$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n \}$$

$$x \in \mathbb{R}^n \quad \text{and} \quad x = (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

$x$ is an n-tuple.

We can add, subtract and scalar multiply

$$\left.\begin{aligned} (x+y)_j &= x_j + y_j \\ (x-y)_j &= x_j - y_j \\ (cx)_j &= c\, x_j \end{aligned}\right\} \text{ define } +, -, \cdot \text{ for } \mathbb{R}^n$$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and the standard basis is

$$e_1 = (1, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$e_n = (0, 0, \ldots, 1)$$

In short, $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

And we can write any $v \in \mathbb{R}^n$ as a "<u>linear</u> <u>combination</u>" of the standard basis $\{e_1, e_2, \ldots, e_n\}$

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

Geometry in $\mathbb{R}^n$ is nicely described with the help of the dot-product

If $x, y \in \mathbb{R}^n$ then $x \cdot y \in \mathbb{R}$ s.t.

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \quad (\text{def}^{\underline{n}})$$

~~We can show $x \cdot y =$~~

Furthermore, the length of a vector $x$ is

$$\|x\| = \sqrt{x \cdot x} \quad (\text{def}^{\underline{n}})$$

You can show that

$$x \cdot y = \|x\| \, \|y\| \cos \theta$$

Or you can define $\theta = \cos^{-1}\left[\frac{x \cdot y}{\|x\| \|y\|}\right]$

(see DR. MAVINGA'S NOTES FROM MODERN GEOM. FOR A BETTER CAREFUL DEVELOPMENT OF THIS)

<u>Einstein Notation</u>: (generally not used in this course until certain tasks)

$$x \cdot y = \sum_i x_i y_i = x_i y_i$$

Levi-Civita Symbol

$$* \; A \times B = \sum_{i,j,k} \epsilon_{ijk} A_i B_j e_k = \underbrace{\epsilon_{ijk} A_i B_j}_{\text{repeated index summed over in Einstein notation.}} e_k$$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

# We need to think about subsets of $\mathbb{R}^n$ with special topological properties

## Set Theory

$$A \subseteq B \quad \text{iff} \quad x \in A \Rightarrow x \in B \quad \forall x \in A.$$

$$A \cap B = \{x \mid x \in A \ \underline{\text{and}} \ x \in B\}$$

$$A \cup B = \{x \mid x \in A \ \underline{\text{or}} \ x \in B\}$$

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Lambda\}$$

$$\bigcup_{j=1}^{n} A_j = \{x \mid \underbrace{x \in A_j \text{ for some } j \in \mathbb{N}}_{\text{Countable union.}}\}$$

## Topology in $\mathbb{R}^n$

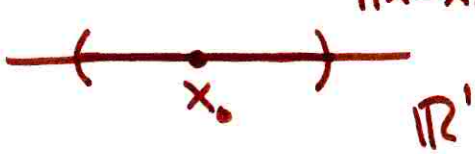
1.) $B_\delta(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \delta\}$

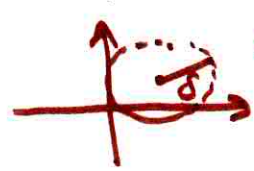
open-ball of radius $\delta$ which is centered at $x_0$.

- Note, later for limits we like to exclude the center so we use deleted open balls $B_\delta(x_0) - \{x_0\} = (B_\delta(x_0))_0$

- $B_\delta(x_0)$ these are our basic <u>open sets</u>

$\|x - x_0\| = |x - x_0|$




open disk

# Topology continued

Euclidean Topology on $\mathbb{R}^n$

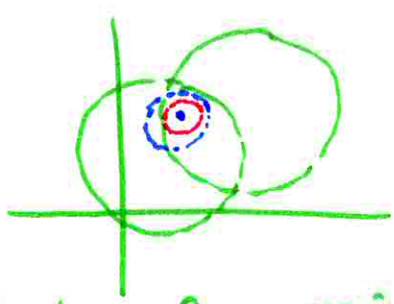
**Def$^n$/** $U \subseteq \mathbb{R}^n$ is said to be <u>open</u> iff all its points are interior points. By <u>interior point</u> we mean $x \in U$ then $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

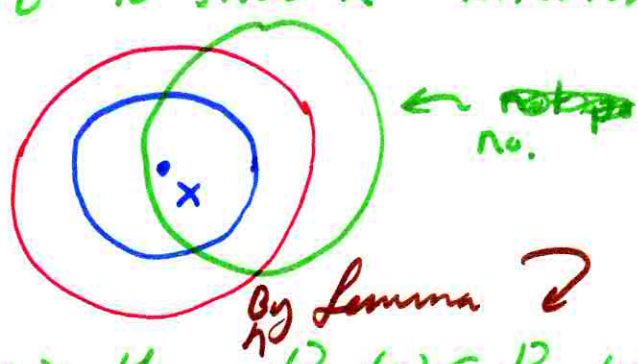
<u>Proposition</u>: the union of $A, B$ open is open.

<u>Proof</u>: Let $x \in A \cup B$ then $x \in A$ or $x \in B$ in either case $\exists\, \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq A$ or $B_\epsilon(x) \subseteq B$ hence $B_\epsilon(x) \subseteq A \cup B$ $\therefore$ $x$ is an interior pt. But $x$ was arbitrary $\therefore$ all pts. in $A \cup B$ are int. Hence $A \cup B$ is open.

<u>Prop</u>: the intersection of $A, B$ open is open

<u>Proof</u>: $x \in A \cap B$ then $\exists\, \epsilon_A, \epsilon_B > 0$ such that $B_{\epsilon_A}(x) \subseteq A$ and $B_{\epsilon_B}(x) \subseteq B$. But, we need to find $B_\delta(x) \subseteq A \cap B$ to show $x$ interior.






Let $\delta = \min(\epsilon_A, \epsilon_B)$ then $B_\delta(x) \subseteq B_{\epsilon_A}(x) \subseteq A$ and $B_\delta(x) \subseteq B_{\epsilon_B}(x) \subseteq B$ $\therefore$ $B_\delta(x) \subseteq A \cap B$

$\therefore$ $A \cap B$ open.

