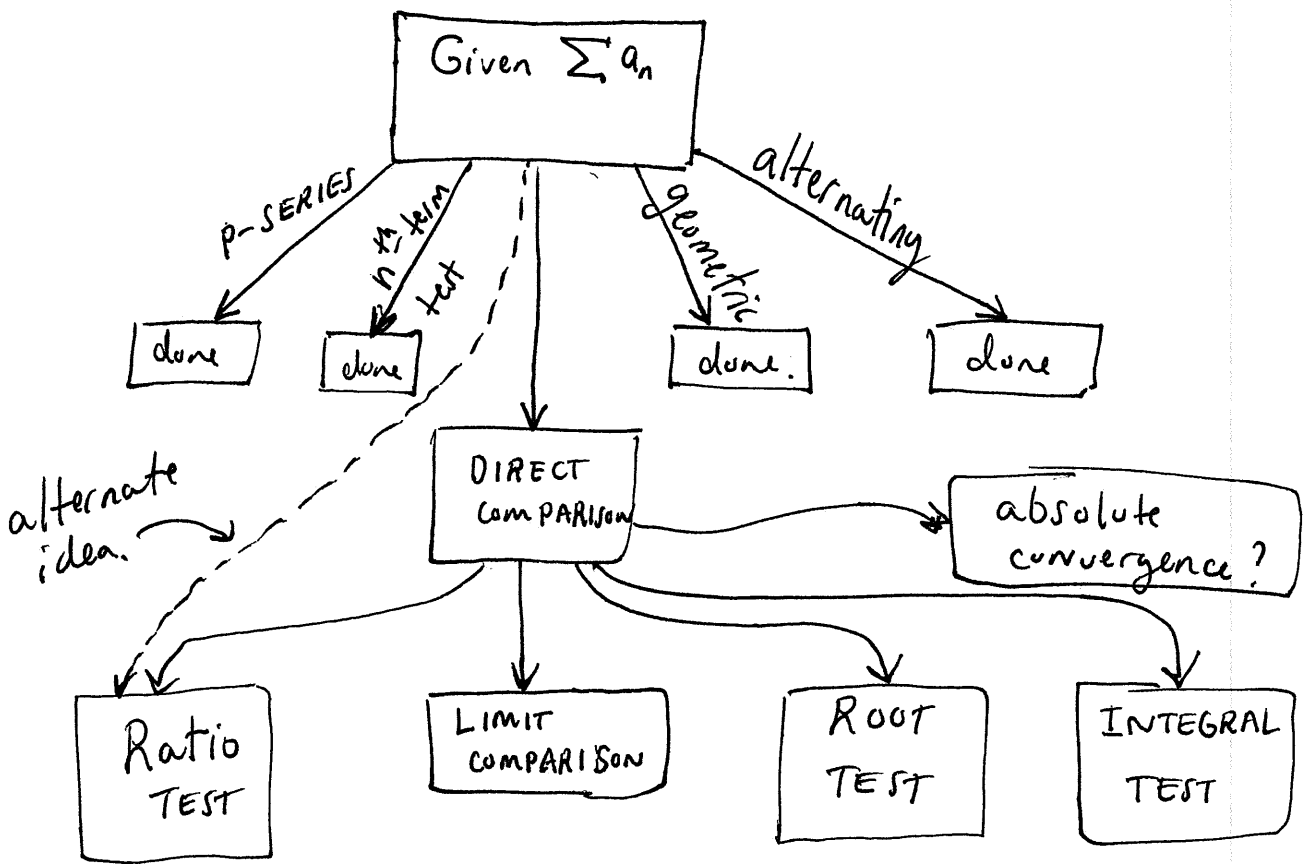
Summary of Convergence Tests

- usually problems come w/o guidance as to which test to use. Therefore, we need to gain global idea or strategy for which test to try. For example:



- THIS IS ROUGHLY MY PLAN.
- 1^{st} attach is identifying if p-series, n^{th} term test, geometric or alternating fits the given series.
- next I think a little about Direct Comparison
- then I think about Ratio, LCT, ROOT and last the $\int$-TEST.
- HONESTLY, the Ratio Test probably deserves more priority in my scheme.

1.) $S = \sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

Observe $\frac{n-1}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$. Thus S converges by Direct Comparison test to $p = 2$ series.

2.) $S = \sum_{n=2}^{\infty} \frac{n+1}{n^3-1}$ will not allow D.C.T. like 1.

Instead intuition suggests S is like $\frac{n}{n^3} = \frac{1}{n^2}$ for $n \gg 1$

Consider $\lim_{n\to\infty} \left[\frac{\frac{n+1}{n^3-1}}{\frac{1}{n^2}} \right] = \lim_{n\to\infty} \left[\frac{n^3+n^2}{n^3-1} \right] = 1$

thus S converges by LCT with convergent $p=2$ series.

3.) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ clearly diverges by n^{th} term test

as $\frac{n^2-1}{n^2+1} \longrightarrow 1$ as $n \to \infty$ hence $\lim_{n\to\infty} (-1)^n\left(\frac{n^2-1}{n^2+1}\right) \neq 0$.

4.) $S = \sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ can be studied via <u>absolute</u> convergence.

Notice $\left| \frac{\sin(2n)}{1+2^n} \right| \leq \frac{1}{1+2^n} < \frac{1}{2^n}$ thus by D.C.T we find $\sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{1+2^n} \right|$ converges as $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2}$ by geometric series result. Thus $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ is abs. convergent hence S is convergent.

