = z^2 + 3z + 2$ centered at $z = 0$; in the same way, the function is defined by a *Laurent polynomial* centered at $z = 0$, there's nothing to find. The major effort of the next Chapter is to develop theory to understand the structure of these Laurent series.

Example 8.5.5. *Let $f(z) = \frac{z^2}{z^2-1}$ consider $g(w) = f(1/w) = \frac{1/w^2}{1/w^2-1} = \frac{1}{1-w^2} = \sum_{k=0}^\infty w^{2k}$. Hence $f(z)$ is analytic at $z = \infty$. Notice, the power series centered at $w = 0$ converges normally on $|w| < 1$ hence the series below converges normally to $f(z)$ for $|z| > 1$*

$$f(z) = \sum_{k=0}^\infty \left(\frac{1}{z}\right)^{2k} = 1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots .$$

Example 8.5.6. *Let $f(z) = \sin(1/z^2)$. Notice $g(w) = \sin(w^2) = w^2 - \frac{1}{3!}(w^2)^3 + \cdots$ for $w \in \mathbb{C}$. Thus $f(z)$ is analytic at $z = \infty$ and $f(z)$ is represented normally on the punctured plane by:*

$$f(z) = \frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^6} + \frac{1}{5!}\frac{1}{z^{10}} + \cdots = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}\frac{1}{z^{4k+2}}.$$

In summary, we have seen that a function which is analytic at $z = z_o \neq \infty$ allows a power series representation $\sum_{k=0}^\infty a_k (z - z_o)^k$ on disk of radius $0 < R \leq \infty$. On the other hand, a function which is analytic at $z = \infty$ has a representation of the form $\sum_{-\infty}^{k=0} a_k z^k = a_o + a_{-1}/z + a_{-2}/z^2 + \cdots$ on an annulus $|z| > R$ where $0 \leq R < \infty$.

Theorem 8.5.7. *If f is analytic at ∞ then there exists $\rho > 0$ such that for $|z - z_o| > \rho$*

$$f(z) = \sum_{k=-\infty}^{0} a_k (z - z_o)^k = a_o + \frac{a_{-1}}{z - z_o} + \frac{a_{-1}}{(z - z_o)^2} + \cdots .$$

I should mention, if you wish a more careful treatment, you might meditate on the arguments offered on page 348 of [R91].

8.6 Manipulation of Power Series

The sum, difference, scalar multiple, product and quotient of power series are discussed in this section.

Theorem 8.6.1. *Suppose $\sum_{k=0}^{\infty} a_k(z - z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_o)^k$ are convergent power series on a domain D then*

$$\sum_{k=0}^{\infty} a_k(z - z_o)^k + c\sum_{k=0}^{\infty} b_k(z - z_o)^k = \sum_{k=0}^{\infty} (a_k + cb_k)(z - z_o)^k$$

for all $z \in D$.

Proof: suppose f, g are analytic on D where $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k(z-z_o)^k$. Let $c \in \mathbb{C}$ and define $h(z) = f(z) + cg(z)$ for each $z \in D$. Observe,

$$h^{(k)}(z_o) = f^{(k)}(z_o) + cg^{(k)}(z_o) \quad \Rightarrow \quad \frac{h^{(k)}(z_o)}{k!} = \frac{f^{(k)}(z_o)}{k!} + c\frac{g^{(k)}(z_o)}{k!} = a_k + cb_k$$

by Theorem 8.4.2. Thus, $h(z) = \sum_{k=0}^{\infty} (a_k + cb_k)(z - z_o)^k$ by Corollary 8.4.4. □

The method of proof is essentially the same for the product of series theorem. We use Corollary 8.4.4 to obtain equality of functions by comparing derivatives. I suppose we should define the product of series:

Definition 8.6.2. Cauchy Product: *Let $\sum_{k=0}^{\infty} a_k(z - z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_o)^k$ then*

$$\left(\sum_{k=0}^{\infty} a_k(z - z_o)^k\right)\left(\sum_{k=0}^{\infty} b_k(z - z_o)^k\right) = \sum_{k=0}^{\infty} c_k(z - z_o)^k$$

where we define $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ for each $k \geq 0$.

Technically, we ought to wait until we prove the theorem below to make the definition above. I hope you can forgive me.

Theorem 8.6.3. *Suppose $\sum_{k=0}^{\infty} a_k(z - z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_o)^k$ are convergent power series on an open disk D with center $z_o \in D$ then*

$$\left(\sum_{k=0}^{\infty} a_k(z - z_o)^k\right)\left(\sum_{k=0}^{\infty} b_k(z - z_o)^k\right) = \sum_{k=0}^{\infty} c_k(z - z_o)^k$$

for all $z \in D$ where c_k is defined by the Cauchy Product; $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ for each $k \geq 0$.

Proof: I follow the proof on page 217 of [R91]. Let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k(z-z_o)^k$ for each $z \in D$. By Theorem 8.3.12 both f and g are holomorphic on D. Therefore, $h = fg$ is holomorphic on D as $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ for each $z \in D$. Theorem 8.4.2 then shows fg is analytic at z_o hence there exist c_k such that $h(z) = f(z)g(z) = \sum_k c_k(z-z_o)^k$. It remains to show that c_k is as given by the Cauchy product. We proceed via Corollary 8.4.4. We need to show $\frac{h^{(k)}(z_o)}{k!} = c_k$ for $k \geq 0$. Begin with $k = 0$,

$$h(z_o) = f(z_o)g(z_o) = a_o b_o = c_o.$$

Continuing, for $k = 1$,

$$h'(z_o) = f'(z_o)g(z_o) + f(z_o)g'(z_o) = a_1 b_0 + a_0 b_1 = c_1.$$

Differentiating once again we find $k = 2$, note $f''(z_o)/2 = a_2$,

$$\begin{aligned} h''(z_o) &= f''(z_o)g(z_o) + f'(z_o)g'(z_o) + g'(z_o)f'(z_o) + f(z_o)g''(z_o) \\ &= 2a_2b_0 + 2a_1b_1 + 2a_0b_2 \\ &= 2c_2. \end{aligned}$$

To treat the k-th coefficient in general it is useful for us to observe the Leibniz k-th derivative rule:

$$(fg)^{(k)}(z) = \sum_{i+j=k} \frac{k!}{i!j!} f^{(i)}(z)g^{(j)}(z) = f^{(k)}(z)g(z) + kf^{(k-1)}(z)g'(z)\cdots + f(z)g^{(k)}(z)$$

Observe, $f^{(i)}(z_o)/i! = a_i$ and $g^{(j)}(z_o)/j! = b_j$ hence:

$$(fg)^{(k)}(z_o) = \sum_{i+j=k} k! a_i b_j = k!(a_o b_k + \cdots + a_k b_o) = k! c_k.$$

Thus, $(fg)^{(k)}(z_o)/k! = c_k$ and the theorem by Corollary 8.4.4. $\square$

I offered the argument for $k = 0, 1$ and 2 explicitly to take the mystery out of the Leibniz rule argument. I leave the proof of the Leibniz rule to the reader. There are other proofs of the product theorem which are just given in terms of the explicit analysis of the series. For example, see Theorem 3.50d of [R76] where the product of a convergent and an absolutely convergent series is shown to converge to an absolutely convergent series defined by the Cauchy Product.

Example 8.6.4. *Find the power series to order* 5 *centered at* $z = 0$ *for* $2\sin z \cos z$

$$\begin{aligned} 2\sin z\cos z &= 2\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots\right) \\ &= 2\left(z - \left[\frac{1}{2} + \frac{1}{6}\right]z^3 + \left[\frac{1}{24} + \frac{1}{12} + \frac{1}{120}\right]z^5 + \cdots\right) \\ &= 2z - \frac{4}{3}z^3 + \frac{4}{15}z^5 + \cdots \end{aligned}$$

Of course, as $2\sin z\cos z = \sin(2z) = 2z - \frac{1}{3!}(2z)^3 + \frac{1}{5!}(2z)^5 + \cdots$ *we can avoid the calculation above. I merely illustrate the consistency.*

The example below is typical of the type of calculation we wish to master:

Example 8.6.5. *Calculate the product below to second order in z:*

$$\begin{aligned}
e^z\cos(2z+1) &= e^z\left(\cos(2z)\cos(1)-\sin(2z)\sin(1)\right)\\
&= \left(1+z+\frac{1}{2}z^2\right)\left(\cos(1)(1-\frac{1}{2}(2z)^2)-2z\sin(1)\right)+\cdots\\
&= \left(1+z+\frac{1}{2}z^2\right)\left(\cos(1)-2\sin(1)z-2\cos(1)z^2\right)+\cdots\\
&= \cos(1)+[\cos(1)-2\sin(1)]\,z+\left(\frac{\cos(1)}{2}-2\sin(1)-2\cos(1)\right)z^2+\cdots
\end{aligned}$$

Stop and ponder why I did not directly expand $\cos(2z+1)$ as $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(2z+1)^{2k+1}$. If you did that, then you would need to gather infinitely many terms together to form the sines and cosines we derived with relative ease from the adding-angles formula for cosine.

The geometric series allows fascinating calculation:

Example 8.6.6. *Multiply* $1+z+z^2+\cdots$ *and* $1-z+z^2+\cdots$.

$$(1+z+z^2+\cdots)(1-z+z^2+\cdots)=\frac{1}{1-z}\cdot\frac{1}{1+z}=\frac{1}{1-z^2}=1+z^2+z^4+\cdots.$$

I probably could add some insight here by merging the calculations I cover in calculus II here, however, I'll stop at this point and turn to the question of division.

Suppose $\sum_{k=0}^{\infty}a_k(z-z_o)^k$ where $a_o\neq 0$. Calculation of $\frac{1}{\sum_{k=0}^{\infty}a_k(z-z_o)^k}$ amounts to calculation of coefficients b_k for $k\geq 0$ such that $\left(\sum_{k=0}^{\infty}a_k(z-z_o)^k\right)\left(\sum_{k=0}^{\infty}b_k(z-z_o)^k\right)=1$. The Cauchy product provides a sequence of equations we must solve:

$$\begin{aligned}
a_ob_o=1 \qquad &\Rightarrow \qquad b_o=1/a_o.\\
a_ob_1+a_1b_o=0, \qquad &\Rightarrow \qquad b_1=\frac{-a_1b_o}{a_o}=\frac{-a_1}{a_o^2}.\\
a_ob_2+a_1b_1+a_2b_o=0, \qquad &\Rightarrow \qquad b_2=-\frac{a_1b_1+a_2b_o}{a_o}=\frac{a_1^2}{a_o^3}-\frac{a_2}{a_o^2}.\\
a_ob_3+a_1b_2+a_2b_1+a_3b_0=0 \qquad &\Rightarrow \qquad b_3=-\frac{a_1b_2+a_2b_1+a_3b_o}{a_o}.
\end{aligned}$$

The calculation above can clearly be extended to higher order. Recursively, we have solution:

$$b_k=-\frac{a_1b_{k-1}+a_2b_{k-2}+\cdots+a_{k-1}b_1+a_kb_o}{a_o}$$

for $k\geq 0$.

Example 8.6.7. *Consider* $2-4z+8z^2-16z^3\cdots$ *identify* $a_o=2, a_1=-4, a_2=8$ *and* $a_3=-16$. *Using the general calculation above this example, calculate*

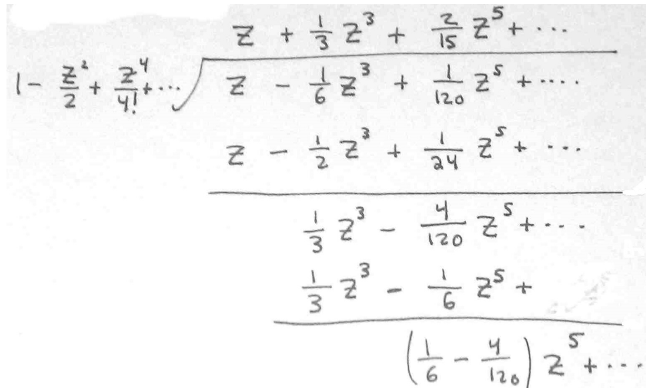
$$b_o=\frac{1}{2},\quad b_1=\frac{4}{4}=1,\quad b_2=\frac{-(-4)(1)-(8)(1/2)}{2}=0,\quad b_3=-\frac{-4(0)+(8)(1)+(-16)(1/2)}{2}=0.$$

Hence,

$$\frac{1}{2-4z+8z^2-16z^3\cdots}=\frac{1}{2}+z+\cdots.$$

I can check our work as $2-4z+8z^2-16z^3\cdots=2(1-2z+(-2z)^2+(-2z)^3\cdots)=\frac{2}{1+2z}$ *hence* $\frac{1}{2-4z+8z^2-16z^3\cdots}=\frac{1+2z}{2}=\frac{1}{2}+z$. *Apparently, we could calculate* $b_k=0$ *for* $k\geq 2$.

We next illustrate how to find the power series for $\tan(z)$ by *long-division*:



The calculation above shows that $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$ divided by $\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots$ yields:

$$\tan z = \frac{\sin z}{\cos z} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots .$$

It should be fairly clear how to obtain higher-order terms by the method of long-division.

We now consider a different method to calculate the power series for $\tan z$ which uses the geometric series to obtain the reciprocal of the cosine series. Consider,

$$\begin{aligned}
\frac{1}{\cos z} &= \frac{1}{1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots} \\
&= \frac{1}{1 - \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right)} \\
&= 1 + \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right) + \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right)^2 + \cdots \\
&= 1 + \frac{1}{2}z^2 + \left(-\frac{1}{24} + \frac{1}{2}\cdot\frac{1}{2}\right)z^4 + \cdots \\
&= 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots .
\end{aligned}$$

Then, to find $\tan(z)$ we simply multiply by the sine series,

$$\begin{aligned}
\sin z \cdot \frac{1}{\cos z} &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots\right) \\
&= z + \left(\frac{1}{2} - \frac{1}{6}\right)z^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)z^5 + \cdots \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots .
\end{aligned}$$

The recursive technique, long-division and geometric series manipulation are all excellent tools which we use freely throughout the remainder of our study. Some additional techniques are elucidated in §8.8. There I show my standard bag of tricks for recentering series.

8.7 The Zeros of an Analytic Function

Power series are, according to Dr. Monty Kester, *Texas sized polynomials*. With all due respect to Texas, it's not *that* big. That said, power series and polynomials do share much in common. In particular, we find a meaningful and interesting generalization of the *factor theorem*.

Definition 8.7.1. *Let f be an analytic function which is not identically zero near $z = z_o$ then we say f has a zero of order N at z_o if*

$$f(z_o) = 0, \quad f'(z_o) = 0, \ \cdots \ f^{(N-1)}(z_o) = 0, \quad f^{(N)}(z_o) \neq 0.$$

A zero of order $N = 1$ is called a **simple zero***. A zero of order $N = 2$ is called a* **double zero***.*

Suppose $f(z)$ has a zero of order N at z_o. If $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ then as $a_k = \frac{f^{(k)}}{k!} = 0$ for $k = 0, 1, \dots, N-1$ we have

$$f(z) = \sum_{k=N}^{\infty} a_k(z-z_o)^k = (z-z_o)^N \sum_{k=N}^{\infty} a_k(z-z_o)^{k-N} = (z-z_o)^N \underbrace{\sum_{j=0}^{\infty} a_{j+N}(z-z_o)^j}_{h(z)}$$

Observe that $h(z)$ is also analytic at z_o and $h(z_o) = a_N = \frac{f^{(N)}(z_o)}{N!} \neq 0$. It follows that there exists $\rho > 0$ for which $0 < |z - z_o| < \rho$ implies $f(z) \neq 0$. In other words, the zero of an analytic function is **isolated**.