**Lemma:** If $\delta_1 \le \delta_2$ then $B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$

Proof) Let $y \in B_{\delta_1}(x) \Rightarrow \|y - x\| < \delta_1 \le \delta_2$
thus $y \in B_{\delta_2}(x)$ $\therefore$ $B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$. //

## Mappings vs. functions

In Math 200, an abstract idea of function is sometimes presented $f : A \to B$ means the inputs of $f$ are taken from $A$ and result in a single value in $B$. The term "function" is used. We forsake that general idea, if I want to use it I'll say "abstract function"

- In this course a function has a codomain which is in $\mathbb{R}$
- A mapping is an abstract function whose codomain is in $\mathbb{R}^n$.

You might say a mapping is a vector-valued function.

Mappings have component functions

Ex) $f(x) = (f_1(x), f_2(x), f_3(x))$ for $x \in \mathbb{R}$.
$f : \mathbb{R} \to \mathbb{R}^3$

$f$ is a mapping with component functions $f_1, f_2, f_3$
We write $f = (f_1, f_2, f_3)$ with this understanding.

# Matrices, Linear Transformations etc...

A matrix $A \in \mathbb{R}^{m \times n}$ is an array of $m$-rows and $n$-columns.

$$A = [A_{ij}] = \left[ \mathrm{col}_1(A) \,\middle|\, \mathrm{col}_2(A) \,\middle|\, \cdots \,\middle|\, \mathrm{col}_n(A) \right]$$

row column

$$= \begin{bmatrix} \mathrm{row}_1(A) \\ \hline \mathrm{row}_2(A) \\ \hline \vdots \\ \hline \mathrm{row}_m(A) \end{bmatrix}$$

(columns)

Notice $\mathrm{col}_1(A) = (A_{11}, A_{21}, \cdots, A_{m1}) \in \mathbb{R}^m$

whereas $\mathrm{row}_1(A) = [A_{11}, A_{12}, \ldots, A_{1n}] \in \mathbb{R}^{1 \times n}$.

Moreover, $(\mathrm{col}_i(A))_j = A_{ji}$

and $(\mathrm{row}_i(A))_j = A_{ij}$

---

We can add, subtract, multiply, scalar mult...

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(cA)_{ij} = c A_{ij}$$

$$(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj} \quad \text{for } A \in \mathbb{R}^{m \times p}, \; B \in \mathbb{R}^{p \times n}, \; AB \in \mathbb{R}^{m \times n}.$$

A <u>linear transformation</u> is a mapping $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$L(x+y) = L(x) + L(y)$$
$$L(cx) = cL(x)$$

$\forall x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

## Matrix of Linear Transformation $L : \mathbb{R}^n \to \mathbb{R}^m$

$$L(e_i) = \sum_{j=1}^{m} A_{ji}\, \bar{e}_j$$

Because $\{\bar{e}_j\}_{j=1}^{m}$ is a basis for $\mathbb{R}^m$ and $L(e_i) \in \mathbb{R}^m$ $\exists!$ set of coeff. $\{A_{ji}\}$ such that the linear comb. above holds.

Def$^{n}$ $[L] = [A_{ji}]$.

<u>Can calculate</u>

$$\begin{aligned} L(v) &= L\left(\sum_{i=1}^{n} v_i e_i\right) \\ &= \sum_{i=1}^{n} v_i\, L(e_i) \\ &= \sum_{i=1}^{n} v_i \sum_{j=1}^{m} A_{ji}\, \bar{e}_j \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ji} v_i\right) \bar{e}_j = \sum_{j=1}^{m} (Av)_j\, \bar{e}_j = Av. \end{aligned}$$

# Matrix Terminology

**Def$^{n}$/** $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ ← standard basis matrices.

$A \in \mathbb{R}^{m\times n}$ then $A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} E_{ij}$

**Ex/** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$

**Th$^{m}$/** If $L_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $L_2 : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$ then $L_2 \circ L_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_3}$ has $[L_2 \circ L_1] = [L_2][L_1]$

**Proof:** see linear.

## Useful Facts

$e_i^T A = \text{row}_i(A)$

$Ae_i = \text{col}_i(A)$

$E_{ij}E_{kl} = \delta_{jk}E_{il}$

$\underbrace{E_{ij} = e_i e_j^T}_{\text{w/o qualification } I \text{ \& } E_{ij} \text{ square.}}$

$e_i^T e_j = \delta_{ij} = e_i \cdot e_j$
$(1\times n)(n\times 1)$

$A_{ij} = (e_i)^T A e_j$

maybe $\underset{m\times n}{E_{ij}} = \underset{m\times 1}{e_i}\ \underset{1\times n}{\bar{e}_j^T}$

# Functions & Mappings, deep thoughts



$$f(A) = \{ f(a) \mid a \in A \}$$

$$\underbrace{f^{-1}(V)}_{} = \{ x \in \mathrm{dom}(f) \mid \exists v \in V \text{ and } f(x) = v \}$$

set of all things in the domain which map to $V$.

Examples:

- level curve in $\mathbb{R}^2$

  $F(x,y) = k$

  $C = \{ (x,y) \in \mathbb{R}^2 \mid F(x,y) = k \} = F^{-1}(\{k\})$

- parametrized curve in $\mathbb{R}^n$

  $$\begin{aligned} C &= \{ \vec{r}(t) \mid t \in J \subseteq \mathbb{R} \} \\ &= \{ (x_1(t), x_2(t), \ldots, x_n(t)) \mid t \in J \} \\ &= \vec{r}(J) \end{aligned}$$

Let $f : U \longrightarrow V$ then

Def$^{\underline{n}}$/ $f$ is 1-1 or injective iff $f^{-1}\{y\}$ is a singleton for each $y \in \mathrm{range}(f)$.

$f(a) = f(b) \Rightarrow a = b \quad \forall a, b \in \mathrm{dom}(f)$.

$f$ onto $T \subseteq V$ iff $\exists U_1 \subseteq U$ s.t. $f(U_1) = T$

<u>Special Map</u>: the projection $\pi_j : \mathbb{R}^n \to \mathbb{R}$

$$\pi_j(x) = x_j = x \cdot e_j$$

~~Also $\pi_V : \mathbb{R}^n \to V \leq \mathbb{R}^n$~~

~~$$\pi_V(x) = \begin{cases} x & \text{if } x \in V \end{cases}$$~~

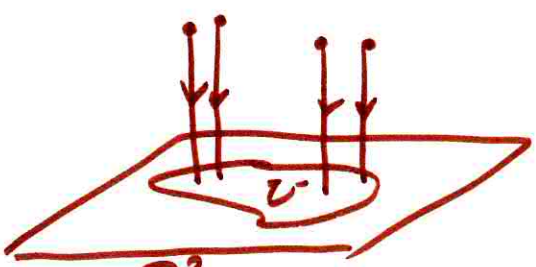
Def$^{\underline{n}}$ 3.1.9.

