LECTURE ON COMPLEX EIGENVALUES & DIAGONALIZATION

PROPOSITION 1: Let A be a square matrix with eigenvector v with eigenvalue λ then v is likewise an e-vector for A^2 with e-value λ^2

Proof: Given $Av = \lambda v$ thus $AAv = A\lambda v \Rightarrow A^2 v = \lambda A v = \lambda^2 v$ thus v is e-vector with e-value λ^2 for A.

PROPOSITION 2: Suppose $Av = \lambda v$ for $v \neq 0$ then v is an e-vector of $B = I + A$.

Proof: Suppose $Av = \lambda v$. Consider $B = I + A$ has $Bv = (I+A)v = v + Av = v + \lambda v$. Therefore, $Bv = (1+\lambda)v$. This shows v is e-vector for B with e-value $1+\lambda$.

PROPOSITION 3: If v is e-vector for A with e-value λ then v is also an e-vector for $M = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n$ where $c_0, c_1, \dots, c_n$ are constants (scalars).

Proof: Suppose $Av = \lambda v$ then notice $A^k v = \underbrace{AA\cdots A}_{k\text{-fold}} v = \underbrace{A \cdots A}_{k-1} \lambda v = \lambda^k v$.

Then consider M as in prop,

$$\begin{aligned} Mv &= c_0 I v + c_1 A v + c_2 A^2 v + \cdots + c_n A^n v \\ &= c_0 v + c_1 \lambda v + c_2 \lambda^2 v + \cdots + c_n \lambda^n v \\ &= (\underbrace{c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots + c_n \lambda^n}_{\sigma}) v \end{aligned}$$

$\therefore$ v is e-vector for M with e-value σ.

PROPOSITION 4: If v_1, v_2 are e-vectors with distinct e-values λ_1, λ_2 then $\{v_1, v_2\}$ forms a LI set of vectors.

Assume $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Suppose

$$c_1 v_1 + c_2 v_2 = 0 \quad (*)$$

Then notice multiplying by $(A - \lambda_2 I)$ gives

$$c_1 (A - \lambda_2 I)v_1 + c_2 \cancel{\underbrace{(A - \lambda_2 I) v_2}_{\lambda_2 v_2 - \lambda_2 v_2}}^{\,0} = (A - \lambda_2 I)0 = 0$$

$$c_1 (Av_1 - \lambda_2 v_1) \overset{*}{=} c_1(\lambda_1 - \lambda_2)v_1 = 0 \Rightarrow \boxed{c_1 = 0}$$

since $\lambda_1 - \lambda_2 \neq 0$ for λ_1, λ_2 are distinct.

Returning to $*$ gives $c_2 v_2 = 0 \Rightarrow c_2 = 0$

since $v_2 \neq 0$ as v_2 is e-vector.

Thus $\{v_1, v_2\}$ is LI.//

PROPOSITION 5: Let $A \in \mathbb{R}^{n\times n}$ and suppose $\lambda = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$ is a complex eigenvalue of A. Then

(a.) $\bar{\lambda} = \alpha - i\beta$ is also complex e-value of A

(b.) If $\beta \neq 0$ and $a, b \in \mathbb{R}^n$ for which $v = a + ib \in \mathbb{C}^n$ is a complex e-vector with $\lambda = \alpha + i\beta$ then $\{a, b\}$ is LI

(a.) Given $Av = \lambda v = (\alpha + i\beta)v$ then take the complex conjugate of the equation to find $\overline{Av} = \overline{(\alpha+i\beta)v} \Rightarrow \bar{A}\bar{v} = \overline{\alpha + i\beta}\,\bar{v}$ and as $A \in \mathbb{R}^{n\times n}$ we have $\bar{A} = A$ thus $\underbrace{A\bar{v} = (\alpha - i\beta)\bar{v}}$

thus $\bar{v}$ is e-vector of A with e-value $\bar{\lambda} = \alpha - i\beta$

(b.) Given $Av = \lambda v$ and $A\bar{v} = \bar{\lambda}\bar{v}$ where $\lambda = \alpha + i\beta \neq \alpha - i\beta = \bar{\lambda}$ hence $\{v, \bar{v}\}$ are LI by Prop. 4. Assume $v = a + ib$ thus $\bar{v} = a - ib$