Definition 8.7.2. *Let $U \subseteq \mathbb{C}$ then $z_o \in U$ is an* **isolated point of** U *if there exists some $\rho > 0$ such that $\{z \in U \mid |z - z_o| < \rho\} = \{z_o\}$.*

We prove that all zeros of an analytic function are isolated a bit later in this section. However, first let me record the content of our calculations thus far:

Theorem 8.7.3. Factor Theorem for Power Series: *If $f(z)$ is an analytic function with a zero of order N at z_o then there exists $h(z)$ analytic at z_o with $h(z_o) \neq 0$ and $f(z) = (z - z_o)^N h(z)$.*

Example 8.7.4. *The prototypical example is simply the monomial $f(z) = (z - z_o)^n$. You can easily check f has a zero $z = z_o$ of order n.*

Example 8.7.5. *Consider $f(z) = \sin(z^2) = z^2 - \frac{1}{6}z^6 + \frac{1}{120}z^{10} + \cdots$. Notice $f(0) = f'(0) = 0$ and $f''(0) = 2$ thus $f(z)$ as a double zero of $z = 0$ and we can factor out z^2 from the power series centered at $z = 0$ for $f(z)$:*

$$f(z) = z^2 \left(1 - \frac{1}{6}z^4 + \frac{1}{120}z^8 + \cdots\right).$$

Example 8.7.6. *Consider $f(z) = \sin(z^2) = z^2 - \frac{1}{6}z^6 + \frac{1}{120}z^{10} + \cdots$ once again. Let us consider the zero for $f(z)$ which is given by $z^2 = n\pi$ for some $n \in \mathbb{Z}$ with $n \neq 0$. This has solutions $z = \pm\sqrt{n\pi}$. In each case, $f(\pm\sqrt{n\pi}) = \sin n\pi = 0$ and $f'(\pm\sqrt{n\pi}) = \pm 2\sqrt{n\pi}\cos \pm\sqrt{n\pi} \neq 0$. Therefore, every other zero of $f(z)$ is simple. Only $z = 0$ is a double zero for $f(z)$. Although the arguments offered* *thus far suffice, I find explicit calculation of the power series centered at $\sqrt{n\pi}$ a worthwhile exercise:*

$$\begin{aligned}
\sin(z^2) &= \sin\left([z-\sqrt{n\pi}+\sqrt{n\pi}\,]^2\right) \\
&= \sin\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})+n\pi\right) \\
&= (-1)^n\sin\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})\right) \\
&= (-1)^n\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})-\frac{1}{6}\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})\right)^3+\cdots\right) \\
&= (z-\sqrt{n\pi})(-1)^n\left(2\sqrt{n\pi}+(z-\sqrt{n\pi})-\frac{4n\pi\sqrt{n\pi}}{3}(z-\sqrt{n\pi})^2+\cdots\right)
\end{aligned}$$

Example 8.7.7. *Consider $f(z)=1-\cosh(z)$ once again $f(0)=1-1=0$ and $f'(0)=\sinh(0)=0$ whereas $f''(0)=-\cosh(0)=-1\neq 0$ hence $f(z)$ has a double zero at $z=0$. The power series for hyperbolic cosine is $\cosh(z)=1+z^2/2+z^4/4!+\cdots$ and thus*

$$f(z)=\frac{1}{2}z^2+\frac{1}{4!}z^4+\cdots=z^2\left(\frac{1}{2}+\frac{1}{4!}z^2+\cdots\right)$$

Definition 8.7.8. *Let f be an analytic function on an exterior domain $|z|>R$ for some $R>0$. If f is not identically zero for $|z|>R$ then we say f has a zero of order N at ∞ if $g(w)=f(1/w)$ has a zero of order N at $w=0$.*

Theorem 8.7.9 translates to the following result for Laurent series[10]:

Theorem 8.7.9. *If $f(z)$ is an analytic function with a zero of order N at ∞ then*

$$f(z)=\frac{a_N}{(z-z_o)^N}+\frac{a_{N+1}}{(z-z_o)^{N+1}}+\frac{a_{N+2}}{(z-z_o)^{N+2}}+\cdots.$$

Example 8.7.10. *Let $f(z)=\frac{1}{1+z^3}$ has*

$$g(w)=\frac{1}{1+1/w^3}=\frac{w^3}{w^3+1}=w^3-w^6+w^9+\cdots$$

hence $g(w)$ has a triple zero at $w=0$ which means $f(z)$ has a triple zero at ∞. We could also have seen this simply by expressing f as a function of $1/z$:

$$f(z)=\frac{1}{1+z^3}=\frac{1/z^3}{1+1/z^3}=\frac{1}{z^3}-\frac{1}{z^6}+\frac{1}{z^9}+\cdots.$$

Example 8.7.11. *Consider $f(z)=e^z$ notice $g(w)=f(1/w)=e^{1/w}=1+\frac{1}{w}+\frac{1}{2}\frac{1}{w^2}+\cdots$ is not analytic at $w=0$ hence we cannot even hope to ask if there is a zero at ∞ for $f(z)$ or what its order is.*

Following Gamelin, we include this nice example.

[10] I will get around to properly defining this term in the next chaper

Example 8.7.12. *Let* $f(z) = \frac{1}{(z-z_o)^n}$ *then observe*

$$\begin{aligned} f(z) &= \frac{1}{z^n - nz^{n-1}z_o + \cdots - nzz_o^{n-1} + z_o^n} \\ &= \frac{1}{z^n}\left(\frac{1}{1 - \frac{nz^{n-1}z_o + \cdots + nzz_o^{n-1} - z_o^n}{z^n}}\right) \\ &= \frac{1}{z^n}\left(\frac{1}{1 - \frac{nz_o}{z} + \cdots + \frac{nz_o^{n-1}}{z^{n-1}} - \frac{z_o^n}{z^n}}\right) \\ &= \frac{1}{z^n}\left(1 + \frac{nz_o}{z} + \cdots - \frac{nz_o^{n-1}}{z^{n-1}} + \frac{z_o^n}{z^n} + \cdots\right). \end{aligned}$$

This serves to show $f(z)$ *has* $z = \infty$ *as a zero of order* n.

Statements as above may be understood literally on the extended complex plane $\mathbb{C}\cup\{\infty\}$ or simply as a shorthand for facts about exterior domains in $\mathbb{C}$.

If you survey the examples we have covered so far in this section you might have noticed that when $f(z)$ is analytic at z_o then $f(z)$ has a zero at z_o iff the zero has finite order. If we were to discuss a zero of infinite order then intuitively that would produce the zero function since all the coefficients in the Taylor series would vanish. Intuition is not always the best guide on such matters, therefore, let us establish the result carefully:

Theorem 8.7.13. *If* D *is a domain and* f *is an analytic function on* D*, which is not identically zero, then the zeros of* f *are isolated points in* D*.*

Proof: let $U = \{z \in D \mid f^{(m)}(z) = 0 \text{ for all } m \geq 0\}$. Suppose $z_o \in U$ then $f^{(k)}(z_o)/k! = 0$ for all $k \geq 0$ hence $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k = 0$. Thus, $f(z)$ vanishes on an open disk $D(z_o)$ centered at z_o and it follows $f^{(k)}(z) = 0$ for each $z \in D(z_o)$ and $k \geq 0$. Thus $D(z_o) \subseteq U$. Hence z_o is an interior point of U, but, as z_o was arbitrary, it follows U is open.

Next, consider $V = D - U$ and let $z_o \in V$. There must exist $n \geq 0$ such that $f^{(n)}(z_o) \neq 0$ thus $a_n \neq 0$ and consequently $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k \neq 0$. It follows there is a disk $D(z_o)$ centered at z_o on which $f(z) \neq 0$ for each $z \in D(z_o)$. Thus $D(z_o) \subseteq V$ and this shows V is an open set.

Consider then, $D = U \cup (D - U)$ hence as D is connected we can only have $U = \emptyset$ or $U = D$. If $U = D$ then we find $f(z) = 0$ for all $z \in D$ and that is not possible by the preconditions of the theorem. Therefore $U = \emptyset$. In simple terms, we have shown that every zero of an non-indentically-vanishing analytic function must have finite order.

To complete the argument, we must show the zeros are isolated. Notice that z_o a zero of $f(z)$ has finite order N hence, by Theorem 8.7.9, $f(z) = (z-z_o)^n h(z)$ where h is analytic at z_o with $h(z_o) \neq 0$. Therefore, there exists $\rho > 0$ for which the series for $h(z)$ centered at z_o represents $h(z)$ for each $|z - z_o| < \rho$. Moreover, observe $h(z) \neq 0$ for all $|z - z_o| < \rho$. Consider $|f(z)| = |z - z_o|^N |h(z)|$, this cannot be zero except at the point $z = z_o$ hence there is no other zero for $f(z)$ on $|z - z_o| < \rho$ hence z_o is isolated. $\square$.

The theorem above has interesting consequences.

Theorem 8.7.14. *If f and g are analytic on a domain D, and if $f(z) = g(z)$ for each z belonging to a set with a nonisolated point, then $f(z) = g(z)$ for all $z \in D$.*

Proof: let $C = \{z \in D \mid f(z) = g(z)\}$ and suppose the **coincidence** set C has a nonisolated point. Consider $h(z) = f(z) - g(z)$ for $z \in D$. If $h(z)$ is not identically zero on D then the existence of C contradicts Theorem 8.7.13 since C by its definition is a set with non-isolated zeros for $h(z)$. Consequently, $h(z) = f(z) - g(z) = 0$ for all $z \in D$. □

Gamelin points out that if we apply the theorem above twice we obtain:

Theorem 8.7.15. *Let D be a domain, and let E be a subset of D that has a nonisolated point. Let $F(z, w)$ be a function defined for each $z, w \in D$ which is analytic in z with w-fixed and likewise analytic in w when we fix z. If $F(z, w) = 0$ whenever $z, w \in E$, then $F(z, w) = 0$ for all $z, w \in D$.*

Early in this course I made some transitional definitions which you might argue are somewhat adhoc. For example, we defined e^z, $\sin z$, $\sinh z$, $\cos z$ and $\cosh z$ all by simply extending their real formulas in the natural manner in view of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. The pair of theorems above show us an astonishing fact about complex analysis: there is just one way to define it as a natural extension of real calculus. Once Euler found his formula for real θ, there was only one complex extension which could be found.

Example 8.7.16. *Let $f(z) = e^z$. Let $g(z)$ be another entire function. Suppose $f(z) = g(z)$ for all $z \in \mathbb{R}$. Then, as $\mathbb{R}$ has a nonisolated point we find $f(z) = g(z)$ for all $z \in \mathbb{C}$. In other words, there is only one entire function on $\mathbb{C}$ which restricts to the real exponential on $\mathbb{R} \subset \mathbb{C}$.*

The same argument may be repeated for $\sin z$, $\sinh z$, $\cos z$ and $\cosh z$. Each of these functions is the unique entire extension of the corresponding function on $\mathbb{R}$. So, in complex analysis, we fix an analytic function on a domain if we know its values on some set with a nonisolated point. For example, the values of an analytic function on a domain are uniquely prescribed if we are given the values on a line-segment, open or closed disk, or even a sequence with a *cluster-point* in the domain. For further insight and some history on the topic of the identity theorem you can read pages 227-232 of [R91].

You might constrast this situation to that of linear algebra; if we are given the **finite** set of values to which a given **basis** in the domain must map then there is a **unique** linear transformation which is the extension from the finite set to the infinite set of points which forms the vector space. On the other side, a smooth function on an interval of $\mathbb{R}$ may be extended smoothly in infinitely many ways. Thus, the structure of complex analysis is stronger than that of real analysis and weaker than that of linear algebra.

One last thought, I have discussed extensions of functions to entire functions on $\mathbb{C}$. However, there may not exist an entire function to which we may extend. For example, $\ln(x)$ for $x \in (0, \infty)$ does not permit an extension to an entire function. Worse yet, we know this extends most naturally to $\log(z)$ which is a *multiply-valued* function. Remmert explains that 18-th century mathematicians wrestled with this issue. The temptation to assume by the *principle of permanence* there was a unique extension for the natural log led to considerable confusion. Euler wrote this in 1749 (page 159 [R91])

> We see therefore that is is essential to the nature of logarithms that each number have an infinity of logarithms and that all these logarithms be different, not only from one another, but also[different] from all the logarithms of every other number.

Ok, to be fair, this is a translation.

8.8 Analytic Continuation

Suppose we have a function $f(z)$ which is holomorphic on a domain D. If we consider $z_o \in D$ then there exist a_k for $k \geq 0$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ for all $z \in D(z_o) \subseteq D$. However, if we define $g(z)$ by the power series for $f(z)$ at z_o then the natural domain of $g(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ is the disk of convergence $D_R(z_o)$ where generally $D(z_o) \subseteq D_R(z_o)$. The function g is an **analytic continuation** of f.

Example 8.8.1. *Consider $f(z) = e^z$ for $z \in A = \{z \in \mathbb{C} \mid 1/2 < |z| < 2\}$. If we note $f(z) = e^{z-1+1} = ee^{z-1} = \sum_{k=0}^{\infty} \frac{e}{k!}(z-1)^k$ for all $z \in A$. However, $D_R(1) = \mathbb{C}$ thus the function defined by the series is an analytic continuation of the exponential from the given annulus to the entire plane.*

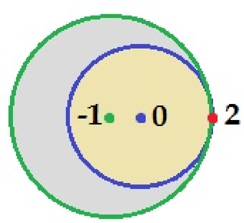
Analytic continuation is most interesting when there are singular points to work around. We can also begin with a function defined by a power series as in the next example.

Example 8.8.2. *Let $f(z) = \displaystyle\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$ for $|z| < 2$. Notice that $f(z) = \frac{1}{1-z/2} = \frac{2}{2-z}$ and we can expand the function as a power series centered at $z = -1$,*

$$f(z) = \frac{2}{2-(z+1-1)} = \frac{2}{3-(z+1)} = \frac{2}{3}\cdot\frac{1}{1-(z+1)/3} = \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{3^k}(z+1)^k.$$

for each z with $|z+1|/3 < 1$ or $|z+1| < 3$. In this case, the power series centered at $z = -1$ extends past $|z| < 2$. If we define $g(z) = \displaystyle\sum_{k=0}^{\infty}\frac{2}{3^{k+1}}(z+1)^k$ then $R = 3$ and the natural domain is $|z+1| < 3$.

The example above is easy to understand in the picture below:



Recentering the given series moves the center further from the singularity of the underlying function $z \mapsto \frac{2}{2-z}$ for $z \neq 2$. We know what will happen if we move the center somewhere else, the new radius of convergence will simply be the distance from the new center to $z = 2$.

In Gamelin §V.8 problem 2 you will see that the analytic continuation of a given holomorphic function need not match the function. It is possible to continue from one branch of a multiply-valued function to another branch. This is also shown on page 160 of Gamelin where he continues the principal branch of the squareroot mapping to the negative branch.

If we study the analytic continuation of a function defined by a series the main question which we face is the nature of the function on the boundary of the disk of convergence. There must be at least one point of divergence. See our Corollary 8.3.4 or look at page 234 of [R91] for a careful argument. Given $f(z) = \sum a_k(z - z_o)^k$ with disk $D_R(z_o)$ of convergence, a point $z_1 \in \partial D_R(z_o)$ is a **singular point** of f if there does not exist a holomorphic function $g(z)$ on $D_s(z_1)$ for which $f(z) = g(z)$ for all $z \in D_R(z_o) \cap D_s(z_1)$. The set of all singular points for f is called the **natural boundary of** f and the disk $D_R(z_o)$ is called the **region of holomorphy for** f. On page 150 of $[R91]$ the following example is offered:

Example 8.8.3. *Set $g(z) = z + z^2 + z^4 + z^8 + \cdots$. The radius of convergence is found to be $R = 1$. Furthermore, we can argue that $g(z) \to \infty$ as z approaches any even root of unity. Remmert shows on the page before that the even (or odd) roots of unity are* **dense** *on the unit circle hence the function $g(z)$ is unbounded at each point on $|z| = 1$ and it follows that the unit-circle is the natural boundary of this series.*

Certainly, many other things can happen on the boundary.

Example 8.8.4. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k = z - \frac{z}{2} + \frac{z}{3} + \cdots$ *converges for each z with $|z| = 1$* **except** *the single singular point $z = -1$.*

Remmert informs that Lusin in 1911 found a series with coefficients $c_k \to 0$ yet $\sum c_k z^k$ diverges at each $|z| = 1$. Then Sierpinski in 1912 produced a series which diverges at every point on the unit-circle **except** $z = 1$. See pages 120-121 [R91] for further details.

In summary, the problem of analytic continuation is subtle. When given a series presentation of an analytic function it may not be immediately obvious where the natural boundary of the given function resides. On the other hand, when the given function is captured by an algebraic expression or a formula in terms of sine, cosine etc. then through essentially precalculus-type domain considerations we can find see the natural boundary arise from the nature of the formula. Any series which represents the function will face the same natural boundaries. Well, I have tried not to overstate anything here, I hope I was successful. The full appreciation of analytic continuation is far beyond this course. For an attack similar to what I have done in examples here, see this MSE question. For a still bigger picture, see Wikipedia article on analytic continuation where it is mentioned that trouble with analytic continuation for functions of several complex variables prompted the invention of sheaf cohomology.

Let me collect a few main points from Gamelin. If D is a disk and f is analytic on D and $R(z_1)$ is the radius of convergence of the power series at $z_1 \in D$ and $R(z_2)$ is the radius of convergence of the power series at $z_2 \in D$, then $|R(z_1) - R(z_2)| \leq |z_1 - z_2|$. This inequality shows the radius of convergence is a continuous function on the domain of an analytic function.

Definition 8.8.5. *We say that f is* **analytically continuable along** *γ if for each t there is a convergent power series*

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n, \qquad |z - \gamma(t)| < r(t),$$

such that $f_a(z)$ is the power series representing $f(z)$ at z_o, and such that when s is near t, then $f_s(z) = f_t(z)$ for all z in the intersection of the disks of convergence for $f_s(z)$ and $f_t(z)$.

It turns out that when we analytically continue a given function from one initial point to a final point it could be the continuations do not match. However, there is a simple condition which assures the continuations do coincide. The idea here is quite like our deformation theorem for closed forms.

Theorem 8.8.6. Monodromy Theorem: *Let $f(z)$ be analytic at z_o. Let $\gamma_o(t)$ and $\gamma_1(t)$ for $a \leq t \leq b$ be paths from z_o to z_1 along which $f(z)$ can be continued analytically. Suppose γ_o can be continuously deformed to γ_1 along paths γ_s which begin at z_o and end at z_1 and allow $f(z)$ to be continued analytically. Then the analytic continuations of $f(z)$ along γ_o and γ_1 coincide at z_1.*

If there is a singularity, that is a point near the domain where the function cannot be analytically extended, then the curves of continuation might not be able to be smoothly deformed. The deformation could get snagged on a singularity. Of course, there is more to learn from Gamelin on this point. I will not attempt to add to his treatment further here.

Chapter IX

Laurent Series and Isolated Singularities

Laurent was a French engineer who lived from 1813 to 1854. He extended Cauchy's work on disks to annuli by introducing reciprocal terms centered about the center of the annulus. His original work was not published. However, Cauchy was aware of the result and has this to say about Laurent's work in his report to the French Academy of 1843:

> the extension given by M. Laurent $\cdots$ seems to us worthy of note

In this chapter we extend Cauchy's theorems on power series for analytic functions. In particular, we study how we can reproduce any analytic function on an annulus by simply adjoining reciprocal powers to the power series. A series built, in general, from both positive and negative power functions centered about some point z_o is called a Laurent series centered at z_o. The annulus we consider can reduce to a deleted disk or extend to ∞. Most of these results are fairly clean extensions of what we have done in previous chapters. Excitingly, we shall see the generalized Cauchy integral formula naturally extends. The extended theorem naturally ties coefficients of a given Laurent series to integrals around a circle in the annulus of convergence. That simple connection lays the foundation for the residue calculus of the next chapter. In terms of explicit calculation, we continue to use the same techniques as in our previous work. However, the domain of consideration is markedly different. We must keep in mind our study is about some annulus.