$\pi_V$ surpicious.

(Ex 3.1.10 good.)

$$\pi_V(x) = \pi_V(x_V + x_\perp)$$
$$= x_V \quad \text{where } x_V \in V, \; x_\perp \in V^\perp$$

Assumes $V \leq \mathbb{R}^n$




$g : \mathbb{R}^3 \to \mathbb{R}^2$

$$g^{-1}\{a, b\} = \{V \leq \mathbb{R}^3 \mid g(V) = \{a, b\}$$

$$g_1(V) = a, \; g_2(V) = b$$

# Continuity of Mappings

Limit DEF$^n$ $f: U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ we say $f$ has limit $b \in \mathbb{R}^m$ at limit point ~~a~~ $a$ of $U$ iff for each $\epsilon > 0$, $\exists \delta > 0$ s.t. $x \in \mathbb{R}^n$ with $0 < \|x - a\| < \delta$ implies $\|f(x) - b\| < \epsilon$. In this case

$$\lim_{x \to a} f(x) = b$$

Def$^n$

Let $f: U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ be a mapping. If $a \in U$ is a limit point of $f$ then we say $f$ is continuous at $a$ iff

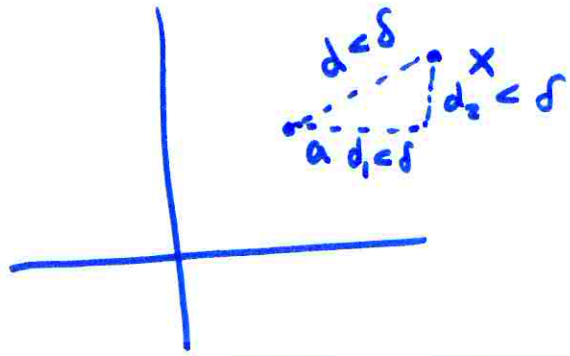
$$\lim_{x \to a} f(x) = f(a).$$

If $a$ is isolated point then by def$^n$ $f$ is cont. at $a$. Then <u>$f$ is cont. on $S$</u> $\subseteq U$ iff $f$ is cont. at ~~$y \in S$ & by~~ each point in $S$. If $f$ is cont. on dom $(f)$ then <u>$f$ is continuous</u>.

<u>Proposition</u>: $\lim_{x \to a} [f(x)] = b$ iff $\lim_{x \to a} f_j(x) = b_j$ $\forall j = 1, 2, \ldots, m$

for $f: \mathbb{R}^n \to \mathbb{R}^m$

<u>Proof</u>: <u>Edwards 7.2</u>.




## Derivative and Differential

Suppose that $U$ is open and $F: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a mapping then we say $F$ is <u>differentiable</u> at $a \in U$ iff $\exists$ a linear mapping $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{h \to 0} \left[ \frac{F(a+h) - F(a) - L(h)}{\|h\|} \right] = 0$$

In such a case we call $L$ the <u>differential at $a$</u> and we denote $L = dF_a$. The <u>matrix</u> of $dF_a$ is called the <u>derivative of $F$ at $a$</u> and we denote $[dF_a] = F'(a)$.

Note: $F'(a) \in \mathbb{R}^{m \times n}$ and $dF_a(v) = F'(a)v$ for all $v \in \mathbb{R}^n$.

where's this from?

$$f'(a) = \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{h} \right) = \lim_{h \to 0} \left( \frac{h f'(a)}{h} \right)$$

$$\hookrightarrow \lim_{h \to 0} \left( \frac{f(a+h) - f(a) - h f'(a)}{h} \right) = 0$$

Ex) $B : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $B(v) = Av$

<u>Claim</u>: $DB_a(v) = Av$ or $DB_a = B$

---

$$\lim_{h \to 0} \left[ \frac{B(a+h) - B(a) - DB_a(h)}{\|h\|} \right]$$

$$\lim_{h \to 0} \left[ \frac{A(a+h) - \cancel{Aa} - Ah}{\|h\|} \right] = \lim_{h \to 0} \left[ \frac{0}{\|h\|} \right] = 0.$$

- $DB_a$ is the best linear approx. to the change in $B$ near $a$.

3/22/2011

# Metric Spaces

The study of $(\mathbb{R}, |\cdot|)$ is real analysis. This is just one example of a general family of theories called metric spaces.

**Def$^{n}$/** A <u>metric space</u> $E$ is a set $E$ together with a <u>distance function</u> $d: E \times E \longrightarrow \mathbb{R}$ such that

(1) $d(p,q) \geq 0 \quad \forall p, q \in E.$

(2.) $d(p,q) = 0$ iff $p = q$.

(3.) $d(p,q) = d(q,p) \quad \forall p, q \in E.$

(4.) $d(p,r) \leq d(p,q) + d(q,r) \quad \forall p,q,r \in E$ ($\Delta$-inequality)

Ex) $\mathbb{R}$ with $d(a,b) = |b-a| = \sqrt{(b-a)^2}$

Ex) $\mathbb{R}^2$ with $d(\vec{v},\vec{w}) = \sqrt{(v_1-w_1)^2 + (v_2-w_2)^2}$

Ex) $\mathbb{R}^n$ with $d(\vec{v},\vec{w}) = \sqrt{(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})} = \underbrace{\|\vec{v}-\vec{w}\|^2}_{\text{squared norm}}$

(see prob. # 22 on pg. 63 Rosenlicht)

Ex) $E_1 \subset E \leftarrow$ metric space is a <u>subspace</u> of $E$. Notice any subset of $E$ will do. We don't face the severe restrictions imposed by other contexts (like linear algebra)

Ex) $\mathbb{R}^2$ with $\|\vec{v}\|_1 = |v_1| + |v_2|$ is called 1-norm. This <u>induces</u> a metric $d_1(\vec{v},\vec{w}) = \|\vec{v}-\vec{w}\|_1$. This can also be done for $\mathbb{R}^n$. Moreover, one studies $\|\vec{v}\|_p = \sqrt[p]{v_1^p + v_2^p + \cdots + v_n^p}$ the $p$-norm. The cases $p = 2$ and $p = 1$ are most common.

# Open & Closed in Metric Space

Def$^{n}$/ ① $B_r(P_0) = \{ P \in E \mid d(P, P_0) < r \}$ open ball
② $\overline{B_r(P_0)} = \{ P \in E \mid d(P, P_0) \leq r \}$ closed ball
③ $S \subseteq E$ is open if, for each $p \in S$, $\exists r > 0$ and $B_r(P) \subset S$.

