Name:

MATH 332-001, MAR. 4, 2010, TEST I

Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Closed book, open mind. Thanks and enjoy. There are 150 bonus points on this exam.

Problem 1 [100pts] Define the items listed below either by supplying a formula or by completing the sentence or paragraph as appropriate. (In each case below $z = (x+iy) \in \mathbb{C}$.)

(a.) $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$.

(b.) $\bar{z} = x - iy$

(c.) $|z| = \sqrt{x^2+y^2}$ or $\sqrt{z\bar{z}}$

(d.) open disk centered at z_o of radius $\epsilon > 0$

$$D_\epsilon(z_o) = \{ z \in \mathbb{C} \mid |z - z_o| < \epsilon \}.$$

(e.) open subset of $\mathbb{C}$

$U \subseteq \mathbb{C}$ is open iff $\forall u \in U$ $\exists \epsilon > 0$ such that $D_\epsilon(u) \subset U$.

(f.) connected subset of $\mathbb{C}$

is one for which any two points are connected by a polygonal path which starts and ends at those points and remains inside the subset in question.

(g.) domain of $\mathbb{C}$

is an open connected subset.

(h.) a complex function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is continuous at a limit point $z_o \in U$ iff

$\lim_{z \to z_o} f(z) = f(z_o)$. (this implicitly requires $z_o \in \text{dom}(f$.

(i.) a complex function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable at $z_o \in U$ iff

$\lim_{h \to 0} \left[\frac{f(z_o + h) - f(z_o)}{h} \right]$ exists.

(j.) a complex function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is analytic at $z_o \in U$ iff

$\exists$ an open disk about ~~each~~ z_o such that f is differentiable at each point in the disk.

Problem 2 [100pts] Find the polar form of $z = 1/(1+i)$. Also, find $Arg(z)$ and $arg(z)$

$$z = \frac{1}{1+i}\left(\frac{1-i}{1-i}\right) = \frac{1-i}{2} \quad \therefore \quad Re(z) = \frac{1}{2} \;\&\; Im(z) = \frac{-1}{2}.$$

Thus z is in quad. IV $\therefore$ $\boxed{Arg(z) = -\pi/4}$

also $\boxed{arg(z) = \left\{ -\frac{\pi}{4} + 2\pi k \;\middle|\; k \in \mathbb{Z} \right\}}$

$$z = \sqrt{\frac{1}{4} + \frac{1}{4}}\, e^{i\left(\frac{-\pi}{4}\right)} = \boxed{\frac{1}{\sqrt{2}} \exp\left(\frac{-i\pi}{4}\right)} \leftarrow \text{polar form.}$$

Problem 3 [100pts] Calculate the Cartesian form of $(\,2 + 2i\,)^{30}$.

Note, $2 + 2i = 2(1+i) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \underline{2\sqrt{2}\, e^{i\pi/4}}$.

$$\begin{aligned}
(2+2i)^{30} &= \left(2\sqrt{2}\, e^{i\pi/4}\right)^{30} \\
&= (2\sqrt{2})^{30}\, e^{30 i\pi/4} \\
&= 2^{45}\left(\cos\left(\frac{15\pi}{2}\right) + i\sin\left(\frac{15\pi}{2}\right)\right) \\
&= 2^{45}\left(0 + i\sin\left(6\pi + \frac{3\pi}{2}\right)\right) \\
&= 2^{45}(0 - i) = \boxed{-2^{45}\, i}
\end{aligned}$$

$30 \cdot \frac{3}{2} = 45$

$\frac{30}{4} = \frac{15}{2}$

$\frac{15\pi}{2} = \frac{12\pi}{2} + \frac{3\pi}{2}$

Problem 4 [125pts] Calculate $(\,1 + i\,)^{\frac{1}{4}}$.

$$(1+i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\pi/4} \qquad (\sqrt{2})^{1/4} = 2^{1/8}$$

$$(1+i)^{1/4} = \Big\{ \underbrace{2^{1/8} e^{i\pi/16}}_{C_0},\ 2^{1/8} e^{i 9\pi/16},\ 2^{1/8} e^{i 17\pi/16},\ 2^{1/8} e^{i 25\pi/16} \Big\}$$