Laurent's proof of the Laurent series development can be found in a publication which his widow published in his honor in 1863. Apparently both Cauchy and Weierstrauss also has similar results in terms of mean values around 1840-1841. As Remmert explains (page 350-355 [R91]), all known proofs of the Laurent decomposition involve integration. Well, apparently, Pringsheim wrote a 1223 page work which avoided integration and instead did everything in terms of mean values. So, we should say, no efficient proof without integrals is known. Also of note, Laurent's Theorem can be derived from the Cauchy-Taylor theorem by direct calculational attack; this difficult proof due to Scheffer in 1884 (which also implicitly uses integral theory) is reproduced on p. 352-355 of [R91].

We could have made the definition some time ago, but, I give it here since I found myself using the term at various points in my exposition of this chapter.

Definition 9.0.1. *If $f \in \mathcal{O}(z_o)$ then there exists some $r > 0$ such that f is holomorphic on $|z - z_o| < r$. In other words, $\mathcal{O}(z_o)$ is the set of holomorphic functions at z_o.*

9.1 The Laurent Decomposition

If a function f is analytic on an annulus then the function can be written as the sum of two analytic functions f_o, f_1 on the annulus. Where, f_o is analytic from the outer circle of the annulus to the center and f_1 is analytic from the inner circle of the annulus to ∞.

Theorem 9.1.1. Laurent Decomposition: *Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Then $f(z)$ can be decomposed as a sum*

$$f(z) = f_o(z) + f_1(z),$$

where f_o is analytic for $|z - z_o| < \sigma$ and f_1 is analytic for $|z - z_o| > \rho$ and at ∞. If we normalize the decomposition such that $f_1(\infty) = 0$ then the decomposition is unique.

Let us examine a few examples and then we will offer a proof of the general assertion.

Example 9.1.2. *Let $f(z) = \dfrac{z^3 + z + 1}{z} = z^2 + 1 + \dfrac{1}{z}$ for $z \neq 0$. In this example $\rho = 0$ and $\sigma = \infty$ and $f_o(z) = z^2 + 1$ whereas $f_1(z) = 1/z$.*

Example 9.1.3. *Let $f(z)$ be an entire function. For example, $e^z, \sin z, \sinh z, \cos z$ or $\cosh z$. Then $f(z) = f_o(z)$ and $f_1(z) = 0$. The function f_o is analytic on any disk, but, we do not assume it is analytic at ∞. On the other hand, notice that $f_1 = 0$ is analytic at ∞ as claimed.*

Example 9.1.4. *Suppose $f(z)$ is analytic at $z_o = \infty$ then there exists some exterior domain $|z - z_o| > \rho$ for which $f(z)$ is analytic. In this case, $f(z) = f_1(z)$ and $f_o(z) = 0$ for all $z \in \mathbb{C} \cup \{\infty\}$.*

Proof: Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Furthermore, suppose $f(z) = f_o(z) + f_1(z)$ where f_o is analyic for $|z - z_o| < \sigma$ and f_1 is analytic for $|z - z_o| > \rho$ and at ∞. Suppose g_o, g_1 form another Laurent decomposition with $f(z) = g_o(z) + g_1(z)$. Notice,

$$g_o(z) - f_o(z) = g_1(z) - f_1(z)$$

for $\rho < |z - z_o| < \sigma$. In view of the above overlap condition we are free to define:

$$h(z) = \begin{cases} g_o(z) - f_o(z), & \text{for } |z - z_o| < \sigma \\ g_1(z) - f_1(z), & \text{for } |z - z_o| > \rho \end{cases}$$

Notice h is entire and $h(z) \to 0$ as $z \to \infty$. Thus h is bounded and entire and we apply Liouville's Theorem to conclude $h(z) = c$ for all $z \in \mathbb{C}$. In particular, $h(z) = 0$ on the annulus $\rho < |z - z_o| < \sigma$ and we conclude that if a Laurent decomposition exists then it must be unique.

The existence of the Laurent Decomposition is due to Cauchy's Integral formula on an annulus. Technically, we have not shown this result explicitly[1], to derive it we simply need to use the cross-cut idea which is illustrated in the discussion preceding Theorem **??**. Once more, suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Consider some subannulus $\rho < r < |z - z_o| < s < \sigma$. Cauchy's Integral formula gives

$$f(z) = \underbrace{\frac{1}{2\pi i} \oint_{|w - z_o| = s} \frac{f(w)}{w - z} dw}_{f_o(z)} - \underbrace{\frac{1}{2\pi i} \oint_{|w - z_o| = r} \frac{f(w)}{w - z} dw}_{-f_1(z)}.$$

[1] see pages 344-346 of [R91] for careful proofs of these results

Notice f_o is analytic for $|z - z_o| < s$ and f_1 is analytic for $|z - z_o| > r$ and $f_1(z) \to 0$ as $z \to \infty$. As Gameline points out here, our current formulation would seem to depend on r, s but we already showed the decomposition is unique if it exists thus f_o and f_1 must be defined for $\rho < |z - z_o| < \sigma$. $\square$

If you wish to read a different formulation of essentially the same proof, I recommend page 347 of [R91].

Example 9.1.5. *Consider* $f(z) = \frac{2z-i}{z(z-i)}$. *This function is analytic on* $\mathbb{C} - \{0, i\}$. *A simple calculation reveals:*

$$f(z) = \frac{1}{z} + \frac{1}{z-i}$$

With respect to the annulus $0 < |z| < 1$ *we have* $f_o(z) = \frac{1}{z-i}$ *and* $f_1(z) = \frac{1}{z}$. *On the other hand, for the annulus* $0 < |z - i| < 1$ *we have* $f_1(z) = \frac{1}{z-i}$ *and* $f_0(z) = \frac{1}{z}$. *If we study disks centered at any point in* $\mathbb{C} - \{0, i\}$ *then* $f_o(z) = f(z)$ *and* $f_1(z) = 0$.

We sometimes call the set such as $0 < |z - i| < 1$ an *annulus*, but, we would do equally well to call it a *punctured disk* centered at $i = 1$.

Example 9.1.6. *Consider* $f(z) = \frac{1}{\sin z}$ *this has a Laurent decomposition on the annuli which fit between the successive zeros of* $\sin z$. *That is, on* $n\pi < |z| < (n + 1)\pi$. *For example, when* $n = 0$ *we have* $\sin z = z - \frac{1}{6}z^3 + \cdots$ *hence, using our geometric series reciprocal technique,*

$$f(z) = \frac{1}{\sin z} = \frac{1}{z - \frac{1}{6}z^3 + \cdots} = \frac{1}{z(1 - \frac{1}{6}z^2 + \cdots)} = \frac{1}{z}\left(1 + (z^2/6 + \cdots)^2 + \cdots\right) = \frac{1}{z} + \frac{1}{36}z^3 + \cdots$$

Hence $f_1(z) = 1/z$ *whereas* $f_o(z) = z^3/36 + \cdots$ *for the punctured disk of raduis* π *centered about* $z = 0$.

Suppose $f(z) = f_o(z) + f_1(z)$ is the Laurent decomposition on $\rho < |z - z_o| < \sigma$. By Theorem 8.4.2 there exists a power series representation of f_o

$$f_o(z) = \sum_{k=0}^{\infty} a_k (z - z_o)^k$$

for $|z - z_o| < \sigma$. Next, by Theorem 8.5.7, noting that $a_o = f_1(\infty) = 0$ gives

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z - z_o)^k$$

for $|z - z_o| > \rho$. Notice both the series for f_o and f_1 converge normally and summing both together gives:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_o)^k$$

which is normally convergen on $\rho < |z - z_o| < \sigma$. In this context, normally convergent means we have uniform convergence for each $s \leq |z - z_o| \leq t$ where $\rho < s < t < \sigma$.

Given a function $f(z)$ defined by a Laurent series centered at z_o:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_o)^k \qquad \star$$

for $\rho < |z - z_o| < \sigma$. We naturally wish to characterize the meaning of the coefficients[2] a_k. This is accomplished by integration. In particular, we begin by integration over the circle $|z - z_o| = r$ where $\rho < r < \sigma$:

$$\begin{aligned}
\int_{|z-z_o|=r} f(z)\, dz &= \int_{|z-z_o|=r} \left(\sum_{k=-\infty}^{\infty} a_k (z - z_o)^k \right) dz \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_{|z-z_o|=r} (z - z_o)^k dz \right) \\
&= \sum_{k=-\infty}^{\infty} a_k \left(2\pi i \delta_{k,-1} \right) \\
&= 2\pi i a_{-1}
\end{aligned}$$

We have used the uniform convergence of the given series which allows term-by-term integration. In addition, the integration was before discussed in Example 7.1.9. In summary, we find the $k = -1$ coefficient has a rather beautiful significance:

$$\boxed{a_{-1} = \frac{1}{2\pi i} \int_{|z-z_o|=r} f(z)\, dz}$$

where the circle of integration can be taken as any circle in the annulus of convergence for the Laurent series. What does this formula mean?

We can integrate by finding a Laurent expansion of the integrand!

Example 9.1.7. *Let* $f(z) = \frac{\sin z}{1-z}$. *Observe,*

$$\frac{\sin z}{1 - z} = \frac{\sin(z - 1 + 1)}{1 - z} = \frac{\cos(1)\sin(z - 1) + \sin(1)\cos(z - 1)}{z - 1} = \frac{\sin 1}{z - 1} + \cos(1) - \frac{\sin 1}{2}(z - 1) + \cdots$$

thus $a_{-1} = \sin 1$ *and we find:*

$$\int_{|z-1|=2} \frac{\sin z}{1 - z}\, dz = 2\pi i \sin 1.$$

We now continue our derivation of the values for the coefficients in $\star$, we divide by $(z - z_o)^{n+1}$ and once more integrate over the circle $|z - z_o| = r$ where $\rho < r < \sigma$:

$$\begin{aligned}
\int_{|z-z_o|=r} \frac{f(z)}{(z - z_o)^{n+1}}\, dz &= \int_{|z-z_o|=r} \left(\sum_{k=-\infty}^{\infty} a_k (z - z_o)^{k-n-1} \right) dz \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_{|z-z_o|=r} (z - z_o)^{k-n-1} dz \right) \\
&= \sum_{k=-\infty}^{\infty} a_k \left(2\pi i \delta_{k-n-1,-1} \right) \\
&= 2\pi i a_n
\end{aligned}$$

[2] We already know for power series on a disk the coefficients are tied to the derivatives of the function at the center of the expansion. However, in the case of the Laurent expansion we only have knowledge about the function on the annulus centered at z_o and z_o may not even be in the domain of the function.

Once again, we have used the uniform convergence of the given series which allows term-by-term integration and the integral identity shown in Example 7.1.9. Notice the Kronecker delta $\delta_{k-n-1,-1} = \begin{cases} 1 & \text{if } k-n-1=-1 \\ 0 & \text{if } k-n-1 \neq -1 \end{cases}$ which means the only nonzero term occurs when $k-n-1=-1$ which is simply $k=n$. Of course, the integral is familar to us. We saw this identity for $k \geq 0$ in our previous study of power series. In particular, Theorem 7.3.2 where we proved the generalized Cauchy integral formula: adapted to our current notation

$$\frac{1}{2\pi i}\int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz = \frac{f^{(n)}(z_o)}{n!}.$$

For the Laurent series we study on $\rho < |z-z_o| < \sigma$ we cannot in general calculate $f^{(n)}(z_o)$. However, in the case $\rho = 0$, we have $f(z)$ analytic on the disk $|z-z_o| < \sigma$ and then we are able to either calculate, for $n \geq 0$ a_n by differentiation or integration. Let us collect our results for future reference:

Theorem 9.1.8. Laurent Series Decomposition: *Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z-z_o| < \sigma$. Then $f(z)$ can be decomposed as a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_o)^n$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i}\int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz$$

for $r > 0$ with $\rho < r < \sigma$.

Notice the deformation theorem goes to show there is no hidden dependence on r in the formulation of the coefficient a_n. The function f is assumed holomorphic between the inner and outer circles of the annulus of convergence hence $\frac{f(z)}{(z-z_o)^{n+1}}$ is holomorpic on the annulus as well and the complex integral is unchanged as we alter the value of r on (ρ, σ).

9.2 Isolated Singularities of an Analytic Function

A singularity of a function is some point which is nearly in the domain, and yet, is not. An isolated singularity is a singular point which is also isolated. A careful definition is given below:

Definition 9.2.1. *A function f has an* **isolated singularity at** *z_o if there exists $r > 0$ such that f is analytic on the punctured disk $0 < |z-z_o| < r$.*

We describe in this section how isolated singularity fall into three classes where each class has a particular type of Laurent series about the singular point. Let me define these now and we will explain the terms as the section continues. Notice Theorem 9.1.8 implies $f(z)$ has a Laurent series in a punctured disk about singularity hence the definition below covers all possible isolated singularities.

Definition 9.2.2. *Suppose f has an isolated singularity at z_o.*

(i.) *If $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ then z_o is a* **removable singularity**.

(ii.) *Let $N \in \mathbb{N}$. If $f(z) = \sum_{k=-N}^{\infty} a_k(z-z_o)^k$ with $a_{-N} \neq 0$ then z_o is a* **pole of order** N.

(iii.) *If $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_o)^k$ where $a_k \neq 0$ for infinitely many $k < 0$ then z_o is an* **essential singularity**.

We begin by studying the case of removable singularity. This is essentially the generalization of a hole in the graph you studied a few years ago.

Theorem 9.2.3. Riemann's Theorem on Removable Singularities: *let z_o be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_o then $f(z)$ has a removable singularity.*

Proof: expand $f(z)$ in a Laurent series about the punctured disk at z_o:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_o)^n$$

for $0 < |z-z_o| < \sigma$. If $|f(z)| < M$ for $0 < |z-z_o| < r$ then for $r < min(\sigma, r)$ we may apply the ML-theorem to the formula for the n-th coefficient of the Laurent series as given by Theorem 9.1.8

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\, dz \right| \leq \frac{M(2\pi r)}{2\pi r^{n+1}} = \frac{M}{r^n}.$$

As $r \to 0$ we find $|a_n| \to 0$ for $n < 0$. Thus $a_n = 0$ for all $n = -1, -2, \dots$. Thus, the Laurent series for $f(z)$ reduces to a power series for $f(z)$ on the deleted disk $0 < |z-z_o| < \sigma$ and it follows we may extend $f(z)$ to the disk $|z-z_o| < \sigma$ by simply defining $f(z_o) = a_o$. $\square$

Example 9.2.4. *Let $f(z) = \frac{\sin z}{z}$ on the punctured plane $\mathbb{C}^\times$. Notice,*

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j} = 1 - \frac{1}{3!}z^2 + \cdots.$$

We can extend f to $\mathbb{C}$ by defining $f(0) = 1$.

To be a bit more pedantic, $\tilde{f}$ is the extension of f defined by $\tilde{f}(z) = f(z)$ for $z \neq 0$ and $\tilde{f}(0) = 1$. The point ? The extension $\tilde{f}$ is a new function which is distinct from f.

We now study poles of order N. Let us begin by making a definition:

Definition 9.2.5. *Suppose f has a pole of order N at z_o. If*

$$f(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o} + \sum_{k=0}^{\infty} a_k(z-z_o)^k$$

then $P(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o}$ is the **principal part of $f(z)$ about** z_o. *When $N = 1$ then z_o is called a* **simple pole**, *when $N = 2$ then z_o is called a* **double pole**.

Notice $f(z) - P(z)$ is analytic.

Theorem 9.2.6. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $f(z) = g(z)/(z - z_o)^N$ where g is analytic at z_o with $g(z_o) \neq 0$.*

Proof: suppose f has a pole of order N at z_o then by definition it has a Laurent series which begins at $n = -N$. We calculate, for $|z - z_o| < r$,

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_o)^k = \frac{1}{(z - z_o)^N} \sum_{k=-N}^{\infty} a_k (z - z_o)^{k+N} = \frac{1}{(z - z_o)^N} \sum_{j=0}^{\infty} a_{j-N} (z - z_o)^j.$$

Define $g(z) = \sum_{j=0}^{\infty} a_{j-N}(z - z_o)^j$ and note that g is analytic at z_o with $g(z_o) = a_{-N} \neq 0$. We know $a_{-N} \neq 0$ by the definition of a pole of order N. Thus $f(z) = g(z)/(z - z_o)^N$ as claimed.

Conversely, suppose there exists g analytic at z_o with $g(z_o) \neq 0$ and $f(z) = g(z)/(z - z_o)^N$. There exist $b_o, b_1, \dots$ with $g(z_o) = b_o \neq 0$ such that

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_o)^k$$

divide by $(z - z_o)^N$ to obtain:

$$f(z) = \frac{1}{(z - z_o)^N} \sum_{k=0}^{\infty} b_k (z - z_o)^k = \sum_{k=0}^{\infty} b_k (z - z_o)^{k-N} = \sum_{j=-N}^{\infty} b_{j+N} (z - z_o)^j$$

identify that the coefficient of the Laurent series at order $-N$ is precisely $b_o \neq 0$ and thus we have shown f has a pole of order N at z_o. $\square$

Example 9.2.7. *Consider $f(z) = \dfrac{e^z}{(z-1)^5}$. Notice e^z is analytic on $\mathbb{C}$ hence by Theorem 9.2.6 the function f has a pole of order $N = 5$ at $z_o = 1$.*

Example 9.2.8. *Consider $f(z) = \dfrac{\sin(z+2)^5}{(z+2)^2}$ notice*

$$f(z) = \frac{1}{(z+2)^5} \left((z+2)^3 - \frac{1}{3!}(z+2)^9 + \frac{1}{5!}(z+2)^{15} + \cdots \right) =$$

simplifying yields

$$f(z) = \frac{1}{(z+2)^2} \underbrace{\left(1 - \frac{1}{3!}(z+2)^6 + \frac{1}{5!}(z+2)^{12} + \cdots \right)}_{g(z)}$$

which shows, by Theorem 9.2.6, the function f has a pole of order $N = 2$ at $z_o = -2$.