We can show $\phi$, $E$ and any union of open sets (finite or countable) is open. The intersection of finitely many open sets is open. Also, the open ball is open (see pg. 40).

Def$^{n}$/ $S \subset E$ is closed iff $cS$ is open.

Here we denote $cS = \{ P \in E \mid P \notin S \}$.
Closed balls are closed in the sense of the def$^{n}$ above.
We can show $E$, $\phi$ and the intersection of arbitrarily many closed sets is closed. However, only the <u>finite</u> union of closed sets is necessarily closed.

Def$^{n}$/ $S \subset E$ is <u>bounded</u> iff $\exists r > 0$, $P_0 \in E$ such that $S \subseteq B_r(P_0)$.

# Sequences

Def$^{n}$/ A sequence in $E$ is an ordered list of elements in $E$. Usually it is a function from $\mathbb{N} \to E$ (although other subsets of $\mathbb{Z}$ work, provided they have a smallest element and passes the successor property). Consider $P_1, P_2, P_3, \ldots$ in $E$. A point $p \in E$ is the <u>limit</u> of $P_1, P_2, \ldots$ if $\forall \epsilon > 0$ there exist $N > 0$ with $N \in \mathbb{N}$ such that for $n > N$ we find $d(P, P_n) < \epsilon$. If $P_1, P_2, \ldots$ has a limit point $p$ then we say its <u>convergent</u>.

Comment, usually $N = N(\epsilon)$. Also, it may be $P_1, P_2, \ldots, P_n, \ldots$ converges to $p$ in $E$. However, $P_1, P_2, \ldots$ in $E' \subset E$ does not converge in $E'$ because $p \notin E'$. We can show the limit of a sequence is unique provided a limit exists.

<u>FACT</u>: Any subsequence of a convergent sequence of points in a metric space converges to the same point. (Prop. on pg. 46)

This FACT is extremely useful for counterexamples. Also it tells us convergence really only cares about the tail ($n \gg 1$) of the sequence.

# Sequences continued (basics)

- Any convergent sequence $P_1, P_2, \ldots$ is bounded. $d_j = d(P_j, P)$

$\epsilon > 0 \Rightarrow \exists N \in \mathbb{N}$ and for $n > N$ $|P_n - P| < \epsilon$, $r = \max\{\epsilon, d_1, d_2, \ldots\}$

**Th 7/** $S \subset E$ is closed iff $\lim_{n\to\infty} P_n \in S$ for any conv. seq. $\{P_n\}$.

$N = 7$

Sequences in metric space also have usual limit theorems $\lim a_n \pm b_n = \lim a_n \pm \lim b_n$ and $\lim a_n b_n = \lim a_n \lim b_n$. Also $\lim a = a$

There are nice, readable proofs in Rosenlicht.

$E = \mathbb{R}$

<u>Comparison Test</u>: If $\{a_n\}, \{b_n\}$ are seq. in $E$ such that $a_n \le b_n$ $\forall n$ then $\lim a_n \le \lim b_n$

**Def 3/** If $a_n \le a_{n+1}$ $\forall n \Rightarrow \{a_n\}$ is increasing.
If $b_n \ge b_{n+1}$ $\forall n \Rightarrow \{b_n\}$ is decreasing.
If $\{a_n\}$ is either inc or dec then $\{a_n\}$ is monotonic

<u>Prop.</u> A bounded monotonic seq. is convergent

<u>Ex)</u> If $|a| < 1$ can show $\lim_{n\to\infty} a^n = 0$
(see pg. 51 very cute)

# Completeness

**Def$^{n}$/** A sequence of points $\{P_n\}$ is CAUCHY iff $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(P_n, P_m) < \epsilon$ for any $n, m > N$.

Naturally $N = N(\epsilon)$ in a typical example. Also it's easy to show convergent $\Rightarrow$ Cauchy. However the converse fails to be true. Basically the problem is our given subspace or metric space could be missing some points. Th$^{m}$'s to follow from pg. 52 of Rosenlicht,

- Prop: any subsequence of CAUCHY is CAUCHY.
- Prop: A CAUCHY sequence of points in a metric space is BOUNDED.
- Prop: CAUCHY with convergent subsequence $\Rightarrow$ convergent. (NICE)

**Def$^{n}$/** A metric space $E$ is complete if every Cauchy sequence of pts. in $E$ converges to a pt. of $E$.

Comments: closed subsets of complete space complete. also $\mathbb{R}$ complete and $\mathbb{R}^n$ complete however $\mathbb{Q}$ not complete. (see pg. 53 of Rosenlicht)

- pages 54 – 61 discuss compact/connected and how they connect with completeness for $E$. We now essentially skip to §6 on p. 83 for our purposes.

# DIGRESSION: (OR NOT) MATRIX EXPONENTIAL

$$\boxed{\text{Def}^{n}/ \quad \text{If } A \in \mathbb{R}^{n\times n} \text{ then } e^{A} = I + A + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}}$$

$\left(\frac{A^n}{n!} = \frac{1}{n!}A^n\right)$

Obvious question why does this series (of matrices!) converge for any $A$?
Let's try to make this Def$^{n}$ meaningful.

1.) $E = \mathbb{R}^{n\times n}$ is a metric space where the distance is induced from the following norm; $\|A\|^2 = |A_{11}|^2 + |A_{12}|^2 + \cdots + |A_{1n}|^2 + |A_{21}|^2 + \cdots + |A_{nn}|^2$

Ex: $d\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\right) = \sqrt{(1-5)^2 + (2-6)^2 + (3-7)^2 + (4-8)^2}$.

2.) <u>Why metric space</u>? Notice

$$\Psi : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n^2} \qquad \Psi(A) = \vec{A}$$

$$\Psi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Notice $\Psi\left(B_\epsilon(A_0)\right) = B_\epsilon(\vec{A}_0)$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}\right) < \epsilon$

$\Psi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ has $\sqrt{(a-a_0)^2 + \cdots + (d-d_0)^2} < \epsilon$

$$d(\vec{A}, \vec{A}_0) < \epsilon$$

3.) If $\{\vec{A}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^{n^2}$ with limit $\vec{A}$ then $\{\Psi^{-1}(\vec{A}_n)\}_{n=1}^{\infty}$ is a sequence of matrices in $\mathbb{R}^{n\times n}$ and

- its Cauchy with limit $\Psi^{-1}(\vec{A})$.