note

$$w_4 = \exp\left(\frac{2\pi i}{4}\right) = e^{i\frac{\pi}{2}} = i$$

or

$$\{C_0,\ iC_0,\ -C_0,\ -iC_0\} = (1+i)^{1/4}$$

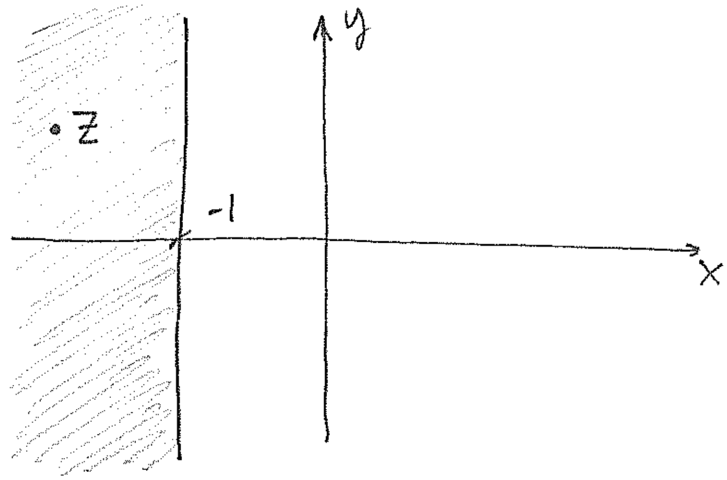
Problem 5 [100pts] State the equation of a circle of radius 4 centered at $z_o = 1 - 3i$.

All $z \in \mathbb{C}$ such that

$$\underline{|z - 1 + 3i| = 4.}$$

Problem 6 [125pts] Sketch the region described by $|z+2| \leq |z|$, support your sketch by providing an equivalent inequality in x, y where $z = x + iy$.

$$|z+2| \leq |z|$$
$$|x+2+iy| \leq |x+iy|$$
$$(x+2)^2 + y^2 \leq x^2 + y^2$$
$$x^2 + 4x + 4 + y^2 \leq x^2 + y^2$$
$$\hookrightarrow 4x + 4 \leq 0$$
$$\hookrightarrow x \leq -1$$



Problem 7 [100pts] Calculate $Log(-2)$. Also, find all solutions to $exp(z) = -2$.

$$\operatorname{Log}(re^{i\theta}) = \ln(r) + i \operatorname{Arg}(re^{i\theta}).$$
$$-2 = 2e^{i\pi} \quad \text{since} \quad e^{i\pi} = \cos\pi + i\sin\pi = -1.$$
$$\operatorname{Log}(-2) = \ln(2) + i \operatorname{Arg}(2e^{i\pi})$$
$$\therefore \boxed{\operatorname{Log}(-2) = \ln(2) + i\pi}$$

$$e^z = -2$$
$$\Rightarrow \log(e^z) = \log(-2)$$
$$\Rightarrow \boxed{z = \log(-2) = \{\ln(2) + i(\pi + 2\pi k) \mid k \in \mathbb{Z}.\}}$$

Problem 8 [100pts] Show that $|zw| = |z||w|$

$$|zw| = \sqrt{(zw)(\overline{zw})}$$
$$= \sqrt{zw\bar{z}\bar{w}}$$
$$= \sqrt{z\bar{z}\, w\bar{w}}$$
$$= \sqrt{z\bar{z}}\,\sqrt{w\bar{w}}$$
$$= |z|\,|w|.$$

Problem 9 [125pts] Use a careful ϵ, δ proof to show that

$$\lim_{z \to a} \overline{3z+1} = \overline{3a+1}$$

Let $\epsilon > 0$ choose $\delta = \epsilon/3$. Suppose $z \in \mathbb{C}$ and $0 < |z-a| < \delta$. Note that

$$\begin{aligned} \left|\overline{3z+1} - (\overline{3a+1})\right| &= |3\bar{z} + 1 - 3\bar{a} - 1| \\ &= |3\bar{z} - 3\bar{a}| \\ &= 3|\bar{z} - \bar{a}| \\ &< 3\delta = \epsilon \end{aligned}$$

$$\therefore \lim_{z \to a} \overline{3z+1} = \overline{3a+1}.$$

Problem 10 [100pts] Let $c \in \mathbb{C}$ and suppose f and g are differentiable complex functions at z_o. Show that $f + cg$ is differentiable at z_o and $(f+cg)'(z_o) = f'(z_o) + cg'(z_o)$.

We are given that

$$\underline{f'(z_o) = \lim_{h \to 0}\left(\frac{f(z_o+h) - f(z_o)}{h}\right) \ \& \ g'(z_o) = \lim_{h \to 0}\left(\frac{g(z_o+h) - g(z_o)}{h}\right)} \ *$$

both exist. Consider then

$$\lim_{h \to 0}\left[\frac{(f+cg)(z_o+h) - (f+cg)(z_o)}{h}\right] =$$

$$= \lim_{h \to 0}\left[\frac{f(z_o+h) - f(z_o) + c\left(g(z_o+h) - g(z_o)\right)}{h}\right]$$

using our given data $*$

$$= \lim_{h \to 0}\left[\frac{f(z_o+h) - f(z_o)}{h}\right] + c\lim_{h \to 0}\left[\frac{g(z_o+h) - g(z_o)}{h}\right]$$

$$= f'(z_o) + c\,g'(z_o).$$

Problem 11 [125pts] Recall that for $c \in \mathbb{C}$ and differentiable functions f, g at z we have $(f+cg)'(z) = f'(z) + cg'(z)$ and $\frac{d}{dz}(e^{cz}) = ce^{cz}$. Use these results to show that