Theorem 9.2.9. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $1/f$ is analytic at z_o with a zero of order N.*

Proof: we know f has pole of order N iff $f(z) = g(z)/(z-z_o)^N$ with $g(z_o) \neq 0$ and $g \in \mathcal{O}(z_o)$. Suppose f has a pole of order N then observe

$$\frac{1}{f(z)} = (z-z_o)^N \cdot \frac{1}{g(z)}.$$

hence $1/f(z)$ has a zero of order N by Theorem 8.7.9. Conversely, if $1/f(z)$ has a zero of order N then by Theorem 8.7.9 we have $\frac{1}{f(z)} = (z-z_o)^N h(z)$ where $h \in \mathcal{O}(z_o)$ and $h(z_o) \neq 0$. Define $g(z) = 1/h(z)$ and note $g \in \mathcal{O}(z_o)$ and $g(z_o) = 1/h(z_o) \neq 0$ moreover,

$$\frac{1}{f(z)} = (z-z_o)^N h(z) \quad \Rightarrow \quad f(z) = \frac{1}{(z-z_o)^N h(z)} = \frac{g(z)}{(z-z_o)^N}$$

and we conclude by Theorem 9.2.6 that f has a pole of order N at z_o. $\square$

The theorem above can be quite useful for quick calculation.

Example 9.2.10. *$f(z) = 1/\sin z$ has a simple pole at $z_o = n\pi$ for $n \in \mathbb{N} \cup \{0\}$ since*

$$\sin(z) = \sin(z - n\pi + n\pi) = \cos(n\pi)\sin(z-n\pi) = (-1)^n(z-n)\pi - \frac{(-1)^n}{3!}(z-n)^3 + \cdots$$

shows $\sin z$ has a simple zero at $z_o = n\pi$ for $n \in \mathbb{N} \cup \{0\}$.

Example 9.2.11. *You should be sure to study the example given by Gamelin on page 173 to 174 where he derives the Laurent expansion which converges on $|z| = 4$ for $f(z) = 1/\sin z$.*

Example 9.2.12. *Let $f(z) = \dfrac{1}{z^3(z-2-3i)^6}$ then f has a pole of order $N = 3$ at $z_o = 0$ and a pole of order $N = 6$ at $z_1 = 2+3i$*

Definition 9.2.13. *We say a function f is* **meromorphic on a domain** *D if f is analytic on D except possibly at isolated singularities of which each is a pole.*

Example 9.2.14. *An entire function is meromorphic on $\mathbb{C}$. However, an entire function may not be analytic at ∞. For example, $\sin z$ is not analytic at ∞ and it has an essential singularity at ∞ so $f(z) = \sin z$ is not meromorphic on $\mathbb{C} \cup \{\infty\}$.*

Example 9.2.15. *A rational function is formed by the quotient of two polynomials $p(z), q(z) \in \mathbb{C}[z]$ where $q(z)$ is not identically zero; $f(z) = p(z)/q(z)$. We will explain in Example 9.3.3 that $f(z)$ is meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$.*

Theorem 9.2.16. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order $N \geq 1$ iff $|f(z)| \to \infty$ as $z \to z_o$.*

Proof: if z_o is a pole of order N then $f(z) = g(z)/(z-z_o)^N$ for $g(z_o) \neq 0$ for $0 < |z-z_o| < r$ for some $r > 0$ where g is analytic at z_o. Since g is analytic at z_o it is continuous and hence bounded on the disk; $|g(z)| \leq M$ for $|z-z_o| < r$. Thus,

$$|f(z)| = |g(z)(z-z_o)^{-N}| \leq M(z-z_o)^{-N} \to \infty$$

as $z \to z_o$. Thus $|f(z)| \to \infty$ as $z \to z_o$.

Conversely, suppose $|f(z)| \to \infty$ as $z \to z_o$. Hence, there exists $r > 0$ such that $f(z) \neq 0$ for $0 < |z - z_o| < r$. It follows that $h(z) = 1/f(z)$ is analytic in for $0 < |z - z_o| < r$. Note that $|f(z)| \to \infty$ as $z \to z_o$ implies $h(z) \to 0$ as $z \to z_o$. Thus $h(z)$ is bounded near z_o and we find by Riemann's removable singularity Theorem 9.2.3 there exist a_n for $n = 0, 1, 2, \dots$ such that:

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

However, $h(z) \to 0$ as $z \to z_o$ hence the extension of $h(z)$ is zero at z_o. If the zero has order N then $h(z) = (z - z_o)^N b(z)$ where $b \in \mathcal{O}(z_o)$ and $b(z_o) \neq 0$. Therefore, we obtain $f(z) = g(z)/(z - z_o)^N$ where $g(z) = 1/b(z)$ where $g \in \mathcal{O}(z_o)$ and $g(z_o) \neq 0$. We conclude z_o is a pole of order N by Theorem 9.2.6.

Example 9.2.17. *Let $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$. Clearly $z_o = 0$ is an essential singularity of f. It has different behaviour than a removable singularity or a pole. First, notice for $z = x > 0$ we have $f(z) = e^{1/x} \to \infty$ as $x \to 0^+$ thus f is not bounded at $z_o = 0$. On the other hand, if we study $z = iy$ for $y > 0$ then $|f(z)| = |e^{\frac{1}{iy}}| = 1$ hence $|f(z)|$ does not tend to ∞ along the imaginary axis.*

Theorem 9.2.18. Casorati-Weierstrauss Theorem: *Let z_o be an essential isolated singularity of $f(z)$. Then for every complex number w_o, there is a sequence $z_n \to z_o$ such that $f(z_n) \to w_o$ as $n \to \infty$.*

Proof: by contrapositive argument. Suppose there exists a complex number w_o for which there does not exist a sequence $z_n \to z_o$ such that $f(z_n) \to w_o$ as $n \to \infty$. It follows there exists $\epsilon > 0$ for which $|f(z) - w_o| > \epsilon$ for all z in a small punctured disk about z_o. Thus, $h(z) = 1/(f(z) - w_o)$ is bounded close to z_o. Consequently, z_o is a removable singularity of $h(z)$ and $h(z) = (z - z_o)^N g(z)$ for some $N \geq 0$ and some analytic function g such that $g(z_o) \neq 0$. But, this gives:

$$\frac{1}{f(z) - w_o} = (z - z_o)^N g(z) \quad \Rightarrow \quad f(z) = w_o + \frac{b(z)}{(z - z_o)^N}$$

where $b = 1/g \in \mathcal{O}(z_o)$ and $b(z_o) \neq 0$. If $N = 0$ then f extends to be analytic at z_o. If $N > 0$ then f has a pole of order N at z_o. In all cases we have a contradiction to the given fact that z_o is an essential singularity. The theorem follows. $\square$

Gamelin mentions **Picard's Theorem** which states that for an essential singularity at z_o, for all w_o except possibly one value, there is a sequence $z_n \to z_o$ for which $f(z_n) = w_o$ for each n. In our example $e^{1/z}$ the exceptional value is $w_o = 0$.

9.3 Isolated Singularity at Infinity

As usual, we use the reciprocal function to transfer the definition from zero to infinity.

Definition 9.3.1. *We say f has an isolated singular point at ∞ if there exists $r > 0$ such that f is analytic on $|z| > r$. Equivalently, we say f has an isolated singular point at ∞ if $g(w) = f(1/w)$ has an isolated singularity at $w = 0$. Furthermore, we say that the isolated singular point at ∞ is removable singularity, a pole of order N or an essential singularity if the corresponding singularity* *at $w = 0$ is likewise a removable singularity, pole of order N or an essential singular point of g. In particular, if ∞ is a pole of order N then the Laurent series expansion:*

$$f(z) = b_N z^N + \cdots + b_1 z + b_o + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots$$

has **principal part**

$$P_\infty(z) = b_N z^N + \cdots + b_1 z + b_o$$

hence $f(z) - P_\infty(z)$ is analytic at ∞.

This section is mostly a definition. I now give a few illustrative examples, partly following Gamelin.

Example 9.3.2. *The function $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$ has an essential singularity at ∞. This implies that while e^z is meromorphic on $\mathbb{C}$, it is* **not** *meromorphic on $\mathbb{C} \cup \{\infty\}$ as it has a singularity which is not a pole or removable.*

Example 9.3.3. *Let $p(z), q(z) \in \mathbb{C}[z]$ with deg$(p(z)) = m$ and deg$(q(z)) = n$ such that $m > n$. Notice that long-division gives $d(z), r(z) \in \mathbb{C}[z]$ for which deg$(d(z)) = m - n$ and deg$(r(z)) < m$ such that*

$$f(z) = \frac{p(z)}{q(z)} = d(z) + \frac{r(z)}{q(z)}$$

The function $\frac{r(z)}{q(z)}$ is analytic at ∞ and $d(z)$ serves as the principal part. We identify f has a pole of order $m - n$ at ∞. It follows that any rational function is **meromorphic** *on the extended complex plane $\mathbb{C} \cup \{\infty\}$*

Example 9.3.4. *Following the last example, suppose $m = n$ then $d(z) = 0$ and the singularity at ∞ is seen to be removable. If $p(z) = a_m z^m + \cdots + a_o$ and $q(z) = b_n z^n + \cdots + b_o$ then we can extend f analytically at ∞ by defining $f(\infty) = a_m/b_n$.*

Example 9.3.5. *Consider $f(z) = (e^{1/z} - 1)/z$ for $z > 0$. Observe*

$$f(z) = (e^{1/z} - 1)/z = \left(\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right)$$

hence the singularity at ∞ is removable and we may extend f to be analytic on the extended complex plane by defining $f(\infty) = 0$.

9.4 Partial Fractions Decomposition

In the last section we noticed in Example 9.3.3 that rational functions were meromorphic on the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Furthermore, it is interesting to notice the algebra of meromorphic functions is very nice: sums, products, quotients where the denominator is not identically zero, all of these are once more meromorphic. In terms of abstract algebra, the set of meromorphic functions on a domain forms a subalgebra of the algebra of holomorphic functions on D. See pages 315-320 of [R91] for a discussion which focuses on the algebraic aspects of meromorphic functions.

It turns out that not only are the rational functions meromorphic on $\mathbb{C}^*$, in fact, they are the only meromorphic functions on $\mathbb{C}^*$.

Theorem 9.4.1. *A meromorphic function on $\mathbb{C}^*$ is a rational function.*

Proof: let $f(z)$ be a meromorphic function on $\mathbb{C}^*$. The number of poles of f must be finite otherwise they would acculumate to give a singularity which was not isolated. If f is analytic at ∞ then we define $P_\infty(z) = f(\infty)$. Otherwise, f has a pole of order N and $P_\infty(z)$ is a polynomial of order N. In both cases, $f(z) - P_\infty(z)$ is analytic at ∞ with $f(z) - P_\infty(z) \to 0$ as $z \to \infty$. Let us label the poles in $\mathbb{C}$ as $z_1, z_2, \dots, z_m$. Furthermore, let $P_k(z)$ be the principal part of $f(z)$ at z_k for $k = 1, 2, \dots, m$. Notice, there exist $\alpha_1, \dots, \alpha_{n_k}$ such that

$$P_k(z) = \frac{\alpha_1}{z - z_k} + \frac{\alpha_2}{(z - z_k)^2} + \cdots + \frac{\alpha_{n_k}}{(z - z_k)^{n_k}}$$

for each k. Notice $P_k(z) \to 0$ as $z \to \infty$ and P_k is analytic at ∞. We define (still following Gamelin)

$$g(z) = f(z) - P_\infty(z) - \sum_{k=1}^{m} P_k(z).$$

Notice g is analytic at each of the poles including ∞. Thus g is an entire function and as $g(z) \to 0$ as $z \to \infty$ it follows g is bounded and by Liouville's Theorem we find $g(z) = 0$ for all $z \in \mathbb{C}$. Therefore,

$$\boxed{f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z).}$$

This completes the proof as we already argued the converse direction in Example 9.3.3. $\square$

The boxed formula is the **partial fractions decomposition of** f. In fact, we have shown:

Theorem 9.4.2. *Every rational function has a partial fractions decomposition: in particular, if $z_1, \dots, z_m$ are the poles of f then*

$$f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z)$$

where $P_\infty(z)$ is a polynomial and $P_k(z)$ is the principal part of $f(z)$ around the pole z_k.

The method to obtain the partial fractions decomposition of a given rational function is described algorithmically on pages 180-181. Essentially, the first thing to do is to we can use long-division to discover the principal part at ∞. Once that is done, factor the denominator to discover the poles of $f(z)$ and then we can simply write out a generic form of $\sum_{k=1}^{m} P_k(z)$. Then, we determine the unknown coefficients implicit within the generic form by algebra. I will illustrate with a few examples:

Example 9.4.3. *Let $f(z) = \dfrac{z^3 + z + 1}{z^2 + 1}$. Notice that $z^3 + z + 1 = z(z^2 + 1) + 1$ hence $f(z) = z + \dfrac{1}{z^2 + 1}$. We now focus on $\dfrac{1}{z^2 + 1}$ notice $z^2 + 1 = (z - i)(z + i)$ hence each pole is simple and we seek complex constants A, B such that:*

$$\frac{1}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i}.$$

Multiply by $z^2 + 1$ to obtain:

$$1 = A(z - i) + B(z + i)$$

Next, evaluate at $z = -i$ and $z = i$ to obtain $1 = -2iA$ and $1 = 2iB$ hence $A = -1/2i$ and $B = 1/2i$ and we conclude:

$$f(z) = z - \frac{1}{2i}\frac{1}{z+i} + \frac{1}{2i}\frac{1}{z-i}.$$

Example 9.4.4. *Let $f(z) = \dfrac{2z+1}{z^2-3z-4}$ notice $z^2 - 3z - 4 = (z-4)(z+1)$ hence*

$$\frac{2z+1}{z^2-3z-4} = \frac{A}{z-4} + \frac{B}{z+1} \quad \Rightarrow \quad 2z+1 = A(z+1) + B(z-4)$$

Evaluate at $z = -1$ and $z = 4$ to obtain:

$$-1 = -5B \;\;\&\;\; 9 = 5A \quad \Rightarrow \quad A = 9/5, \quad B = 1/5.$$

Thus,

$$f(z) = \frac{1}{5}\left(\frac{5}{z-4} + \frac{1}{z+1}\right)$$

Example 9.4.5. *Suppose $f(z) = \dfrac{1+z}{z^4-3z^3+3z^2-z}$. Long division is not needed as this is already a proper rational function. Notice*

$$z^4 - 3z^3 + 3z^2 - z = z(z^3 - 3z^2 + 3z - 1) = z(z-1)^3.$$

Thus we seek: complex constants A, B, C, D for which

$$\frac{1+z}{z^4-3z^3+3z^2-z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

Multiplying by the denominator yields,

$$1 + z = A(z-1)^3 + Bz(z-1)^2 + Cz(z-1) + Dz, \quad \star$$

which is nice to write as

$$1 + z = A(z^3 - 3z^2 + 3z - 1) + B(z^3 - 2z^2 + z) + C(z^2 - z) + Dz$$

for what follows. Differentiating gives

$$1 = A(3z^2 - 6z + 3) + B(3z^2 - 4z + 1) + C(2z - 1) + D, \qquad \frac{d\star}{dz}$$

differentiating once more yields

$$0 = A(6z - 6) + B(6z - 4) + C(2), \qquad \frac{d^2\star}{dz^2}$$

differentiating for the third time:

$$0 = 6A + 6B$$

Thus $A = -B$. Set $z = 1$ in $\star$ to obtain $2 = D$. Once again, set $z = 1$ in $\frac{d\star}{dz}$ to obtain $1 = C(2-1)+2$ hence $C = -1$. Finally, set $z = 1$ in $\frac{d^2\star}{dz^2}$ to obtain $0 = 2B - 2$ thus $B = 1$ and we find $A = -1$ as a consequence. In summary:

$$\frac{1+z}{z^4-3z^3+3z^2-z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}.$$

Perhaps you did not see the technique I used in the example above in your previous work with partial fractions. It is a nice addition to the usual algebraic technique.