5.) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ is alternating series

with $b_n = \dfrac{n^4}{4^n}$. It can be shown that

$n^4 < 4^n$ for $n \gg 1$. Observe, extending

n continuously, [small compared to ↓ for $n \gg 1$]

$$\frac{db_n}{dn} = \frac{4n^3 \cdot 4^n - n^4 \ln(4) 4^n}{(4^n)^2} \approx \frac{-n^4 \ln(4) 4^n}{4^{2n}} < 0$$

thus b_n decreases for large enough n. Moreover,

$$\lim_{n\to\infty}\left(\frac{n^4}{4^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{LLLL}{=}} \lim_{n\to\infty}\left[\frac{4\cdot 3\cdot 2}{(\ln(4))^4 4^n}\right] = 0$$

Thus $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ conv. by A.S.T.

6.) $S = \displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n}$. Focus on leading terms

to see $a_n = \dfrac{\sqrt{n^4+1}}{n^3+n} \sim \dfrac{\sqrt{n^4}}{n^3} = \dfrac{n^2}{n^3} = \dfrac{1}{n}$.

We try L.C.T. against $p=1$ (given that ↑ intuition)

$$\lim_{n\to\infty}\left(\frac{a_n}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n^4+1}}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{1+1/n^4}}{1+1/n^2}\right) = 1.$$

Thus S diverges by L.C.T. against $p=1$ series.

7.) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ diverges as $\frac{1}{2+\sin k} \not\to 0$ as $k \to \infty$.

Thus the series diverges by $k^{\underline{th}}$-term test.

8.) $\displaystyle S = \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ ← when I see ! I try Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{((n+1)!)^{n+1}}{(n+1)^{4(n+1)}}\right]\left[\frac{n^{4n}}{(n!)^n}\right]$$

well, this seems like a bad idea... <u>instead</u> try ROOT TEST, for $n \geq 5$,

$$\sqrt[n]{|a_n|} = \left(\frac{(n!)^n}{(n^4)^n}\right)^{1/n} = \frac{n!}{n^4} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{n^4}$$

as $n \to \infty$ note $\frac{n(n-1)(n-2)(n-3)}{n^4} \to 1$ thus

$\sqrt[n]{|a_n|} \sim (n-4)!$ for $n \gg 1$ which

indicates $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ thus S diverges by ROOT TEST.

9.) $$\sum_{n=1}^{\infty} \frac{(3^2)^{n+1}}{2^{4n+3}} = \sum_{n=1}^{\infty} \frac{3^{2n}3^2}{2^{4n}2^3} = \underbrace{\sum_{n=1}^{\infty} \frac{9}{8}\left(\frac{9}{16}\right)^n}_{\text{geometric series}}$$

with $a = 9/8$ and $r = 9/16 < 1$

thus $\displaystyle\sum_{n=1}^{\infty} \frac{(3^2)^{n+1}}{2^{4n+3}} = \frac{9/8}{1-9/16} = \frac{18}{7}$. It converges!

10.) $S = \sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

Let's try the Ratio Test,

$$\lim_{n\to\infty}\left[\left|\frac{a_{n+1}}{a_n}\right|\right] = \lim_{n\to\infty}\left[\frac{(n+1)^{2(n+1)}}{(2+n)^{3(n+1)}}\right]\left[\frac{(1+n)^{3n}}{n^{2n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1+n)^{5n+2}}{n^{2n}(n+2)^{3n+3}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1+n)^2\left((1+n)^5\right)^n}{(n+2)^3\left(n^2(n+2)^3\right)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{(n+2)^3}\left[\frac{(n+1)^5}{n^2(n+2)^3}\right]^n\right]$$

$$= \lim_{n\to\infty}\left[\underbrace{\frac{(n+1)^2}{(n+2)^3}}_{\to 0}\cdot\left[\frac{\underbrace{\left(1+\frac{1}{n}\right)^{5n}}_{\to e^5}}{\underbrace{\left(\frac{2}{n}+1\right)^{3n}}_{\to e^6}}\right]\right]$$

$$= 0.$$

I think the ROOT TEST is easier here,

$$\sqrt[n]{|a_n|} = \left(\left[\frac{n^2}{(1+n)^3}\right]^n\right)^{1/n} = \frac{n^2}{(1+n)^3} \longrightarrow 0$$

YEP. ROOT TEST BETTER HERE, S converges by ROOT TEST.

11.) $\sum_{n=1}^{\infty} \frac{n(-3)^n}{(2n)!}$ note $|a_n| = \frac{n \cdot 3^n}{(2n)!}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{(n+1)\,3^{n+1}}{(2(n+1))!}\right)\Bigg/\left(\frac{(2n)!}{n \cdot 3^n}\right)$$

$$= \frac{(n+1)\cancel{3^n}\,3\,(2n)!}{(2n+2)!\,n\cdot\cancel{3^n}}$$

$$= \frac{3(n+1)\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,n}$$

$$= \frac{3(n+1)}{n(2n+2)(2n+1)} \longrightarrow 0 \text{ as } n \to \infty$$

Thus $\sum_{n=1}^{\infty} \frac{n(-3)^n}{(2n)!}$ converges absolutely by Ratio Test.

Remark: 12. could be done with k^{th} term test directly.

12.) $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$ diverges. We can see this by D.C.T. Consider, $3^k < 4^k$ thus

$$\frac{5^k}{3^k+4^k} \geq \frac{5^k}{4^k+4^k} = \frac{1}{2}\left(\frac{5}{4}\right)^k$$

and $\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{5}{4}\right)^k$ diverges by k^{th} term test as $\frac{1}{2}\left(\frac{5}{4}\right)^k \not\to 0$ as $k \to \infty$. Btw.