<u>Proof</u>: Assume $\{\vec{A}_n\}_{n=1}^{\infty}$ is Cauchy. Let $B_n = \Psi^{-1}(\vec{A}_n)$ for $n \geq 1$. Let $\epsilon > 0$ choose $\tilde{N} > 0$ such that $\forall m,n > \tilde{N}$ we have $d(\vec{A}_m, \vec{A}_n) < \epsilon$. (I can choose such an $\tilde{N}$ b.c. $\{\vec{A}_n\}$ is Cauchy) Let $m,n > \tilde{N}$ then

$$d(\vec{A}_m, \vec{A}_n) < \epsilon$$

$$\Rightarrow d(\Psi^{-1}(\vec{A}_m), \Psi^{-1}(\vec{A}_n)) < \epsilon$$

d for $\mathbb{R}^{n\times n}$

$$d(B_m, B_n) < \epsilon$$

$$\therefore \{B_n\}_{n=1}^{\infty} \text{ is Cauchy.}$$

Thus it follows $\mathbb{R}^n$ complete $\Rightarrow$ <u>$\mathbb{R}^{n\times n}$ complete</u>.

$$\boxed{\text{Def}^{n}\text{/ Let } A_j \in \mathbb{R}^{n\times n} \text{ for all } j \in \mathbb{N}. \text{ Then } A_1 + A_2 + A_3 + \cdots = \lim_{m\to\infty}\left(\sum_{j=1}^{m} A_j\right) = \sum_{j=1}^{\infty} A_j.}$$

(Here, $\lim_{m\to\infty} B_m = B$ iff $\forall \epsilon > 0\ \exists M \in \mathbb{N}$ such that $d(B_m, B) < \epsilon$ for $m > M$.)

Consider

$$I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{m!}A^m = S_m$$

$m > l$)

$$S_m - S_l = I + A + \cdots + \frac{1}{l!}A^l + \frac{1}{(l+1)!}A^{l+1} + \cdots \frac{1}{m!}A^m$$

$$-\left(I + A + \cdots + \frac{1}{l!}A^l\right)$$

$$= \frac{1}{(l+1)!}A^{l+1} + \cdots + \frac{1}{m!}A^m$$

(I'm thinking $d(S_m, S_l)^2 = \|S_m - S_l\|^2$

$$S_m - S_l = \sum_{k=l+1}^{m} \frac{1}{k!}A^k \qquad \|AB\| \le \|A\|\,\|B\|$$

$$\|S_m - S_l\| \le \sum_{k=l+1}^{m} \frac{1}{k!}\|A^k\| \le \overbrace{\sum_{k=l+1}^{m} \frac{1}{k!}\|A\|^k}^{S_m - S_l}$$

$\Delta$-inequality for $\mathbb{R}^{n\times n}$

part of tail of $S_m$

$$S_m = \sum_{m=0}^{\infty} \frac{1}{m!}\|A\|^m$$

Ex) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\qquad e^{At} = ?$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

$$A^4 = A^2 A^2 = I$$
$$A^5 = A^4 A = A$$
$$A^6 = A^4 A^2 = I(-I) = -I$$
$$A^7 = A^4 A^3 = A^3 = -A$$

$$A^{4k} = I, \quad A^{4k+1} = A, \quad A^{4k+2} = -I$$
$$A^{4k+3} = -A \quad \forall k \in \mathbb{N} \cup \{0\}.$$

$$e^{At} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} = \sum_{k=0}^{\infty} \frac{t^{4k}}{(4k)!} I$$

$$+ \sum_{k=0}^{\infty} \frac{t^{4k+1}}{(4k+1)!} A + \sum_{k=0}^{\infty} \frac{t^{4k+2}}{(4k+2)!}(-I) + \sum_{k=0}^{\infty} \frac{t^{4k+3}}{3!}(-A)$$

$$= \left( \sum_{k=0}^{\infty} \frac{t^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{t^{4k+2}}{(4k+2)!} \right) I$$

$$+ \left( \sum_{k=0}^{\infty} \frac{t^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{t^{4k+3}}{(4k+3)!} \right) A$$

$$= \left(1 + \frac{t^4}{4!} + \frac{t^8}{8!} + \cdots - \frac{t^2}{2!} - \frac{t^6}{6!} - \frac{t^{10}}{10!} + \cdots\right) I + \cdots = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

## An indirect method to calculate $e^{At}$:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & -2-\lambda \end{vmatrix} = (\lambda+2)\lambda + 4$$

$$= \lambda^2 + 2\lambda + 4$$

$$= (\lambda+1)^2 + 3$$

---

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \rightsquigarrow \det(A - \lambda I) = -\lambda(-2-\lambda) + 1$$

$$= \lambda^2 + 2\lambda + 1$$

$$= (\lambda+1)^2$$

$\vec{u}_1$ $\quad (A+I)\vec{u}_1 = \begin{bmatrix} +1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

$\vec{u}_2$ $\quad (A+I)\vec{u}_2 = \vec{u}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$u + v = 1 \rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^{At} = e^t\left(I + t(A+I) + \frac{t^2}{2!}(A+I)^2 \cdots\right)$$

$$e^{At}\vec{u}_1 = e^t\vec{u}_1$$

$$e^{At}\vec{u}_2 = e^t(\vec{u}_2 + t\vec{u}_1)$$

$$\boxed{\begin{gathered} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(t) = c_1 e^{At}\vec{u}_1 + \cdots + c_n e^{At}\vec{u}_n \end{gathered}}$$

$$e^{At}[\vec{u}_1 \mid \vec{u}_2] = [e^t\vec{u}_1 \mid e^t(\vec{u}_2 + t\vec{u}_1)]$$

$$\boxed{e^{At} = [e^t\vec{u}_1 \mid e^t(\vec{u}_2 + t\vec{u}_1)][\vec{u}_1 \mid \vec{u}_2]^{-1}}$$

$$t \xmapsto{f} e^{At} \qquad f: \mathbb{R} \to \mathbb{R}^{n\times n}$$

$\frac{d\vec{x}}{dt} = A\vec{x}$ has sol$^{\underline{n}}$ matrix $e^{At}$

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right)$$

$$\frac{d}{dt}[\vec{c}_1|\cdots|\vec{c}_n] \qquad = \sum_{n=0}^{\infty} \frac{d}{dt}\left[\frac{t^n}{n!}A^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{n\,t^{n-1}}{n!}A^n \qquad \frac{n}{n!} = \frac{1}{(n-1)!}$$

$$= A\underbrace{e^{At}}_{A[\vec{c}_1|\vec{c}_2|\vec{c}_3|\cdots|\vec{c}_n]}$$

$$[\vec{c}_1'|\cdots|\vec{c}_n'] = [A\vec{c}_1|A\vec{c}_2|\cdots|A\vec{c}_n]$$

$$\frac{d\vec{c}_j}{dt} = A\vec{c}_j \quad \forall j$$ each column of $e^{At}$ is sol$^{\underline{n}}$.