Example 9.4.6. On how partial fractions helps us find Laurent Series *in the last example we found:*

$$f(z) = \frac{1+z}{z^4 - 3z^3 + 3z^2 - z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}.$$

If we want the explicit Laurent series about $z = 1$ we simply need to expand the analytic function $-1/z$ as a power series:

$$\frac{-1}{z} = \frac{-1}{1 + (z-1)} = \sum_{n=0}^{\infty} (-1)^{n+1}(z-1)^n$$

thus for $0 < |z - 1| < 1$

$$f(z) = \frac{2}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^{n+1}(z-1)^n.$$

This is the Laurent series of f about $z_o = 1$. The other singular point is $z_1 = 0$. To find the Laurent series about z_1 we need to expand $\frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$ as a power series about $z_1 = 0$. To begin,

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Let $g(z) = -\frac{1}{(z-1)^2}$ and notice $\int g(z)dz = C + \frac{1}{z-1} = C - \sum_{n=0}^{\infty} z^n$ thus

$$g(z) = \frac{d}{dz}\left[\int g(z)dz\right] = \frac{d}{dz}\left[C - \sum_{n=0}^{\infty} z^n\right] = -\sum_{n=1}^{\infty} nz^{n-1} = -\sum_{j=0}^{\infty} (j+1)z^j.$$

Let $h(z) = 2/(z-1)^3$ notice $\int h(z)dz = -1/(z-1)^2$ and $\int(\int h(z)dz)dz = 1/(z-1) = -\sum_{n=0}^{\infty} z^n$. I have ignored the constants of integration (why is this ok?). Observe,

$$h(z) = \frac{d}{dz}\frac{d}{dz}\left[\int\left(\int h(z)dz\right)dz\right] = \frac{d}{dz}\frac{d}{dz}\left[-\sum_{n=0}^{\infty} z^n\right] = \frac{d}{dz}\left[-\sum_{n=1}^{\infty} nz^{n-1}\right]$$

$$= -\sum_{n=2}^{\infty} n(n-1)z^{n-2}$$

$$= -\sum_{j=0}^{\infty} (j+2)(j+1)z^j.$$

Thus, noting $f(z) = -1/z + 1/(z-1) + g(z) + h(z)$ we collect our calculations above to obtain:

$$f(z) = \frac{-1}{z} - \sum_{j=0}^{\infty} \left(1 + (j+1) + (j+2)(j+1)\right) z^j = \frac{-1}{z} - \sum_{j=0}^{\infty} \left(j^2 + 4j + 4\right) z^j.$$

Neat, $j^2 + 4j + 4 = (j+2)^2$ hence:

$$f(z) = \frac{-1}{z} - \sum_{j=0}^{\infty} (j+2)^2 z^j = \frac{-1}{z} + 4 + 9z + 16z^2 + 25z^3 + 36z^4 + \cdots.$$

Term-by-term integration and differentiation allowed us to use geometric series to expand the basic rational functions which appear in the partial fractal decomposition. I hope you see the method I used in the example above allows us a technique to go from a given partial fractal decomposition to the Laurent series about any point we wish. Of course, singular points are most fun.

Chapter X

The Residue Calculus

In this chapter we collect the essential tools of the residue calculus. Then, we solve a variety of real integrals by relating the integral of interest to the residue of a complex function. The method we present here is not general. Much like second semester calculus, we show some typical examples and hold out hope the reader can generalize to similar examples. These examples date back to the early nineteenth or late eighteenth centuries. Laplace, Poisson and ,of course, Cauchy were able to use complex analysis to solve a myriad of real integrals. That said, according to Remmert [R91] page 395:

> Nevertheless there is no cannonical method of finding, for a given integrand and interval of integration, the best path γ in $\mathbb{C}$ to use.

And if that isn't sobering enough, from Ahlfors:

> even complete mastery does not guarantee success

Ahlfors was a master so this comment is perhaps troubling. Generally, complex integration is an art. For example, if you peruse the answers of Ron Gordon on the *Math Stackexchange Website* you'll see some truly difficult problems solved by one such artist.

Some of the examples solved in this chapter are also solved by techinques of real second semester calculus. I include such examples to illustrate the complex technique with minimal difficulty.

Keep in mind I have additional examples posted in NotesWithE100toE117. I will lecture some from those examples and some from these notes.

10.1 The Residue Theorem

In Theorem 9.1.8 we learned that a function with an isolated singularity has a Laurent expansion: in particular, if $0 \leq \rho < \sigma \leq \infty$, and $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Then $f(z)$ can be decomposed as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)}{(z - z_o)^{n+1}}\, dz$$

for $r > 0$ with $\rho < r < \sigma$. The $n = -1$ coefficient has special significance when we focus on the expansion in a deleted disk about z_o.

Definition 10.1.1. *Suppose $f(z)$ has an isolated singularity z_o and Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

for $0 < |z - z_o| < \rho$ then we define the **residue of f at z_o** *by*

$$Res\,[f(z), z_o] = a_{-1}.$$

Notice, the $n = -1$ coefficient is only the residue when we consider the deleted disk around the singularity. Furthermore, by Theorem 9.1.8, for the Laurent series in the definition above we have

$$a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_o|=r} f(z)\, dz$$

where r is any fixed radius with $0 < r < \rho$.

Example 10.1.2. *Suppose $n \neq 1$,*

$$Res\left[\frac{1}{z - z_o}, z_o\right] = 1 \qquad \& \qquad Res\left[\frac{1}{(z - z_o)^n}, z_o\right] = 0.$$

Example 10.1.3. *In Example 9.4.3 we found*

$$f(z) = \frac{z^3 + z + 1}{z^2 + 1} = z - \frac{1}{2i}\frac{1}{z + i} + \frac{1}{2i}\frac{1}{z - i}.$$

From this partial fractions decomposition we are free to read that

$$Res\,[f(z), i] = \frac{1}{2i} \qquad \& \qquad Res\,[f(z), -i] = \frac{-1}{2i}.$$

Do you understand why there is no hidden $1/(z - i)$ term in $f(z) - \frac{1}{2i}\frac{1}{z-i}$? If you don't then you ought to read §$VI.4$ again.

Example 10.1.4. *In Example 9.4.4 we derived:*

$$f(z) = \frac{2z+1}{z^2-3z-4} = \frac{1}{5}\left(\frac{5}{z-4} + \frac{1}{z+1}\right)$$

From the above we can read:

$$Res\,[f(z), 4] = 1 \qquad \& \qquad Res\,[f(z), -1] = \frac{1}{5}.$$

Example 10.1.5. *In Example 9.4.5 we derived:*

$$f(z) = \frac{1+z}{z^4-3z^3+3z^2-z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

By inspection of the above partial fractal decomposition we find:

$$Res\,[f(z), 0] = -1 \qquad \& \qquad Res\,[f(z), 1] = 1.$$

Example 10.1.6. *Consider* $(\sin z)/z^6$ *observe*

$$\frac{1}{z^6}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right) = \frac{1}{z^5} - \frac{1}{6z^3} + \frac{1}{120z} + \cdots.$$

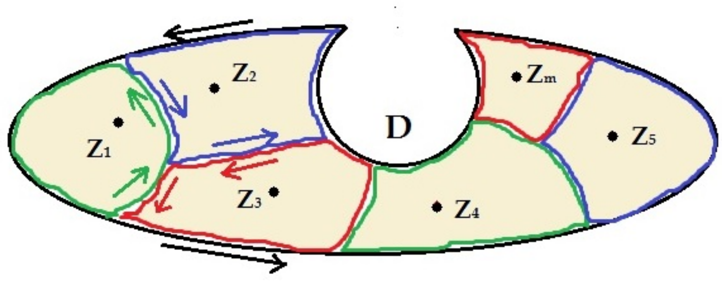
In view of the expansion above, we find:

$$Res\left[\frac{\sin z}{z^6}, 0\right] = \frac{1}{120}$$

Theorem 10.1.7. Cauchy's Residue Theorem: *let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D. Suppose that f is analytic on $D \cup \partial D$, except for a finite number of isolated singularities $z_1, \dots, z_m$ in D. Then*

$$\int_{\partial D} f(z)\, dz = 2\pi i \sum_{j=1}^{m} Res\,[f(z), z_j]\,.$$

Proof: this follows immediately from m-applications of Theorem 9.1.8. We simply parse D into m simply connected regions each of which contains just one singular point. The net-integration only gives the boundary as the cross-cuts cancel. The picture below easily generalizes for $m > 3$.



Of course, we could also just envision little circles around each singularity and apply the deformation theorem to reach the ∂D. $\square$

Our focus has shifted from finding the whole Laurent series to just finding the coefficient of the reciprocal term. In the remainder of this section we examine some useful *rules* to find residues.

Proposition 10.1.8. Rule 1: *if $f(z)$ has a simple pole at z_o, then*

$$Res\,[f(z), z_o] = \lim_{z \to z_o} (z - z_o) f(z).$$

Proof: since f has a simple pole at z_o we have:

$$f(z) = \frac{a_{-1}}{z - z_o} + g(z)$$

where $g \in \mathcal{O}(z_o)$. Hence,

$$\lim_{z \to z_o} [(z - z_o) f(z)] = \lim_{z \to z_o} [a_{-1} + (z - z_o) g(z)] = a_{-1}. \qquad \square$$

Example 10.1.9.

$$Res\left[\frac{z^3 + z + 1}{z^2 + 1}, i\right] = \lim_{z \to i} (z - i) \frac{z^3 + z + 1}{(z - i)(z + i)} = \lim_{z \to i} \frac{z^3 + z + 1}{z + i} = \frac{-i + i + 1}{i + i} = \frac{1}{2i}.$$

You can contrast the work above with that which was required in Example 10.2.2.

Example 10.1.10. *Following Example 10.1.4, let's see how Rule 1 helps:*

$$Res\left[\frac{2z + 1}{z^2 - 3z - 4}, -1\right] = \lim_{z \to -1} (z + 1) \frac{2z + 1}{(z + 1)(z - 4)} = \frac{2(-1) + 1}{-1 - 4} = \frac{1}{5}.$$

Proposition 10.1.11. Rule 2: *if $f(z)$ has a double pole at z_o, then*

$$Res\,[f(z), z_o] = \lim_{z \to z_o} \frac{d}{dz} \left[(z - z_o)^2 f(z)\right].$$

Proof: since f has a double pole at z_o we have:

$$f(z) = \frac{a_{-2}}{(z - z_o)^2} + \frac{a_{-1}}{z - z_o} + g(z)$$

where $g \in \mathcal{O}(z_o)$. Hence,

$$\begin{aligned}
\lim_{z \to z_o} \frac{d}{dz} \left[(z - z_o)^2 f(z)\right] &= \lim_{z \to z_o} \frac{d}{dz} \left[a_{-2} + (z - z_o) a_{-1} + (z - z_o)^2 g(z)\right] \\
&= \lim_{z \to z_o} \left[a_{-1} + 2(z - z_o) g(z) + (z - z_o)^2 g(z)\right] \\
&= a_{-1}. \qquad \square
\end{aligned}$$

Example 10.1.12.

$$Res\left[\frac{1}{(z^3 + 1) z^2}, 0\right] = \lim_{z \to 0} \frac{d}{dz} \left[\frac{z^2}{(z^3 + 1) z^2}\right] = \lim_{z \to 0} \left[\frac{-3z^2}{(z^3 + 1)^2}\right] = 0.$$

Let me generalize Gamelin's example from page 197. I replace i in Gamelin with a.

Example 10.1.13. *keep in mind $z^2 - a^2 = (z + a)(z - a)$,*

$$Res\left[\frac{1}{(z^2 - a^2)^2}, a\right] = \lim_{z \to a} \frac{d}{dz} \left[\frac{(z - a)^2}{(z^2 - a^2)^2}\right] = \lim_{z \to a} \left[\frac{1}{(z + a)^2}\right] = \frac{2}{(z + a)^3}\bigg|_{z = a} = \frac{-2}{8a^3}.$$

In the classic text of Churchill and Brown, the rule below falls under one of the p, q theorems. See §57 of [C96]. We use the notation of Gamelin here and resist the urge to *mind our p's and q's*.

Proposition 10.1.14. Rule 3: *If $f, g \in \mathcal{O}(z_o)$, and if g has a simple zero at z_o, then*

$$Res\left[\frac{f(z)}{g(z)}, z_o\right] = \frac{f(z_o)}{g'(z_o)}.$$

Proof: if f has a zero of order $N \geq 1$ then $f(z) = (z - z_o)^N h(z)$ and $g(z) = (z - z_o)k(z)$ where $h(z_o), k(z_o) \neq 0$ hence

$$\frac{f(z)}{g(z)} = \frac{(z - z_o)^N h(z)}{(z - z_o)k(z)} = (z - z_o)^{N-1}\frac{h(z)}{k(z)}$$

which shows $\lim_{z \to z_o} \frac{f(z)}{g(z)} = 0$ if $N > 1$ and for $N = 1$ we have $\lim_{z \to z_o} \frac{f(z)}{g(z)} = \frac{h(z_o)}{k(z_o)}$. In either case, for $N \geq 0$ we find $\frac{f(z)}{g(z)}$ has a removable singularity hence the residue is zero which is consistent with the formula of the proposition as $f(z_o) = 0$. Next, suppose $f(z_o) \neq 0$ then by Theorem 9.2.6 we have $f(z)/g(z)$ has a simple pole hence Rule 1 applies:

$$\text{Res}\left[f(z)/g(z), z_o\right] = \lim_{z \to z_o} (z - z_o)\frac{f(z)}{g(z)} = \frac{f(z_o)}{\lim_{z \to z_o}\left(\frac{g(z) - g(z_o)}{z - z_o}\right)} = \frac{f(z_o)}{g'(z_o)}.$$

where in the last step I used that $g(z_o) = 0$ and $g'(z_o), f(z_o) \in \mathbb{C}$ with $g'(z_o) \neq 0$ were given. $\square$

Example 10.1.15. *Observe $g(z) = \sin z$ has simple zero at $z_o = \pi$ since $g(\pi) = \sin \pi = 0$ and $g'(\pi) = \cos \pi = -1 \neq 0$. Rule 3 hence applies as $e^z \in \mathcal{O}(\pi)$,*

$$Res\left[\frac{e^z}{\sin z}, \pi\right] = \frac{e^\pi}{\cos \pi} = -e^\pi.$$

Example 10.1.16. *Notice $g(z) = (z - 3)e^z$ has a simple zero at $z_o = 3$. Thus, noting $\cos z \in \mathcal{O}(3)$ we apply Rule 3.*

$$Res\left[\frac{\cos z}{(z - 3)e^z}, 3\right] = \left.\frac{\cos(z)}{e^z + (z - 3)e^z}\right|_{z=3} = \frac{\cos(3)}{e^3}.$$

One more rule to go:

Proposition 10.1.17. Rule 4: *if $g(z)$ has a simple pole at z_o, then*

$$Res\left[\frac{1}{g(z)}, z_o\right] = \frac{1}{g'(z_o)}.$$

Proof: apply Rule 3 with $f(z) = 1$. $\square$

I'll follow Gamelin and offer this example which does clearly show why Rule 4 is so nice to know:

Example 10.1.18. *note that $g(z) = z^2 + 1$ has $g(i) = 0$ and $g'(i) = 2i \neq 0$ hence g has simple zero at $z_o = i$. Apply Rule 4,*

$$Res\left[\frac{1}{z^2 + 1}, i\right] = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i}.$$

10.2 Integrals Featuring Rational Functions

Let $R > 0$. Consider the curve ∂D which is formed by joining the line-segment $[-R, R]$ to the upper-half of the positively oriented circle $|z| = R$. Let us denote the half-circle by C_R hence $\partial D = [-R, R] \cup C_R$. Notice the domain D is a half-disk region of radius R with the diameter along the real axis. If $f(z)$ is a function which is analytic at all but a finite number of isolated singular points $z_1, \dots, z_k$ in D then Cauchy's Residue Theorem yields:

$$\int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j]$$

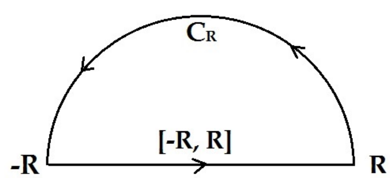
In particular, we find

$$\int_{[-R,R]} f(z)\, dz + \int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j]$$

But, $[-R, R]$ has $z = x$ hence $dz = dx$ and $f(z) = f(x)$ for $-R \leq x \leq R$ and

$$\boxed{\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j].}$$

The formula above connects integrals in the real domain to residues and the contour integral along a half-circle C_R. We can say something interesting in general for rational functions.



Suppose $f(z) = \frac{p(z)}{q(z)}$ where $\deg(q(z)) \geq \deg(p(z)) + 2$. Let $\deg(q(z)) = n$ and $\deg(p(z)) = m$ hence $n - m \geq 2$. Also, assume $q(x) \neq 0$ for all $x \in \mathbb{R}$ so that no[1] singular points fall on $[-R, R]$. In the homework[2], based on an argument from page 131 of [C96], I showed there exists $R > 0$ for which $q(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_o$ is bounded below $|a_n|R^n/2$ for $|z| > R$; that is $|q(z)| \geq \frac{|a_n|}{2} R^n$ for all $|z| > R$. On the other hand, it is easier to argue that $p(z) = b_m z^m + \cdots + b_1 z + b_o$ is bounded for $|z| > R$ by repeated application of the triangle inequality:

$$|p(z)| \leq |b_m z^m| + \cdots + |b_1 z| + |b_o| \leq |b_m|R^m + \cdots + |b_1|R + |b_o|.$$

Therefore, if $|z| > R$ as described above,

$$|f(z)| = \frac{|p(z)|}{|q(z)|} \leq \frac{|b_m|R^m + \cdots + |b_1|R + |b_o|}{\frac{|a_n|}{2} R^n} \leq \frac{M}{R^{n-m}}$$

[1] in §$VII.5$ we study fractional residues which allows us to treat singularities on the boundary in a natural manner, but, for now, they are forbidden

[2] there exists a year where this was Problem 44

where M is a constant which depends on the coefficients of $p(z)$ and $q(z)$. Applying the ML-estimate to C_R for $R > 0$ for which the bound applies we obtain:

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{M(2\pi R)}{R^{n-m}} = \frac{2M\pi}{R^{n-m-1}}$$

This bound applies for all R beyond some positive value hence we deduce:

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\, dz \right| \leq \lim_{R\to\infty} \frac{2M\pi}{R^{n-m-1}} = 0 \quad \Rightarrow \quad \lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

as $n-m \geq 2$ implies $n-m-1 \geq 1$. Therefore, the boxed formula provides a direct link between the so-called *principal value* of the real integral and the sum of the residues over the upper half-plane of $\mathbb{C}$:

$$\boxed{\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right].}$$

Sometimes, for explicit examples, it is expected that you show the details for the construction of M and that you retrace the steps of the general path I sketched above. If I have no interest in that detail then I will tell you to use the Proposition below:

Proposition 10.2.1. *If $f(z)$ is a rational function which has no real-singularities and for which the denominator is of degree at least two higher than the numerator then*

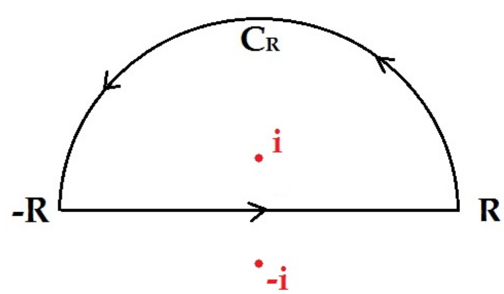
$$\boxed{\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{j=1}^{k} Res\left[f(z), z_j\right].}$$

where $z_1, \dots, z_k$ are singular points of $f(z)$ for which $\mathbf{Im}(z_j) > 0$ for $j = 1, \dots, k$.