(a.) $\frac{d}{dz}(\cos(z)) = -\sin(z)$

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{1}{2}\left[e^{iz} + e^{-iz}\right]\right) \quad : \text{def}^{n} \text{ of } \cos z$$

$$= ie^{iz}/2 - ie^{-iz}/2 \quad : \text{using sum rule \& chain rule \& pulled out constant } 1/2.$$

$$= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= -\sin(z).$$

(b.) $\frac{d}{dz}(\sinh(z)) = \cosh(z)$

$$\frac{d}{dz}(\sinh z) = \frac{d}{dz}\left(\frac{1}{2}\left[e^{z} - e^{-z}\right]\right) \quad : \text{def}^{n} \text{ of } \sinh z.$$

$$= \frac{1}{2}\frac{d}{dz}(e^{z}) - \frac{1}{2}\frac{d}{dz}(e^{-z}) \quad : \text{linearity of } \frac{d}{dz}.$$

$$= \frac{1}{2}e^{z} - \frac{1}{2}(-e^{-z}) = \frac{1}{2}(e^{z} + e^{-z}) = \underline{\cosh z}.$$

Problem 12 [100pts] State the largest complex domain for which the following complex functions are analytic:

(a.) $f(z) = \frac{1}{\sin(z)}$ $\quad \text{dom}(f) = \{z \in \mathbb{C} \mid \sin(z) \neq 0\}.$

$$\sin z = \sin(x+iy) = \sin x \cos(iy) + \underbrace{\sin(iy)}_{} \cos(x) = \sin x \cosh y + i \sinh y \cos x$$

$$\frac{1}{2i}\left(e^{iiy} - e^{-iiy}\right) = \sinh(y)$$

$\Rightarrow \sin z = 0$ gives

$\sin x \cosh y + i \sinh y \cos x = 0$

$\sin x = 0$ and $\sinh y = 0$ only sol^{n}'s since $\cosh y \neq 0$ $\forall y \in \mathbb{R}$ and if $\cos(x) = 0$ it follows $\sin x \neq 0$.

Thus $\sin z = 0$ iff $x = n\pi$ and $y = 0$ for $n \in \mathbb{Z}$.

(b.) $f(z) = \frac{1}{z^3+z} = \frac{1}{z(z^2+1)} = \frac{1}{z(z+i)(z-i)}$

$$\underline{\text{dom}(f) = \mathbb{C} - \{0, i, -i\}}.$$

Problem 13 [150pts] Let $f(z) = 1/z^2$ for $z \neq 0$. Prove that $f'(z) = -2/z^3$ for $z \neq 0$.

refering to the power rule for the case $n = -2$ does not constitute a proof. You should either use the definition or a theorem to give a solid non-circular argument.

$$f'(z) = \lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\frac{1}{(z+h)^2} - \frac{1}{z^2}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{z^2 - (z+h)^2}{h z^2 (z+h)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{z^2} - \cancel{z^2} - 2zh - h^2}{h z^2 (z+h)^2} \right] = \lim_{h \to 0} \left[\frac{-2z - h}{z^2 (z+h)^2} \right] = \frac{-2z}{z^2 (z+0)^2} = \frac{-2}{z^3}. \;//$$

Problem 14 [100pts] Observe that $u(x, y) = y$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$. Find the harmonic conjugate for u and construct an analytic function from u and its harmonic conjugate v.

Need CR - eq^{n}'s.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow \underline{\frac{\partial v}{\partial y} = 0}. \qquad \therefore\ v = f(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow \underline{\frac{\partial v}{\partial x} = -1} \longrightarrow \frac{df}{dx} = -1$$

$$\therefore\ f(x) = -x$$

or $v = -x$

$$\boxed{v(x,y) = -x}.$$

$$f(x,y) = u(x,y) + i\,v(x,y)$$

$$= y + i(-x)$$

$$= y - ix$$

$$= -i(x + iy)$$

$$= -iz$$ ← no surprise this is harmonic.

Problem 15 [100pts] Prove that $\mathbb{C}$ is connected.

Need polygonal path between two arbitrary pts z_1, z_2 in $\mathbb{C}$. Just use line segment $[z_1, z_2]$.

$$\underline{\gamma(t) = z_1 + t(z_2 - z_1)}.$$

for $0 \leq t \leq 1$