---

$$\left(e^{At}\right)^{-1} = e^{-At}$$

This shows $e^{At}$ is fundamental sol$^{\underline{n}}$ matrix for system of ODEs $\frac{d\vec{x}}{dt} = A\vec{x}$. In other word, this solves <u>any</u> linear ODE with constant coefficients.

one independent variable. ordinary differential equation

# A couple examples on $e^{At}$'s utility

**Ex** $y'' + 2y' + y = 0$

$2^{nd}$ order ODE with constant coefficients.

Use reduction of order to convert to system of 2 $1^{st}$ order ODE's,

$$x_1 = y \Rightarrow x_1' = x_2$$
$$x_2 = y' \Rightarrow x_2' = y'' = -y - 2y' = -x_1 - 2x_2$$

Hence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}}_{\text{A is the "complementary" matrix to } y''+2y'+y=0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{a.k.a. } \frac{d\vec{x}}{dt} = A\vec{x}$$

no accident!

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix} = \lambda(\lambda+2)+1 = \lambda^2+2\lambda+1 = 0$$

Hence $(\lambda+1)^2 = 0 \therefore \underline{\lambda_1 = \lambda_2 = -1}$.

<u>e-vector</u>: $(A+I)\vec{u}_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\vec{u}_1 = 0 \therefore \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will do.

generalized e-vector order 2 : $(A+I)\vec{u}_2 = \vec{u}_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

find sol$^{\underline{n}}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ works.

Then $\vec{x}(t) = c_1 e^{-t}\vec{u}_1 + c_2 e^{-t}(\vec{u}_2 + t\vec{u}_1)$

$$= \begin{bmatrix} c_1 e^{-t} + c_2(e^{-t} + te^{-t}) \\ -c_1 e^{-t} - c_2 t e^{-t} \end{bmatrix} \begin{matrix} \leftarrow y \\ \leftarrow y' \end{matrix}$$

$y = (c_1 + c_2)e^{-t} + c_2 t e^{-t} = \bar{c}_1 e^{-t} + \bar{c}_2 t e^{-t}$ (double root)

$y' = -(c_1 + \cancel{c_2})e^{-t} + \cancel{c_2}(e^{-t} - te^{-t}) = -c_1 e^{-t} - c_2 t e^{-t}$

INCE Diff. Eq$^{\underline{n}}$'s

(one way to see need for $t$ in double root case)

Ex)

$$A = \begin{bmatrix} -1 & 1 & 0 & & & & \\ 0 & -1 & 1 & & & & \\ 0 & 0 & -1 & & & & \\ & & & 3 & 1 & & \\ & & & 0 & 3 & & \\ & & & & & 2 & 0 \\ & & & & & 0 & 2 \end{bmatrix}$$

Jordan Block matrix
Has standard basis as generalized e-vectors.

we can read e-values $-1, -1, -1, 3, 3, 2, 2$ off the diagonal since $A$ is triangular. More over, it's easy to verify that

$$(A+I)e_1 = 0$$
$$(A+I)e_2 = e_1 \longrightarrow (A+I)^2 e_2 = 0$$
$$(A+I)e_3 = e_2 \longrightarrow (A+I)^3 e_3 = 0$$

$$(A-3I)e_4 = 0$$
$$(A-3I)e_5 = e_4$$

$$(A-2I)e_6 = (A-2I)e_7 = 0$$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has what e-vectors?

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} e_1 = e_1$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} e_2 = 2e_2$

$e_1$ & $e_2$

(diagonal Blocks ~ e-vectors)
non diag. Blocks ~ e-vector & g.e.-vectors.

Th$^{m}$/ For <u>any</u> matrix $B$ with real e-values $\exists P \in GL(n)$ such that $A = P^{-1}BP$ is Jordan Block. The matrix $P$ is comprized of gen. e-vectors concatenated together.

$$\frac{d\vec{x}}{dt} = B\vec{x} \xrightarrow[\text{change}]{\text{coord.}} \frac{d\vec{y}}{dt} = A\vec{y} \xrightarrow[\text{via } e^{At}]{\text{Solve}} \text{sol}^{\underline{n}} \text{ in } \vec{y} \xrightarrow{\text{change to } \vec{x}.}$$

# Sequences of Functions

- Note: §2 and §3 of Rosenlicht we're already covered for us in my notes or Edwards. He has some abstraction to metric spaces but the essential argument is same. Use composition, product, sum, projection to build continuity for polynomials etc....

- §4 & 5 he discusses compactness, uniform continuity etc... we'll add def$^{n}$'s as we need. But let's start on §6.

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Def$^{n}$/ Let $E, E'$ be metric spaces for $n = 1, 2, 3 \ldots$ let $f_n : E \longrightarrow E'$ be a function. If $p \in E$ we say sequence $f_1, f_2, f_3, \ldots$ conv. at $p$ if the seq. $f_1(p), f_2(p), f_3(p), \ldots$ converges in $E'$. We say $f_1, f_2, \ldots$ converges on $E$ (or converges) if the sequences converges at each $p \in E$. We write

$$f(p) = \lim_{n \to \infty} f_n(p).$$

for all $p \in E$. Or $f = \lim_{n \to \infty} f_n$.

---

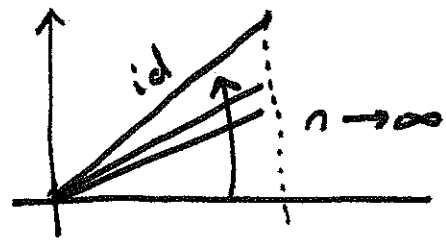
Note to say $\{f_n(p)\}_{n=1}^{\infty}$ converges in $E'$ means for each $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. for all $n > N$ we have $d'(f_n(p), f(p)) < \epsilon$.

At the moment, the necessity for the metric space structure of $E$ is unclear.