Example 10.2.2. *We calculate $\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1}$ by noting the complex extension of the integrand $f(z) = \frac{1}{z^2+1}$ satisfies the conditions of Proposition 10.2.1. Thus,*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = 2\pi i Res\left[\frac{1}{z^2+1}, i\right] = \left.\frac{2\pi i}{2z}\right|_{z=i} = \frac{2\pi i}{2i} = \pi.$$

Thus[3] $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi.$



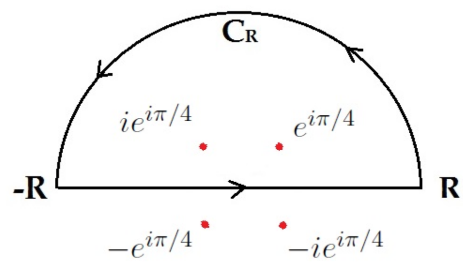
[3] so, technically, the double infinite double integral is defined by distinct parameters tending to ∞ and $-\infty$ independent of one another, however, for this integrand there is no difference between $\int_a^b \frac{dx}{x^2+1}$ with $a \to \infty$ and $b \to -\infty$ verses $a = -b = R$ tending to ∞. Gamelin starts to discuss this issue in $\S VII.6$

You can contrast the way I did the previous example with how Gamelin presents the work.

Example 10.2.3. *Consider* $f(z) = \frac{1}{z^4+1}$ *notice singularities of this function are the fourth roots of* -1*;* $z^4+1=0$ *implies* $z \in (-1)^{1/4} = \{e^{i\pi/4}, ie^{i\pi/4}, -e^{i\pi/4}, -ie^{i\pi/4}\}$*. Only two of these fall in the upper-half plane. Thus, by Proposition 10.2.1*

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^4+1} &= 2\pi i Res\left[\frac{1}{z^4+1}, e^{i\pi/4}\right] + 2\pi i Res\left[\frac{1}{z^4+1}, ie^{i\pi/4}\right].\\ &= \frac{2\pi i}{4z^3}\bigg|_{e^{i\pi/4}} + \frac{2\pi i}{4z^3}\bigg|_{ie^{i\pi/4}}\\ &= \frac{2\pi i}{4e^{i3\pi/4}} + \frac{2\pi i}{4i^3e^{3i\pi/4}}\\ &= \frac{\pi}{2e^{i3\pi/4}}\left[i + \frac{i}{i^3}\right] = \frac{-\pi}{2e^{i3\pi/4}}[1-i] = \frac{-\pi}{2e^{i3\pi/4}}\sqrt{2}e^{-i\pi/4} = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

where we noted $e^{-i\pi/4}/e^{i3\pi/4} = 1/e^{i\pi} = -1$ *to cancel the* -1*. It follows that:* $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$.



Wolfram Alpha reveals the antiderivative for the previous example can be directly calculated:

$$\int\frac{dx}{x^4+1} = (-\log(x^2-\sqrt{2}x+1)+\log(x^2+\sqrt{2}x+1)-2\tan^{-1}(1-\sqrt{2}x)+2\tan^{-1}(\sqrt{2}x+1))/(4\sqrt{2})+C.$$

Then to calculate the improper integral you just have to calculate the limit of the expression above at $\pm\infty$ and take the difference. That said, I think I prefer the method which is more *complex*.

The method used to justify Proposition 10.2.1 applies to non-rational examples as well. The key question is how to bound, or more generally capture, the integral along the half-circle as $R \to \infty$. Sometimes the direct complex extension of the real integral is not wise. For example, for $a > 0$, when faced with

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos(ax)dx$$

we would not want to use $f(z) = \frac{p(z)\cos(az)}{q(z)}$ since $\cos(aiy) = \cosh(ay)$ is unbounded. Instead, we would consider $f(z) = \frac{p(z)e^{iaz}}{q(z)}$ from which we obtain values for both $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos(ax)dx$ and $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin(ax)dx$. I will not attempt to derive an analog to Proposition 10.2.1. Instead, I consider the example presented by Gamelin.

Example 10.2.4. *Consider $f(z) = \frac{e^{iaz}}{z^2+1}$. Notice f has simple poles at $z = \pm i$, the picture of Example 10.2.2 applies here. By Rule 3,*

$$Res\left[\frac{e^{iaz}}{z^2+1}, i\right] = \frac{e^{iaz}}{2z}\bigg|_i = \frac{e^{-a}}{2i}.$$

Let D be the half disk with $\partial D = [-R,R] \cup C_R$ then by Cauchy's Residue Theorem

$$\int_{[-R,R]} \frac{e^{iaz}}{z^2+1}\, dz + \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = \frac{2\pi i e^{-a}}{2i} = \pi e^{-a} \quad \star\, .$$

For C_R we have $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$ hence for $z \in C_R$ with $R > 1$,

$$|f(z)| = \left|\frac{e^{iaz}}{z^2+1}\right| = \frac{1}{|z^2+1|} \leq \frac{1}{||z|^2-1|} = \frac{1}{R^2-1}$$

Thus, by ML-estimate,

$$\left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz\right| \leq \frac{2\pi R}{1-R^2} \quad \Rightarrow \quad \lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = 0.$$

Returning to $\star$ we find:

$$\lim_{R\to\infty} \int_{[-R,R]} \frac{e^{iax}}{x^2+1}\, dx = \pi e^{-a} \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\, dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\, dx = \pi e^{-a}.$$

The real and imaginary parts of the equation above reveal:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\, dx = \pi e^{-a} \qquad \& \qquad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\, dx = 0.$$

In §$VII.7$ we learn about Jordan's Lemma which provides an estimate which allows for integration of expressions such as $\frac{\sin x}{x}$.

10.3 Integrals of Trigonometric Functions

The idea of this section is fairly simple once you grasp it:

> Given an integral involving sine or cosine find a way to represent it as the formula for the contour integral around the unit-circle, or some appropriate curve, then use residue theory to calculate the complex integral hence calculating the given real integral.

Let us discuss the main algebraic identities to begin: if $z = e^{i\theta} = \cos\theta + i\sin\theta$ then $\bar{z} = e^{-i\theta} = \cos\theta - i\sin\theta$ hence $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. Of course, we've known these from earlier in the course. But, we also can see these as:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \& \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

moreover, $dz = ie^{i\theta}d\theta$ hence $d\theta = dz/iz$. It should be emphasized, the formulas above hold for the unit-circle.

Consider a complex-valued rational function $R(z)$ with singular points $z_1, z_2, \dots z_k$ for which $|z_j| \neq 0$ for all $j = 1, 2, \dots, k$. Then, by Cauchy's Residue Theorem

$$\int_{|z|=1} R(z)\, dz = 2\pi i \sum_{|z_j|<1} \text{Res}\,(R(z), z_j)$$

In particular, as $z = e^{i\theta}$ parametrizes $|z| = 1$ for $0 \leq \theta \leq 2\pi$,

$$\int_0^{2\pi} R(\cos\theta + i\sin\theta)\, ie^{i\theta} d\theta = 2\pi i \sum_{|z_j|<1} \text{Res}\,(R(z), z_j)$$

In examples, we often begin with $\int_0^{2\pi} R(\cos\theta + i\sin\theta)\, ie^{i\theta} d\theta$ and work our way back to $\int_{|z|=1} R(z)\, dz$.

Example 10.3.1.

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_{|z|=1} \frac{dz/iz}{5 - 4 \cdot \frac{i}{2}\left(z - \frac{1}{z}\right)} \\
&= \int_{|z|=1} \frac{1}{i} \cdot \frac{dz}{5z - 2i\,(z^2 - 1)} \\
&= \int_{|z|=1} \frac{dz}{2z^2 - 2 + 5iz}
\end{aligned}$$

Notice $2z^2 + 5iz - 2 = (2z + i)(z + 2i) = 2(z + i/2)(z + 2i)$ *is zero for* $z_o = -i/2$ *or* $z_1 = -2i$. *Only* z_o *falls inside* $|z| = 1$ *therefore, by Cauchy's Residue Theorem,*

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_{|z|=1} \frac{dz}{2z^2 + 5iz - 2} \\
&= 2\pi i\, Res\left[\frac{1}{2z^2 + 5iz - 2}, -i/2\right] \\
&= (2\pi i) \frac{1}{4z + 5i}\bigg|_{z=-i/2} \\
&= \frac{2\pi i}{-2i + 5i} \\
&= \frac{2\pi}{3}.
\end{aligned}$$

The example below is approximately borrowed from Remmert page 397 [R91].

Example 10.3.2. *Suppose* $p \in \mathbb{C}$ *with* $|p| \neq 1$. *We wish to calculate:*

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta.$$

Converting the integrand and measure to $|z| = 1$ *yields:*

$$\frac{1}{1 - p\left(z + \frac{1}{z}\right) + p^2} \frac{dz}{iz} = \left[\frac{1}{z - pz^2 - p + p^2 z}\right] \frac{dz}{i} = \left[\frac{1}{(z - p)(1 - pz)}\right] \frac{dz}{i}.$$

Hence, if $|p| < 1$ then $z = p$ is in $|z| \leq 1$ and it follows $1 - pz \neq 0$ for all points z on the unit-circle $|z| = 1$. Thus, we have only one singular point as we apply the Residue Theorem:

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \int_{|z|=1} \left[\frac{1}{(z-p)(1-pz)}\right] \frac{dz}{i} = 2\pi Res\left[\frac{1}{(z-p)(1-pz)}, p\right]$$

By Rule 1,

$$Res\left[\frac{1}{(z-p)(1-pz)}, p\right] = \lim_{z \to p}(z-p)\frac{1}{(z-p)(1-pz)} = \frac{1}{1-p^2}$$

and we conclude: if $|p| < 1$ then

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \frac{2\pi}{1-p^2}.$$

Suppose $|p| > 1$ then $z - p \neq 0$ for $|z| = 1$ and $1 - pz = 0$ for $z_o = 1/p$ for which $|z_o| = 1/|p| < 1$. Thus the Residue Theorem faces just one singularity within $|z| = 1$ for the $|p| > 1$ case:

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \int_{|z|=1} \left[\frac{1}{(z-p)(1-pz)}\right] \frac{dz}{i} = 2\pi Res\left[\frac{1}{(z-p)(1-pz)}, 1/p\right]$$

By Rule 1,

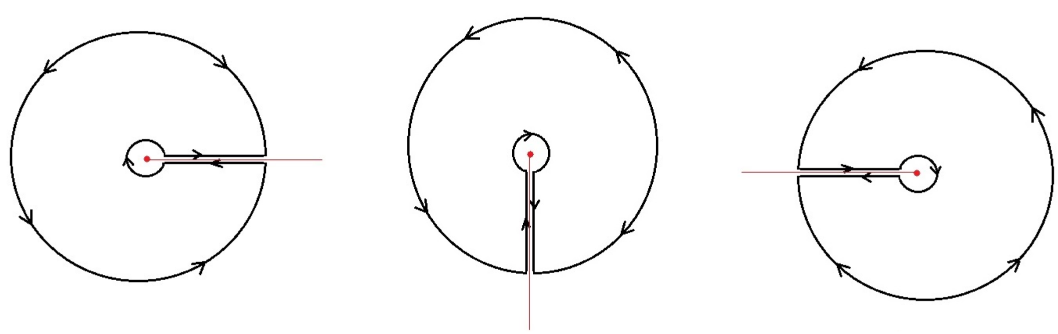
$$Res\left[\frac{1}{(z-p)(1-pz)}, 1/p\right] = \lim_{z \to 1/p}(z-1/p)\frac{1}{(z-p)(z-1/p)(-p)} = \frac{1}{(1/p-p)(-p)} = \frac{1}{p^2-1},$$

neat. Thus, we conclude, for $|p| > 1$,

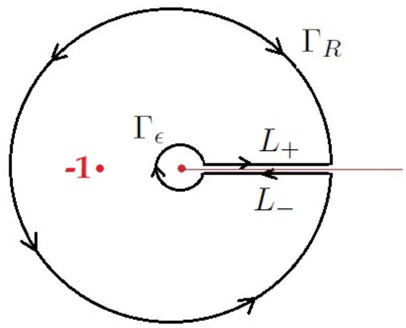
$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \frac{2\pi}{p^2-1}.$$

10.4 Integrands with Branch Points

Cauchy's Residue Theorem directly applies to functions with isolated singularities. If we wish to study functions with branch cuts then some additional ingenuity is required. In particular, the **keyhole contour** is often useful. For example, the following template could be used for branch cuts along the positive real, negative imaginary and negative real axis.



Example 10.4.1. *Consider* $\displaystyle\int_0^\infty \frac{x^a}{(1+x)^2}dx$ *where* $a \neq 0$ *and* $-1 < a < 1$. *To capture this integral we study* $f(z) = \frac{z^a}{(1+z)^2}$ *where* $z^a = |z|^a exp(aLog_0(z))$ *is the branch of* z^a *which has a jump-discontinuity along* $\theta = 0$ *which is also at* $\theta = 2\pi$. *Let* Γ_R *be the outside circle in the contour below. Let* Γ_ϵ *be the small circle encircling* $z = 0$. *Furthermore, let* $L_+ = [\epsilon + i\delta, R + i\delta]$ *and* $L_- = [R - i\delta, \epsilon - i\delta]$ *where* δ *is a small positive constant*[4] *for which* $\delta \to 0$ *and* $\epsilon \to 0$. *Notice, in the limits* $\epsilon \to 0$ *and* $R \to \infty$, *we have* $L_+ \to [0, \infty]$ *and* $L_- \to [\infty, 0]$



The singularity $z_o = -1$ *falls within the contour for* $R > 1$ *and* $\epsilon < 1$. *By Rule 2 for residues,*

$$Res\left(\frac{z^a}{(1+z)^2}, -1\right) = \lim_{z\to -1} \frac{d}{dz}\left[z^a\right] = \lim_{z\to -1}\left(az^{a-1}\right) = a(-1)^{a-1} = -a(e^{i\pi})^a = -ae^{i\pi a}.$$

Cauchy's Residue Theorem applied to the contour thus yields:

$$\int_{\Gamma_R} f(z)\,dz + \int_{L_-} f(z)\,dz + \int_{\Gamma_\epsilon} f(z)\,dz + \int_{L_+} f(z)\,dz = -2\pi i a e^{i\pi a}$$

If $|z| = R$ *then notice:*

$$\left|\frac{z^a}{(1+z)^2}\right| \leq \frac{R^a}{(R-1)^2}.$$

Also, if $|z| = \epsilon$ *then*

$$\left|\frac{z^a}{(1+z)^2}\right| \leq \frac{\epsilon^a}{(1-\epsilon)^2}.$$

In the limits $\epsilon \to 0$ *and* $R \to \infty$ *we find by the* ML*-estimate*

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \frac{R^a}{(R-1)^2}(2\pi R) = \frac{2\pi R^{a-1}}{(1-1/R)^2} \to 0$$

as $-1 < a < 1$ *implies* $a - 1 < 0$. *Likewise,as* $a + 1 > 0$ *we find:*

$$\left|\int_{\Gamma_\epsilon} f(z)\,dz\right| \leq \frac{\epsilon^a}{(1-\epsilon)^2}(2\pi\epsilon) = \frac{2\pi\epsilon^{a+1}}{(1-\epsilon)^2} \to 0.$$

We now turn to unravel the integrals along $L_\pm$. *For* $z \in L_+$ *we have* $Arg_0(z) = 0$ *whereas* $z \in L_-$ *we have* $Arg_0(z) = 2\pi$. *In the limit* $\epsilon \to 0$ *and* $R \to \infty$ *we have:*

$$\int_{L_+} \frac{z^a}{(1+z)^2}\,dz = \int_0^\infty \frac{x^a}{(1+x)^2}\,dx \qquad \& \qquad -\int_{L_-} \frac{z^a}{(1+z)^2}\,dz = \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,dx$$

[4] we choose δ as to connect $L_\pm$ and the inner and outer circles

where the phase factor on L_- arises from the definition of z^a by the $Arg_0(z)$ branch of the argument. Bringing it all together,

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx - e^{2\pi i a}\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = -2\pi i a e^{i\pi a}.$$

Solving for the integral of interest yields:

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = \frac{-2\pi i a e^{i\pi a}}{1-e^{2\pi i a}} = \frac{\pi a}{\frac{1}{2i}\left(e^{i\pi a}-e^{-i\pi a}\right)} = \frac{\pi a}{\sin(\pi a)}$$

At this point, Gamein remarks that the function $g(w) = \int_0^\infty \frac{x^w\,dx}{(1+x)^2}$ is analytic on the strip $-1 < \mathbf{Re}(w) < 1$ as is the function $\frac{\pi w}{\sin \pi w}$ thus by the identity princple we find the integral identity holds for $-1 < \mathbf{Re}(w) < 1$.