Notice $v + \bar{v} = 2a$ and $v - \bar{v} = 2ib$ thus $a = \frac{1}{2}(v + \bar{v})$ and $b = \frac{1}{2i}(v - \bar{v})$. Suppose $c_1 a + c_2 b = 0$ thus

$$c_1\left(\frac{v+\bar{v}}{2}\right) + c_2\left(\frac{v-\bar{v}}{2i}\right) = 0 \Rightarrow \left(\frac{c_1}{2} + \frac{c_2}{2i}\right)v + \left(\frac{c_1}{2} - \frac{c_2}{2i}\right)\bar{v} = 0$$

Thus, by LI of $\{v, \bar{v}\}$ we find $\frac{c_1}{2} + \frac{c_2}{2i} = 0$ and $\frac{c_1}{2} - \frac{c_2}{2i} = 0$ and it follows that $c_1 = c_2 = 0$ thus $\{a, b\}$ is LI.

E1 Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$

(a.) Find e-values of A
(b.) Find basis $\{v_1, v_2\}$ of complex e-vectors for $\mathbb{C}^2$
(c.) Show $[\beta]^{-1} A [\beta]$ is diagonalized in the complex sense
(d.) Let $\gamma = \{Re(v_1), Im(v_1)\}$ form basis for $\mathbb{R}^2$ and calculate $[\gamma]^{-1} A [\gamma]$

(a.) $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{bmatrix} = \lambda(\lambda - 2) + 2 = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0$

$$(\lambda - 1)^2 = -1$$

$$\boxed{\lambda = 1 \pm i}.$$

(b.) $\underline{\lambda = 1 + i}$

$$(A - (1+i)I)\vec{v} = \begin{bmatrix} -1-i & 1 \\ -2 & 2-(+1+i) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-1-i)u + v = 0$$

$$v = (1+i)u \quad \underline{\text{choose}}\ u = 1$$

Find $v_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ then $\bar{v}_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$$\boxed{\beta = \left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \right\}} \text{ is basis for } \mathbb{C}^2$$

$$\mathbb{C}^2 = \{ (z, w) \mid z, w \in \mathbb{C} \}$$

(c.) $[\beta] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1+i & 1-i \end{array}\right]$ $\qquad \frac{1}{i} = -i \quad (i^2 = ii = -1)$

$$[\beta]^{-1} = \frac{1}{1(1-i) - 1(1+i)} \left[\begin{array}{c|c} 1-i & -1 \\ \hline -1-i & 1 \end{array}\right] = \frac{1}{-2i} \left[\begin{array}{c|c} 1-i & -1 \\ \hline -1-i & 1 \end{array}\right] = \frac{i}{2} \left[\begin{array}{c|c} 1-i & -1 \\ \hline -1-i & 1 \end{array}\right]$$

$$[\beta]^{-1} A [\beta] = [\beta]^{-1} A [V_1 | \overline{V_1}]$$
$$= [\beta]^{-1} [AV_1 | A\overline{V_1}]$$
$$= [\beta]^{-1} [(1+i)V_1 | (1-i)\overline{V_1}]$$
$$= [\beta]^{-1} \left[\begin{array}{c|c} 1+i & 1-i \\ \hline (1+i)^2 & (1-i)^2 \end{array}\right]$$
$$= \frac{i}{2} \left[\begin{array}{c|c} 1-i & -1 \\ \hline -1-i & 1 \end{array}\right] \left[\begin{array}{c|c} 1+i & 1-i \\ \hline 2i & -2i \end{array}\right]$$