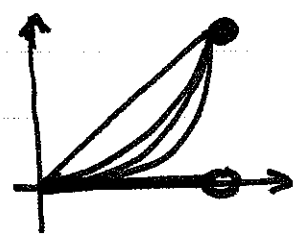
Ex) $f_n : [0,1] \longrightarrow \mathbb{R}$ with $f_n(x) = x - \frac{1}{n}x$

$f_n \longrightarrow id$ as $n \to \infty$, $f_n(x) \longrightarrow x$ as $n \to \infty$.

$id(x) = x$




Ex) $f_n(x) = x^n$ for $f_n : [0,1] \longrightarrow \mathbb{R}$



$$\lim_{n\to\infty} (x^n) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$\{ \underbrace{f_1, f_2, f_3, \ldots}_{\text{all continuous functions}} \}$

BUT discontinuous limit fnct.

Def$^n$/ Let $E, E'$ be metric spaces for $n = 1, 2, 3, \ldots$ let $f_n : E \longrightarrow E'$ and $f : E \longrightarrow E'$ another function. Then the seq $f_1, f_2, f_3 \ldots$ is said to <u>converge uniformly</u> to $f$ if, for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d'(f(p), f_n(p)) < \epsilon$ whenever $n > N$ and $\forall p \in E$.

Ex) $\{x - \frac{1}{n}x\}_{n=1}^{\infty}$ where domain is $[0,1]$ $f_n(x) = x - \frac{1}{n}x$

$$d\left(x - \tfrac{1}{n}x, x\right) = \left|x - \left(x - \tfrac{1}{n}x\right)\right| = \frac{|x|}{n} \le \frac{1}{n} < \epsilon$$

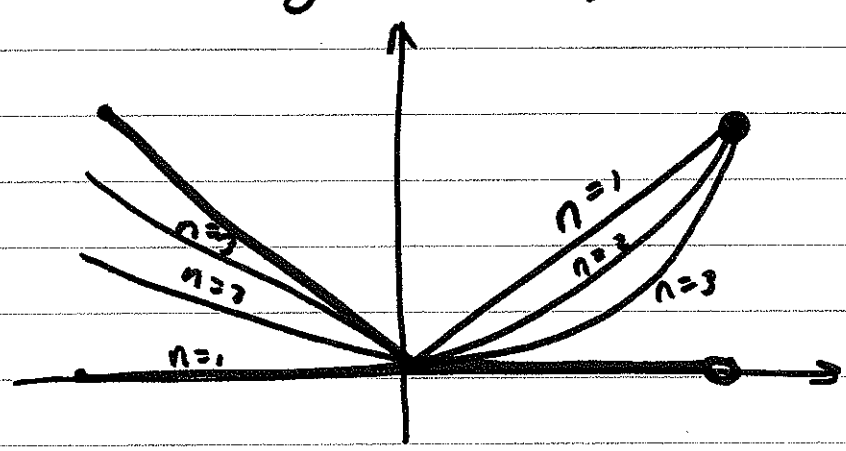
Let $\epsilon > 0$ choose $N = 1/\epsilon$.

**Prop** $E, E'$ be metric spaces, with $E'$ complete for $n = 1, 2, 3, \ldots$ the seq of functions $f_1, f_2, \ldots$ is <u>uniformly</u> <u>convergent</u> iff for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m, n > N \Rightarrow d'(f_m(p), f_n(p)) < \epsilon \ \forall p \in E$.

aka. a sequence of functions is uniformly convergent iff it is a <u>Cauchy</u> <u>Sequence</u> of <u>functions</u>

**Th$^{\underline{m}}$** Let $E, E'$ be metric spaces and $f_1, f_2, \ldots$ be uniformly conv. seq of ~~function~~ continuous funcs from $E \to E'$. Then $\lim_{n \to \infty} f_n$ is continuous funct. from $E \to E'$

Comment about generalizing Th$^{\underline{m}}$ (from §7)




$$f_n(x) = \begin{cases} -x + \frac{x}{n} & -1 \le x \le 0 \\ x^n & 0 \le x \le 1 \end{cases}$$

$\{f_n\}$ conv. uniformly on $(-1, 0)$ but no $(0, 1]$

next, Chapter 7

## Concerning the metric structure of function space (p. 87-90) Rosenlicht

**Lemma:** Let $E$ and $E'$ be metric spaces and let $f, g : E \to E'$ be continuous functions. Then the function $P \longmapsto \underbrace{d'(f(P), g(P))}_{h(P)}$ is continuous on $E$. $\lim_{P \to P_0} h(P) = h(P_0)$

**Proof:** Let $\epsilon > 0$, we need to find $\delta > 0$ s.t. whenever $d(P, P_0) < \delta \Rightarrow |d'(f(P), g(P)) - d'(f(P_0), g(P_0))| < \epsilon$.

Note $f, g$ continuous $\therefore \exists\, \delta_1, \delta_2 > 0$ such that

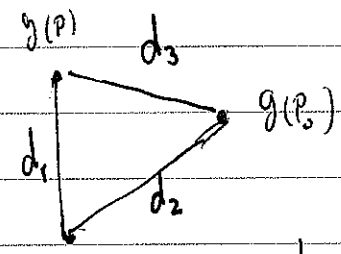
$d(P, P_0) < \delta_1 \Rightarrow d'(f(P), f(P_0)) < \epsilon/2$

$d(P, P_0) < \delta_2 \Rightarrow d'(g(P), g(P_0)) < \epsilon/2$

Let $\delta = \min(\delta_1, \delta_2)$ and consider

$\cancel{|d'(f(P), g(P)) - d'(f(P_0), g(P_0))|}$

$\cancel{\leq |d'(f(P), g(P_0)) + d'(g(P_0), g(P))| + |d'(f(P), f(P_0))|}$




$d_1 - d_2 \leq d_3$

Consider,

$|d'(f(P), g(P)) - d'(f(P_0), g(P_0))| \leq$ $\Delta$-inequality

$\leq |d'(f(P), g(P) - d'(f(P), g(P_0))| + |d(f(P), g(P_0) - d(f(P_0), g(P_0))|$

$\leq |d'(g(P), g(P_0)| + |d(f(P), f(P_0))|$

$\leq \epsilon/2 + \epsilon/2$

$\leq \epsilon \quad \forall P \in E$ s.t. $d(P, P_0) < \delta$. //

**Remark:** in case $E = E' = \mathbb{R}$ this Lemma simply states $P \longmapsto |f(P) - g(P)|$ is continuous fnct on $\mathbb{R}$.