The following example appears as a homework problem on page 227 of [C96].

Example 10.4.2. *Show that* $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$

Let $f(z) = \dfrac{z^{-1/2}}{z^2+1}$ *where the root-function has a branch cut along $[0,\infty]$. We use the keyhole contour introduced in the previous example. Notice $z = \pm i$ are simple poles of $f(z)$. We consider $z^{-1/2} = |z|^{-1/2}exp\left(\frac{-1}{2}Log_0(z)\right)$. In other words, if $z = re^{-\theta}$ for $0 < \theta \leq 2\pi$ then $z^{-1/2} = \frac{1}{\sqrt{r}e^{i\theta/2}}$. Thus, for $z = x$ in L_+ we have $z^{-1/2} = 1/\sqrt{x}$. On the other hand for $z = x$ in L_- we have $z^{-1/2} = -1/\sqrt{x}$ as $e^{i(2\pi)/2} = e^{i\pi} = -1$. Notice, $z^2+1 = (z-i)(z+i)$ and apply Rule 3 to see*

$$Res\,(f(z), i) = \frac{i^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i} \qquad \& \qquad Res\,(f(z), -i) = \frac{(-i)^{-1/2}}{-2i} = \frac{e^{-3\pi i/4}}{-2i}$$

Consequently, assuming[5] *the integrals along Γ_R and Γ_ϵ vanish as $R \to \infty$ and $\epsilon \to 0$ we find:*

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} - \int_0^\infty \frac{dx}{-\sqrt{x}(x^2+1)} = 2\pi i\left(\frac{e^{-i\pi/4}}{2i} + \frac{e^{-3\pi i/4}}{-2i}\right)$$

Notice $-1 = e^{i\pi}$ and $e^{i\pi}e^{-3\pi i/4} = e^{\pi i/4}$ hence:

$$2\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = 2\pi\left(\frac{e^{-i\pi/4}}{2} + \frac{e^{\pi i/4}}{2}\right) = 2\pi\cos\pi/4 \quad \Rightarrow \quad \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

The key to success is care with the details of the branch cut. It is a critical detail. I should mention that E116 in the handwritten notes is worthy of study. I believe I have assigned a homework problem of a similar nature. There we consider a rectangular path of integration which tends to infinity and uncovers and interesting integral. There are also fascinating examples of wedge-shaped integrations and many other choices I currently have not included in this set of notes.

[5] I leave these details to the reader, but intuitively it is already clear the antiderivative is something like $\sqrt{x}$ at the origin and $1/\sqrt{x}$ for $x \to \infty$.

10.5 Fractional Residues

In general when a singularity falls on a proposed path of integration then there is no simple method of calculation. Generically, you would make a little indentation and then take the limit as the indentation squeezes down to the point. If that limiting process uniquely produces a value then that gives the integral along such a path. In the case of a simple pole there is a nice reformulation of Cauchy's Residue Theorem.

Theorem 10.5.1. *If z_o is a simple pole of f and C_ϵ is an arc of $|z-z_o|=\epsilon$ of angle α then*

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,dz = \alpha i Res\,(f(z),z_o)\,.$$

Proof: since f has a simple pole we have:

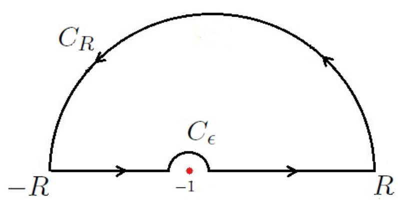
$$f(z)=\frac{A}{z-z_o}+g(z)$$

where, by the definition of residue, $A=\text{Res}\,(f(z),z_o)$. The arc $|z-z_o|=\epsilon$ of angle α is parametrized by $z=z_o+\epsilon e^{i\theta}$ for $\theta_o\le\theta\le\theta_o+\alpha$. As the arc is a bounded subset and g is analytic on the arc it follows there exists $M>0$ for which $|g(z)|<M$ for $|z-z_o|=\epsilon$. Furthermore, the integral of the singular part is calculated:

$$\int_{C_\epsilon}\frac{Adz}{z-z_o}=\int_{\theta_o}^{\theta_o+\alpha}\frac{Ai\epsilon e^{i\theta}d\theta}{\epsilon e^{i\theta}}=iA\int_{\theta_o}^{\theta_o+\alpha}d\theta=i\alpha A. \qquad \square$$

Of course this result is nicely consistent with the usual residue theorem if we consider $\alpha=2\pi$ and think about the deformation theorem shrinking a circular path to a point.

Example 10.5.2. *Let $\gamma=C_R\cup[-R,-1-\epsilon]\cup C_\epsilon\cup[-1+\epsilon,R]$. This is a half-circular path with an indentation around $z_o=-1$. Here we assume C_ϵ is a half-circle of radius ϵ above the real axis.*



The aperature is π hence the fractional residue theorem yields:

$$\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{dz}{(z+1)(z-i)}=-\pi i Res\left(\frac{1}{(z+1)(z-i)},-1\right)=-\pi i\left(\frac{1}{-1-i}\right)=\frac{\pi(1+i)}{2}$$

For $|z|=R>1$ notice $\left|\frac{1}{(z+1)(z-i)}\right|\le\left|\frac{1}{||z|-|1||\cdot||z|-|i||}\right|=\frac{1}{(R-1)^2}=M$. Thus, $|\int_{C_R}\frac{dz}{(z+1)(z-i)}|\le\frac{\pi R}{(R-1)^2}\to 0$ as $R\to\infty$. Cauchy's Residue Theorem applied to the region bounded by γ yields:

$$\int_\gamma\frac{dz}{(z+1)(z-i)}=2\pi i Res\left(\frac{1}{(z+1)(z-i)},-i\right)=\frac{2\pi i}{-i+1}=\pi(i-1)$$

Hence, in the limit $R\to\infty$ and $\epsilon\to 0$ we find:

$$P.V.\int_{-\infty}^{\infty}\frac{dx}{(x+1)(x-i)}+\frac{\pi(1+i)}{2}=\pi(i-1)$$

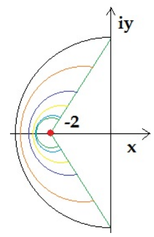
Therefore,

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x+1)(x-i)} = \frac{\pi}{2}(i-3).$$

The quantity above is called the principle value for two reasons: first: it approaches $x = \infty$ and $x = -\infty$ symmetrically, second: it approaches the improper point $x = -1$ from the left and right at the same rate. The integral (which is defined in terms of asymmetric limits) itself is divergent in this case. We define the term **principal value** *in the next section.*

Example 10.5.3. *You may recall: Let $\gamma(t) = 2\sqrt{3}e^{it}$ for $\pi/2 \leq t \leq 3\pi/2$. Calculate $\int_\gamma \frac{dz}{z+2}$. A wandering math ninja stumble across your path an mutters $\tan(\pi/3) = \sqrt{3}$.*

Residue Calculus Solution: *if you imagine deforming the given arc from $z = 2i\sqrt{3}$ to $z = -2i\sqrt{3}$ into curves which begin and end along the rays connecting $z = -2$ to $z = \pm 2i\sqrt{3}$ then eventually we reach tiny arcs C_ϵ centered about $z = -2$ each subtending $4\pi/3$ of arc.*



Now, there must be some reason that this deformation leaves the integral unchanged since the fractional residue theorem applied to the limiting case of the small circles yields:

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{dz}{z+2} = \frac{4\pi}{3} i Res\left(\frac{1}{z+2}, -2\right) = \frac{4\pi i}{3}.$$

Of course, direct calculation by the complex FTC yields the same:

$$\begin{aligned}
\int_\gamma \frac{dz}{z+2} &= Log_0(z+2)\Big|_{2i\sqrt{3}}^{-2i\sqrt{3}} \\
&= Log_0(-2i\sqrt{3}+2) - Log_0(2i\sqrt{3}+2) \\
&= Log_0(2(1-i\sqrt{3})) - Log_0(2(1+i\sqrt{3})) \\
&= \ln|2(1-i\sqrt{3}| + iArg_0(4\exp(5\pi i/3)) - \ln|2(1+i\sqrt{3}| + iArg_0(4\exp(\pi i/3)) \\
&= \frac{5\pi i}{3} - \frac{\pi i}{3} \\
&= \frac{4\pi i}{3}
\end{aligned}$$

It must be that the integral along the line-segments is either zero or cancels. Notice $z = -2 + t(2 \pm 2i\sqrt{3})$ for $\epsilon \leq t \leq 1$ parametrizes the rays $(-2, \pm 2i\sqrt{3}]$ in the limit $\epsilon \to 0$ and $dz = (2 \pm 2i\sqrt{3})dt$ thus

$$\int_{(-2,\pm 2i\sqrt{3}]} \frac{dz}{z+2} = \int_\epsilon^1 \frac{dt}{t} = \ln 1 - \ln\epsilon = -\ln\epsilon.$$

However, the direction of the rays differs to complete the path in a consistent CCW direction. We go from -2 to $2i\sqrt{3}$, but, the lower ray goes from $2i\sqrt{3}$ to -2. Apparently these infinities cancel (gulp). I think the idea of this example is a dangerous game.

I covered the example on page 210 of Gamelin in lecture. There we derive the identity:

$$\int_0^\infty \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{4}.$$

by examining a half-circular path with indentations about $z = 0$ and $z = -1$.

10.6 Principal Values

If $\int_{-\infty}^{\infty} f(x)\,dx$ diverges or $\int_a^b f(x)dx$ diverges due to a singularity for $f(x)$ at $c \in [a,b]$ then it may still be the case that the corresponding *principal values* exist. When the integrals converge absolutely then the principal value agrees with the integral. These have mathematical application as Gamelin describes briefly at the conclusion of the section.

Definition 10.6.1. *We define P.V.* $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$. *Likewise, if f is continuous on $[a,c)$ and $(c,b]$ then we define*

$$P.V. \int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+}\left(\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right)$$

In retrospect, this section is out of place. We would do better to introduce the concept of principal value towards the beginning. For example, in [C96] this is put forth at the outset. Thus I am inspired to present the following example stolen from [C96].

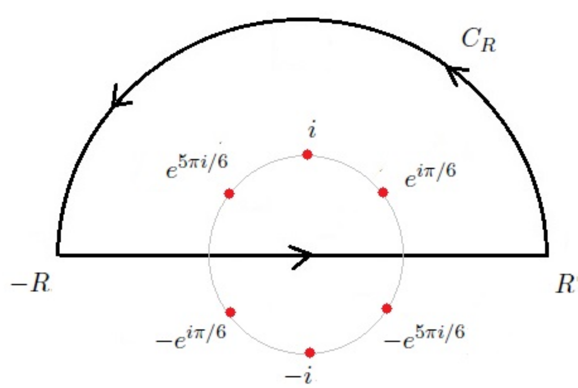
Example 10.6.2. *We wish to calculate* $\int_0^\infty \frac{x^2}{x^6+1}\,dx$. *The integral can be argued to exist by comparison with other convergent integrals and, as the integrand is non-negative, it converges absolutely. Thus we may find P.V.* $\int_0^\infty \frac{x^2}{x^6+1}\,dx$ *to calculate* $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$. *The integrand is even thus:*

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,dx = \frac{1}{2}P.V.\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,dx.$$

Observe $f(z) = \frac{z^2}{z^6+1}$ *has singularities at solutions of* $z^6+1=0$. *In particular,* $z \in (-1)^{1/6}$.

$$\begin{aligned}(-1)^{1/6} &= e^{i\pi/6}\{1, e^{2\pi i/6}, e^{4\pi i/6}, -1, -e^{2\pi i/6}, -e^{4\pi i/6}\}\\ &= \{e^{i\pi/6}, e^{3\pi i/6}, e^{5\pi i/6}, -e^{i\pi/6}, -e^{3\pi i/6}, -e^{5\pi i/6}\}\\ &= \{e^{i\pi/6}, i, e^{5\pi i/6}, -e^{i\pi/6}, -i, -e^{5\pi i/6}\}\end{aligned}$$

We use the half-circle path $\partial D = C_R \cup [-R,R]$ *as illustrated below:*



Application of Cauchy's residue theorem requires we calculate the residue of $\frac{z^2}{1+z^6}$ *at* $w = e^{i\pi/6}, i$ *and* $e^{5\pi i/6}$. *In each case we have a simple pole and Rule 3 applies:*

$$Res\left(\frac{z^2}{1+z^6}, w\right) = \frac{w^2}{6w^5}.$$

Hence,

$$Res\left(\frac{z^2}{1+z^6}, e^{i\pi/6}\right) = \frac{(e^{i\pi/6})^2}{6(e^{i\pi/6})^5} = \frac{1}{6e^{3i\pi/6}} = \frac{1}{6i},$$

and

$$Res\left(\frac{z^2}{1+z^6}, i\right) = \frac{(i)^2}{6(i)^5} = -\frac{1}{6i},$$

and

$$Res\left(\frac{z^2}{1+z^6}, e^{5i\pi/6}\right) = \frac{(e^{5i\pi/6})^2}{6(e^{5i\pi/6})^5} = \frac{1}{6e^{15i\pi/6}} = \frac{1}{6i}.$$

Therefore,

$$\int_{\partial D} \frac{z^2}{z^6+1}\, dz = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3}.$$

Notice if $|z| = R > 1$ *then* $\left|\frac{z^2}{z^6+1}\right| \leq \frac{R^2}{R^6-1}$ *hence the* ML*-estimate provides:*

$$\left|\int_{C_R} \frac{z^2}{z^6+1}\, dz\right| \leq \frac{R^2}{R^6-1}(\pi R) \to 0$$

as $R \to \infty$. *If* $z \in [-R, R]$ *then* $z = x$ *for* $-R \leq x \leq R$ *and* $dz = dx$ *hence*

$$\int_{[-R,R]} \frac{z^2}{z^6+1}\, dz = \int_{-R}^{R} \frac{x^2}{x^6+1}\, dx.$$

Thus, noting $\partial D = C_R \cup [-R, R]$ *we have:*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{3} \quad \Rightarrow \quad P.V. \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{3} \quad \Rightarrow \quad \int_0^{\infty} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{6}.$$

10.7 Jordan's Lemma

Lemma 10.7.1. Jordan's Lemma: *if* C_R *is the semi-circular contour* $z(\theta) = Re^{i\theta}$ *for* $0 \leq \theta \leq \pi$, *in the upper half plane, then* $\int_{C_R} |e^{iz}||dz| < \pi$.

Proof: note $|e^{iz}| = \exp(\mathbf{Re}(iz)) = \exp(\mathbf{Re}(iRe^{i\theta})) = e^{-R\sin\theta}$ and $|dz| = |iRe^{i\theta}d\theta| = Rd\theta$ hence the Lemma is equivalent to the claim:

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}.$$

By definition, a concave down function has a graph that resides above its secant line. Notice $y = \sin\theta$ has $y'' = -\sin\theta < 0$ for $0 \leq \theta \leq \pi/2$. The secant line from $(0, 0)$ to $(\pi/2, 1)$ is $y = 2\theta/\pi$.

Therefore, it is geometrically (and analytically) evident that $\sin\theta \geq 2\theta/\pi$. Consequently, following Gamelin page 216,

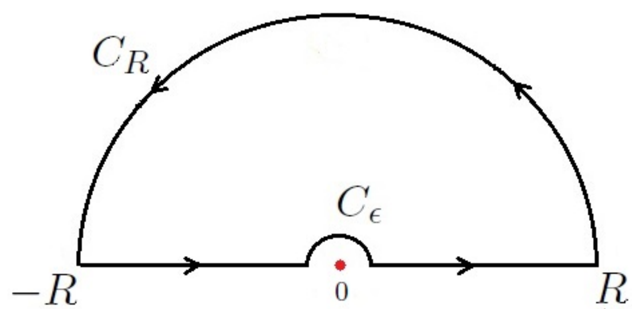
$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta$$

make a $t = 2R\theta/\pi$ substitution to find:

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}\int_0^{1/R} e^{-t} dt < \frac{\pi}{R}\int_0^{\infty} e^{-t}\, dt = \frac{\pi}{R}. \qquad \square$$

Jordan's Lemma allows us to treat integrals of rational functions multiplied by sine or cosine where the rational function has a denominator function with just one higher degree than the numerator. Previously we needed two degrees higher to make the ML-estimate go through nicely. For instance, see Example 10.2.4.