$(1+i)^2 = (1+i)(1+i) =$
$= 1 + i + i + i^2$
$= 2i$

$(1-i)^2 = (1-i)(1-i) = -2i$

$$= \frac{i}{2} \left[\begin{array}{c|c} (1-i)(1+i) - 1(+2i) & \cancel{(1-i)^2 + 2i} \to 0 \\ \hline -\cancel{(1+i)(1+i) + 2i} \to 0 & -(1+i)(1-i) - 2i \end{array}\right] \qquad \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \overline{\lambda} \end{array}\right]$$

$$= \frac{i}{2} \left[\begin{array}{c|c} 2 \ast 2i & 0 \\ \hline 0 & -2-2i \end{array}\right] = \left[\begin{array}{c|c} i+1 & 0 \\ \hline 0 & -i+1 \end{array}\right] = \left[\begin{array}{c|c} 1+i & 0 \\ \hline 0 & 1-i \end{array}\right]$$

(d.) $\gamma = \{ Re(v_1), Im(v_1) \} = $

$= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$v_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ i \end{bmatrix}$

$v_1 = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{a_1} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_1}$

$[\gamma] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad [\gamma]^{-1} = \frac{1}{1-0} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$[\gamma]^{-1} A [\gamma] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = R(1+i)$$

(Real Jordan 2×2 block for $\lambda = 1+i$)

E2 Let $B = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

(a.) find e-values of B

(b.) find basis $\beta = \{v_1, v_2\}$ of complex e-vectors for $\mathbb{C}^2$

(a.) $\det\begin{pmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = (\lambda-3)(\lambda-1)+2 = \lambda^2 - 4\lambda + 5 = \underbrace{(\lambda-2)^2+1}_{\boxed{\lambda = 2 \pm i}}$

(b.) $\lambda = 2+i$ $\quad 3-(2+i) = 1-i$ & $1-(2+i) = -1-i$

$\left(B - (2+i)I\right)v_1 = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1-i)u + v = 0$

$v = (i-1)u$

choose $u=1$, $v = i-1$

Thus $\boxed{\beta = \left\{ \begin{bmatrix} 1 \\ i-1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i-1 \end{bmatrix} \right\}}$

<u>Remark</u>: $[\beta]^{-1} B [\beta] = \left[\begin{array}{c|c} 2+i & 0 \\ \hline 0 & 2-i \end{array}\right]$

$\boxed{E3}$ Let $C = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \left[\begin{array}{c|c} B & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \hline 0\,0 & 3 \end{array}\right]$

(a.) find e-values of C

(b.) find e-basis for $\mathbb{C}^3$ or C

(c.) Is C diagonalizable over $\mathbb{R}$?

(d.) Is C diagonalizable over $\mathbb{C}$?

(a.) $\det(C - \lambda I) = \det \left[\begin{array}{cc|c} 3-\lambda & 1 & 0 \\ -2 & 1-\lambda & 0 \\ \hline 0 & 0 & 3-\lambda \end{array}\right] = \det \begin{bmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} \det(3-\lambda)$

$= [(\lambda-2)^2 + 1](3-\lambda)$

$\boxed{\lambda_{1,2} = 2 \pm i \ \ \& \ \ \lambda_3 = 3}$

(b.) Notice can use $V_1 = \begin{bmatrix} 1 \\ i-1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ -i-1 \end{bmatrix}$ adapted to

$V_1 = \begin{bmatrix} 1 \\ i-1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ -i-1 \\ 0 \end{bmatrix}$ are e-vector for C with $\lambda_1 = 2+i$, $\lambda_2 = 2-i$

$BV_3 = 3V_3 \longrightarrow \underline{\begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 3 \begin{bmatrix} u \\ v \\ w \end{bmatrix}}$ $\quad V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Find $\boxed{\beta = \left\{ \begin{bmatrix} 1 \\ i-1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i-1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

(c.) no not diag. over $\mathbb{R}$

(d.) YES, $[\beta]^{-1} C [\beta] = \begin{bmatrix} 2+i & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\boxed{E4}$ Let $D = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & A \end{array}\right] = \left[\begin{array}{cc|cc} 3 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 2 \end{array}\right]$