Example 10.7.2. *To show $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ we calculate the integral of $f(z) = \frac{e^{iz}}{z}$ along an indented semi-circular path pictured below:*



Notice, for $|z| = R$ we have:

$$\left|\int_{C_R} \frac{e^{iz}}{z} dz\right| \leq \int_{C_R} \left|\frac{e^{iz}}{z}\right| |dz| = \frac{1}{R}\int_{C_R} \left|e^{iz}\right| |dz| < \frac{\pi}{R}$$

where in the last step we used Jordan's Lemma. Thus as $R \to \infty$ we see the integral of $f(z)$ along C_R vanishes. Suppose $R \to \infty$ and $\epsilon \to 0$ then Cauchy's residue and fractional residue theorems combine to yield:

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{x}\, dx - \pi i Res\left(\frac{e^{iz}}{z}, 0\right) + \lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z} dz = 0$$

hence, noting the residue is 1*,*

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{x}\, dx = i\pi \quad \Rightarrow \quad \lim_{R\to\infty}\int_{-R}^{R}\left(\frac{\cos x}{x} + i\frac{\sin x}{x}\right) dx = i\pi.$$

Note, $\frac{\cos x}{x}$ is an odd function hence the principal value of that term vanishes. Thus,

$$\lim_{R\to\infty} i\int_{-R}^{R} \frac{\sin x}{x}\, dx = i\pi \quad \Rightarrow \quad P.V.\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \quad \Rightarrow \quad \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Example 10.7.3. *We can calculate* $\displaystyle\int_0^\infty \frac{x\sin(2x)}{x^2+3}$ *by studying the integral of* $f(z) = \dfrac{ze^{2iz}}{z^2+3}$ *around the curve* $\gamma = C_R \cup [-R, R]$ *where* C_R *is the half-circular path in the CCW-direction. Notice* $z = \pm i\sqrt{3}$ *are simple poles of* f*, but, only* $z = i\sqrt{3}$ *falls within* γ*. Notice, by Rule 3,*

$$Res\left(\frac{ze^{2iz}}{z^2+3}, i\sqrt{3}\right) = \frac{i\sqrt{3}e^{-2\sqrt{3}}}{2i\sqrt{3}} = \frac{e^{-2\sqrt{3}}}{2}.$$

Next, we consider $|z| = R$*, in particular notice:*

$$\left|\int_{C_R} \frac{ze^{2iz}}{z^2+3}\,dz\right| \leq \int_{C_R} \left|\frac{ze^{2iz}}{z^2+3}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{2iz}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \left|e^{iz}\right| |dz|$$

Notice, Jordan's Lemma gives

$$\int_{C_R} \left|e^{iz}\right| |dz| < \pi = \pi \cdot \frac{1}{\pi R}\int_{C_R} |dz| = \int_{C_R} \frac{1}{R}|dz|$$

hence,

$$\frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \left|e^{iz}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \frac{1}{R}|dz| = \frac{1}{R^2-3}\int_{C_R} \left|e^{iz}\right| |dz| < \frac{\pi^2}{R^2-3}.$$

Clearly as $R \to \infty$ *the integral of* $f(z)$ *along* C_R *vanishes. We find the integral along* $[-R, R]$ *where* $z = x$ *and* $dz = dx$ *must match the product of* $2\pi i$ *and the residue by Cauchy's residue theorem*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{2ix}}{x^2+3}\,dx = (2\pi i)\frac{e^{-2\sqrt{3}}}{2} = \pi i e^{-2\sqrt{3}}.$$

Of course, $e^{2ix} = \cos(2x) + i\sin(2x)$ *and the integral of* $\frac{x\cos(2x)}{x^2+3}$ *vanishes as it is an odd function. Cancelling the factor of* i *we derive:*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(2x)}{x^2+3}\,dx = \pi e^{-2\sqrt{3}} \quad \Rightarrow \quad \int_0^\infty \frac{x\sin(2x)}{x^2+3}\,dx = \frac{\pi}{2}e^{-2\sqrt{3}}$$

We have shown the solution of Problem 4 on page 214 of [C96]. The reader will find more useful practice problems there as is often the case.

10.8 Exterior Domains

Exterior domains are interesting. Basically this is Cauchy's residue theorem turned inside out. Interestingly a term appears to account for the residue at ∞. We decided to move on to the next chapter this semester. If you are interested in further reading on this topic, you might look at: this MSE exchange or this MSE exchange or this nice Wikipedia example or this lecture from Michael VanValkenburgh at UC Berkeley. Enjoy.

10.9 Application of Residue Theory to Summation of Series

Chapter XI

The Logarithmic Integral

We just cover the basic part of Gamelin's exposition in this chapter. It is interesting that he provides a proof of the Jordan curve theorem in the smooth case. In addition, there is a nice couple pages on simply connected and equivalent conditions in view of complex analysis. All of these are interesting, but our interests take us elsewhere this semester.

The *argument principle* is yet another interesting application of the residue calculus. In short, it allows us to count the number of zeros and poles of a given complex function in terms of the logarithmic integral of the function. Then, Rouché's Theorem provides a technique for counting zeros of a given function which has been extended by a small *perturbation*. Both of these sections give us tools to analyze zeros of functions in surprising new ways.

11.1 The Argument Principle

Let us begin by defining the main tool for our analysis in this section:

Definition 11.1.1. *Suppose f is analytic on a domain D. For a curve γ in D such that $f(z) \neq 0$ on γ we say:*

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma} d\log f(z)$$

is the **logarithmic integral of $f(z)$ along γ**.

Essentially, the logarithmic integral measures the change of $\log f(z)$ along γ.

Example 11.1.2. *Consider $f(z) = (z - z_o)^n$ where $n \in \mathbb{Z}$. Let $\gamma(z) = z_o + Re^{i\theta}$ for $0 \leq \theta \leq 2\pi k$. Calculate,*

$$\frac{f'(z)}{f(z)} = \frac{n(z - z_o)^{n-1}}{(z - z_o)^n} = \frac{n}{z - z_o}$$

thus,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma} \frac{n\,dz}{z - z_o} = \frac{n}{2\pi i}\int_{0}^{2\pi k} \frac{Rie^{i\theta}d\theta}{Re^{i\theta}} = \frac{n}{2\pi}\int_{0}^{2\pi k} d\theta = nk.$$

The number $k \in \mathbb{Z}$ is the **winding number** *of the curve and n is either ($n > 0$) the* **number of zeros** *or ($n < 0$) $-n$ is the* **number of poles** *inside γ. In the case $n = 0$ then there are neither zeros nor poles inside γ. Our counting here is that a pole of order 5 counts as 5 poles and a zero repeated counts as two zeros etc..*

The example above generalizes to the theorem below:

Theorem 11.1.3. argument principle I: *Let D be a bounded domain with a piecewise smooth boundary ∂D, and let f be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then*

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty,$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.

Proof: Let z_o be a zero of order N for $f(z)$ then $f(z) = (z-z_o)^N h(z)$ where $h(z_o) \neq 0$. Calculate:

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{N(z-z_o)^{N-1}h(z) + (z-z_o)^N h'(z)}{(z-z_o)^N h(z)} \\ &= \frac{N}{z-z_o} + \frac{h'(z)}{h(z)}\end{aligned}$$

likewise, if z_o is a pole of order N then $f(z) = \dfrac{h(z)}{(z-z_o)^N} = (z-z_o)^{-N}h(z)$ hence

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-N(z-z_o)^{-N-1}h(z) + (z-z_o)^{-N} h'(z)}{(z-z_o)^{-N} h(z)} \\ &= \frac{-N}{z-z_o} + \frac{h'(z)}{h(z)}\end{aligned}$$

Thus,

$$\text{Res}\left(\frac{f'(z)}{f(z)}, z_o\right) = \pm N$$

where $(+)$ is for a zero of order N and $(-)$ is for a pole of order N. Let $z_1, \dots, z_j$ be the zeros and poles of f, which are finite in number as we assumed f was meromorphic. Cauchy's residue theorem yields:

$$\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = 2\pi i\sum_{k=1}^{j}\text{Res}\left(\frac{f'(z)}{f(z)}, z_o\right) = 2\pi i\sum_{k=1}^{j}N_k = 2\pi i(N_0 - N_\infty). \qquad \square$$

To better understand the theorem is it useful to break down the logarithmic integral. The calculations below are a shorthand for the local selection of a branch of the logarithm

$$\log(f(z)) = \ln|f(z)| + i\arg(f(z)),$$

hence

$$d\log(f(z)) = d\ln|f(z)| + i d\arg(f(z))$$

for a curve with $f(z) \neq 0$ along the curve it is clear that $ln|f(z)|$ is well-defined along the curve and if $z : [a,b] \to \gamma$ then

$$\int_\gamma d\ln|f(z)| = \ln|f(b)| - \ln|f(a)|.$$

If the curve γ is closed then $f(a) = f(b)$ and clearly

$$\int_\gamma d\ln|f(z)| = 0.$$

However, the argument cannot be defined on an entire circle because we must face the 2π-jump somewhere. The logarithmic integral does not measure the argument of γ directly, rather, the arguments of the image of γ under f:

$$\int_\gamma d\arg(f(z)) = \arg(f(\gamma(b))) - \arg(f(\gamma(a))).$$

For a piecewise smooth curve we simply repeat this calculation along each piece and obtain the net-change in the argument of f as we trace out the curve.

Theorem 11.1.4. argument principle II: *Let D be a bounded domain with a piecewise smooth boundary ∂D, and let f be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then the increase in the argument of $f(z)$ around the boundary of D is 2π times the number of zeros minus the number of poles in D,*

$$\int_{\partial D} d\arg(f(z)) = 2\pi(N_0 - N_\infty).$$

We have shown this is reasonable by our study of $d\log(f(z)) = d\ln|f(z)| + i d\arg(f(z))$. Note,

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)} \quad \Rightarrow \quad d\log(f(z)) = \frac{f'(z)}{f(z)}\, dz.$$

Thus the Theorem 11.1.4 is a just a reformulation of Theorem 11.1.3.

Gamelin's example on page 227-228 is fascinating. I will provide a less sophisticated example of the theorem above in action.

Example 11.1.5. *Consider $f(z) = z^3 + 1$. Let $\gamma(t) = z_o + Re^{it}$ for $R > 0$ and $0 \leq t \leq 2\pi$. Thus $[\gamma]$ is $|z - z_o| = R$ given the positive orientation. If $R = 2$ and $z_o = 0$ then*

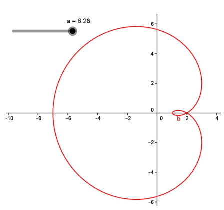
$$f(\gamma(t)) = 8e^{3it} + 1$$

The points traced out by $f(\gamma(t))$ above cover a circle centered at 1 with radius 8 three times. It follows the argument of $f(z)$ has increased by 6π along γ thus revealing $N_0 - N_\infty = 3$ and as f is entire we know $N_\infty = 0$ hence $N_0 = 3$. Of course, this is not surprising, we can solve $z^3 + 1 = 0$ to obtain $z \in (-1)^{1/3}$. All of these zeros fall within the circle $|z| = 2$.

Consider $R = 1$ and $z_o = -1$. Then $\gamma(t) = -1 + e^{it}$ hence

$$f(\gamma(t)) = \left(e^{it} - 1\right)^3 + 1 = e^{3it} - 3e^{2it} + 3e^{it} - 1 + 1$$

If we plot the path above in the complex plane we find:



Which shows $f(\gamma(t))$ increases its argument by 2π hence just one zero falls within $[\gamma]$ in this case. I used Geogebra to create the image above. Notice the slider allows you to animate the path which helps as we study the dynamics of the argument for examples such as this. To plot, as far as I currently know, you'll need to find $\mathbf{Re}(\gamma(t))$ and $\mathbf{Im}(\gamma(t))$ then its pretty straightforward.

11.2 Rouché's Theorem

This is certainly one of my top ten favorite theorems:

Theorem 11.2.1. Rouché's Theorem: *Let D be a bounded domain with a piecewise smooth boundary ∂D. Let f and h be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D, counting multiplicities.*

Proof: by assumption $|h(z)| < |f(z)|$ we cannot have a zero of f on the boundary of D hence $f(z) \neq 0$ for $z \in \partial D$. Moreover, it follows $f(z) + h(z) \neq 0$ on ∂D. Observe, for $z \in \partial D$,

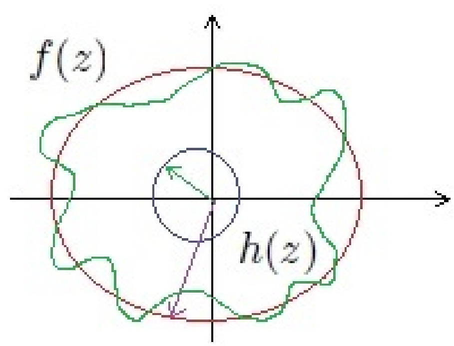
$$f(z) + h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right],$$

We are given $|h(z)| < |f(z)|$ thus $\left|\frac{h(z)}{f(z)}\right| < 1$ and we find $\textbf{Re}\left(1 + \frac{h(z)}{f(z)}\right) > 0$. Thus all the values of $1 + \frac{h(z)}{f(z)}$ on ∂D fall into a half plane which permits a single-valued argument function throughout hence any closed curve gives no gain in argument from $1 + \frac{h(z)}{f(z)}$. Moreover,

$$\arg\left(f(z) + h(z)\right) = \arg\left(f(z)\right) + \arg\left[1 + \frac{h(z)}{f(z)}\right]$$

hence the change in $\arg\left(f(z) + h(z)\right)$ is matched by the change in $\arg\left(f(z)\right)$ and by Theorem 11.1.4, and the observation that there are no poles by assumption, we conclude the number of zeros for f and $f + h$ are the same counting multiplicities. □

Once you understand the picture below it offers a convincing reason to believe:



The red curve we can think of as the image of $f(z)$ for $z \in \partial D$. Note, ∂D is not pictured. Continuing, the green curve is a *perturbation* or *deformation* of the red curve by the blue curve which is the graph of $h(z)$ for $z \in \partial D$. In order for $f(z) + h(z) = 0$ we need for $f(z)$ to be cancelled by $h(z)$. But, that is clearly impossible given the geometry.

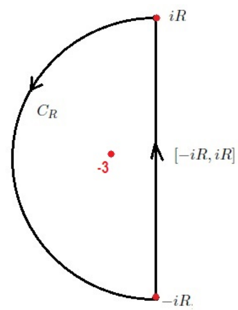
Often the following story is offered: suppose you walk a dog on a path which is between R_1 and R_2 feet from a pole. If your leash is less than R_1 feet then there is no way the dog can get caught on the pole. The function $h(z)$ is like the leash, the path which doesn't cross the origin is the red curve and the green path is formed by the dog wandering about the path while being restricted by the leash.

Example 11.2.2. *Find the number of zeros for $p(z) = z^{11} + 12z^7 - 3z^2 + z + 2$ within the unit circle. Let $f(z) = 12z^7$ and $h(z) = z^{11} - 3z^2 + z + 2$ observe for $|z| = 1$ we have $|h(z)| \leq 1 + 3 + 1 + 2 = 7$ and $|f(z)| = 12|z|^7 = 12$ hence $|h(z)| \leq f(z)$ for all z with $|z| = 1$. Observe $f(z) = 12z^7$ has a zero of multiplicity 7 at $z = 0$ hence by Rouché's Theorem $p(z) = f(z) + h(z) = z^{11} + 12z^7 - 3z^2 + z + 2$ also has seven zeros within the unit-circle.*

Rouché's Theorem also has great application beyond polynomial problems:

Example 11.2.3. *Prove that the equation $z + 3 + 2e^z = 0$ has precisely one solution in the left-half-plane. The idea here is to view $f(z) = z + 3$ as being perturbed by $h(z) = 2e^z$. Clearly $f(-3) = 0$ hence if we can find a curve γ which bounds $\mathbf{Re}(z) < 0$ and for which $|h(\gamma(t))| \leq |f(\gamma(t))|$ for all $t \in dom(\gamma)$ then Rouché's Theorem will provide the conclusion we desire.*

Therefore, consider $\gamma = C_R \cup [-iR, iR]$ where C_R has $z = Re^{it}$ for $\pi/2 \leq t \leq 3\pi/2$.



Consider $z \in [-iR, iR]$ then $z = iy$ for $-R \leq y \leq R$ observe:

$$|f(z)| = |iy + 3| = \sqrt{9 + y^2} \qquad \& \qquad |h(z)| = |2e^{iy}| = 2$$

thus $|h(z)| < |f(z)|$ for all $z \in [-iR, iR]$. Next, suppose $z = x + iy \in C_R$ hence $-R \leq x \leq 0$ and $-R \leq y \leq R$ with $x^2 + y^2 = R^2$. In particular, assume $R > 5$. Note:

$$|f(z)| = |x + iy + 3| \quad \Rightarrow \quad R - 3 \leq |f(z)| \leq \sqrt{9 + R^2}.$$

the claim above is easy to see geometrically as $|z + 3|$ is simply the distance from z to -3 which is smallest when $y = 0$ and largest when $x = 0$. Furthermore, as $-R \leq x \leq 0$ and e^x is a strictly increasing function,

$$|h(z)| = \left|2e^x e^{iy}\right| = 2e^x < 2 < R - 3 < |f(z)|$$

where you now hopefully appreciate why we assumed $R > 5$. Consequently $|h(z)| \leq |f(z)|$ for all $z \in C_R$ with $R > 5$. We find by Rouché's Theorem $f(z)$ and $f(z) + h(z) = z + 3 + 2e^z$ has only one zero in γ for $R > 5$. Thus, suppose $R \to \infty$ and observe γ serves as the boundary of $\mathbf{Re}(z) < 0$ and so the equation $z + 3 + 2e^z = 0$ has just one solution in the left-half plane.

Notice, Rouché's Theorem does not tell us what the solution of $z + 3 + 2e^z = 0$ with $\mathbf{Re}(z) < 0$ is. The theorem merely tells us that the solution uniquely exists.

Example 11.2.4. *Consider $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_o$ where $a_n \neq 0$. Let $f(z) = a_n z^n$ and $h(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_o$ then $p(z) = f(z) + h(z)$. Moreover, if we choose $R > 0$ sufficiently large then $|h(z)| \leq |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_o| < |a_n| R^n = |f(z)|$ for $|z| = R$ hence Rouché's Theorem tells us that there are n-zeros for $p(z)$ inside $|z| = R$ as it is clear that $z = 0$ is a zero of multiplicity n for $f(z) = a_n z^n$. Thus every $p(z) \in \mathbb{C}[z]$ has n-zeros, counting multiplicity, on the complex plane.*

The proof of the Fundamental Theorem of Algebra above is nicely direct in contrast to other proofs by contradiction we saw in previous parts of this course.

Chapter 12

Conformal Mapping

12.1 Fractional Linear Transformations

12.2 the conformal mapping technique

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