(a.) find the e-values of D

(b.) find e-basis $\beta = \{v_1, v_2, v_3, v_4\}$ for D

(c.) find basis γ for which $[\gamma]^{-1} D [\gamma]$ is block diagonal with real Jordan blocks on the diagonal.

using $\boxed{E1}$ & $\boxed{E2}$

(a.) $\det(D - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} \det\begin{pmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix} = 0 \Rightarrow \boxed{\begin{array}{l} \lambda_{1,2} = 2 \pm i \\ \lambda_{3,4} = 1 \pm i \end{array}}$

(b.) adjoining zeros to e-bases from $\boxed{E1}$ & $\boxed{E2}$ gives,

$$\boxed{\beta = \left\{ \begin{bmatrix} 1 \\ i-1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i-1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1-i \end{bmatrix} \right\}}$$

$$D \begin{bmatrix} 1 \\ i-1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i-1 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 + i - 1 \\ -2 + i - 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + i \\ -3 + i \\ 0 \\ 0 \end{bmatrix} = (2+i) \begin{bmatrix} 1 \\ i-1 \\ 0 \\ 0 \end{bmatrix}$$

(c.)

$$\gamma = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$[\gamma]^{-1} D [\gamma] = \left[\begin{array}{c|c} R(2+i) & 0 \\ \hline 0 & R(1+i) \end{array} \right]$$

$$= \underbrace{\left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array} \right]}_{\text{"REAL JORDAN FORM" OF } D.}$$

$\boxed{E5}$ Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(a.) find e-values of A

(b.) find an eigenbasis $\beta = \{V_1, V_2, V_3\}$ for A

(c.) Is A diagonalizable over $\mathbb{R}$?

(d.) Is A diagonalizable over $\mathbb{C}$?

(a.) $$\det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix} = -\lambda\left(-\lambda(1-\lambda)+1\right) - 1\left(0(1-\lambda)-1\right)$$

$$= -\lambda(\lambda^2 - \lambda + 1) + 1$$

$$= -\lambda^3 + \lambda^2 - \lambda + 1 \leftarrow \lambda = 1 \text{ is root}$$

$\therefore (\lambda - 1)$ is factor

$$= -(\lambda^3 - \lambda^2 + \lambda - 1)$$

$$= -(\lambda - 1)(\lambda^2 + 1)$$

$$\therefore \boxed{\lambda_1 = 1 \text{ and } \lambda_{2,3} = \pm i}$$

(b.) $(A-I)\vec{v}_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$u = w$
$v = w$
choose $w = 1$ $\therefore$ $\underline{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ $\quad$ $\underline{\text{check } A\vec{v}_1 = \vec{v}_1}$ ✓

$\underline{\lambda = i}$

$(A - iI)\vec{v}_2 = \begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ 1 & -1 & 1-i \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$i(1-i) = i - i^2 = i + 1$

$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{2i}{2} = i$

$\left[\begin{array}{ccc|c} 1 & -1 & 1-i & 0 \\ -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \end{array}\right] \xrightarrow{r_2 + ir_1} \left[\begin{array}{ccc|c} 1 & -1 & 1-i & 0 \\ 0 & 1-i & 1+i & 0 \\ 0 & -i & 1 & 0 \end{array}\right] \xrightarrow{\frac{r_2}{1-i}} \left[\begin{array}{ccc|c} 1 & -1 & 1-i & 0 \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \end{array}\right]$

$\xrightarrow[r_3 + ir_2]{r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\to u = -w$
$\to v = -iw$
$\underline{w = 1}$ choice.

$\vec{v}_2 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix} \therefore \vec{v}_3 = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$

$\therefore \boxed{\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \right\}}$

(c.) No. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ is <u>not</u> <u>real</u> <u>diagonalizable</u>

(d.) YES, if we calculated,

$$[\beta]^{-1} A [\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

<u>Remark</u>: $\gamma = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ then

$$[\gamma]^{-1} A [\gamma] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & R(i) \end{array}\right]}_{\text{"REAL JORDAN FORM" of } A}$$