**Def$^n$/** ${}^{\alpha}\mathcal{F}(E, E') = {}^{\alpha}\mathcal{F}$ = set of all continuous functions on $E$.

- next we explain how $E'$ compact gives us a nice idea of distance in ${}^{\alpha}\mathcal{F}$

Def$^{\underline{n}}$/ Let $\mathcal{F} = \{ f : E \to E' \mid E, E'$ metric spaces & $f$ continuous & $E$ compact$\}$
we <u>define</u> $D(f,g) = \max\{ d'(f(p), g(p)) \mid p \in E \}$

<u>Claim</u>: $D$ so defined gives $\mathcal{F}$ a metric space structure.
To begin note $d \circ (f \times g)$ is continuous (by the Lemma) function from $E$ (compact) into $\mathbb{R}$ hence $\exists \max$ for $d \circ (f \times g)$ for each $f, g \in \mathcal{F}$ hence $D : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$
~~is into indeed a binary operat~~

2.) $D(f,f) = \max\{ d'(f(p), f(p)) \mid p \in E \}$ since $d'$ metric!
$= \max\{ 0 \mid p \in E \}$
$= 0$

3.) $D(f,g) = \max\{ d'(f(p), g(p)) \mid p \in E \}$ sym. of $d'$ since $d'$ metric
$= \max\{ d'(g(p), f(p)) \mid p \in E \}$
$= D(g,f)$

3I) $D(f,g) = \max\{ d'(f(p), g(p)) \mid p \in E \}$
$= d'(f(p_0), g(p_0))$ for some $p_0 \in E$ by max value obtained by cont. fnct. on <u>compact</u> space th$^{\underline{m}}$ (★

$\therefore D(f,g) \geq 0$ as $d'(f(p_0), g(p_0)) \geq 0$ as $d'$ is metric.

4.) $D(f,h) = \max\{ d'(f(p), h(p)) \mid p \in E \}$
$= d'(f(p_0), h(p_0))$ (★ once more.)
$\leq d'(f(p_0), g(p_0)) + d'(g(p_0), h(p_0))$ ($\Delta$-ineq. on $d'$)
$\leq \max\{ d'(f(p), h(p)) \mid p \in E \} + \max\{ d'(g(p), h(p)) \mid p \in E \}$
$= D(f,h) + D(h,g)$

$\therefore (\mathcal{F}, D)$ is an <u>abstract</u> metric space.
Here the points are <u>functions</u>.

Ok, so $(\mathcal{F}(E,E'), D)$ where $E$ is compact is a metric space of functions... and what?

Let $f_1, f_2, f_3, \ldots$ be a sequence of functions in $\mathcal{F}$ then we say $f_n \longrightarrow f$ as $n \to \infty$ for some $f \in \mathcal{F}$ iff

$$\lim_{n \to \infty} [D(f, f_n)] = 0$$

To unfold this, iff for each $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. whenever $n \geq N$ we have $D(f, f_n) < \epsilon$.
Which means $\max\{ d'(f(P), f_n(P)) \mid P \in E \} < \epsilon$
which says $d'(f(P), f_n(P)) < \epsilon$ $\forall P \in E$ and $n \geq N$.

THUS (!) The sequence $f_1, f_2, f_3, \ldots \longrightarrow f$ in $\mathcal{F}$ iff the sequence of functions $f_1, f_2, f_3, \ldots$ on $E$ converges uniformly to $f$

$$\underbrace{\text{sequential convergence in } \mathcal{F}}_{\text{max-metric } D} \longleftrightarrow \underbrace{\text{uniform convergence for functions on } E.}_{\text{in terms of } d \,\&\, d' \text{ on } E \,\&\, E'}$$

Continuing, $f_1, f_2, \ldots : E \longrightarrow E' \leftarrow$ (complete)
& $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $\mathcal{F}$ then for all $\epsilon > 0$, $\exists M \in \mathbb{N}$ such that whenever $m, n \geq M$ $D(f_n, f_m) < \epsilon$. Hence $\max\{d'(f_n(P), f_m(P)) \mid P \in E\} < \epsilon$
hence $d'(f_n(P), f_m(P)) < \epsilon$ $\forall P \in E$ & $n, m \geq M$.
Which makes $f_1, f_2, \ldots$ uniformly convergent to $f$ by prop. on page 86 of Rosenlicht. Hence
$f_1, f_2, f_3 \longrightarrow f : E \to E'$ as $n \to \infty$ and as the limit of uniformly convergent seq. of continuous functions is continuous it follows $f \in \mathcal{F}$. Therefore cauchy seq. in $\mathcal{F}$ converge in $\mathcal{F}$ $\therefore$ $\mathcal{F}$ is complete.

Th$^{m}$/ Let $E, E'$ be metric spaces with $E$ compact and $E'$ complete then $\mathcal{F} = \{ f : E \to E' \mid \text{continuous } f \}$ is a <u>complete</u> <u>metric</u> <u>space</u> with distance

$$D(f, g) = \max \{ d(f(P), g(P)) \mid P \in E \}.$$

Moreover, a sequence of points in $\mathcal{F}$ is convergent iff it is a uniformly convergent sequence of functions

(Def$^{n}$/ Let $E$ be compact then $C(E) = \{ f : E \to \mathbb{R} \mid f \text{ continuous} \}$ is a complete metric space of real-valued functions on $E$.)

<u>Remark</u>: many interesting exercises worthy of study on pgs. <u>90-95</u>.
Now we move on to Chapter VII where we should find careful proofs of many central Th$^{m}$'s from calculus II.

To summarize uniform ~~continuity~~ convergence vs. continuity of limit

$$\lim_{P \to P_0} \left[ \left( \lim_{n \to \infty} f_n \right)(P) \right] = \left( \lim_{n \to \infty} f_n \right)(P_0) \quad \leftarrow \text{continuity of limiting fnct.}$$

$$= \lim_{n \to \infty} \left( f_n(P_0) \right)$$

$$= \lim_{n \to \infty} \left( \lim_{P \to P_0} f_n(P) \right)$$

Uniform convergence of $f_1, f_2, \ldots$ (each continuous)
$\Rightarrow$ $f_n \longrightarrow f$ ~~uniform~~ as $n \to \infty$ and that limit $f$ is likewise continuous $\left( \lim_{P \to P_0} f(P) = f(P_0) \right)$

# Uniform Continuity Th$^{m}$'s for Calculus

Th$^{m}$ (p. 138 Rosenlicht) Let $a, b \in \mathbb{R}$ with $a < b$ and $f_1, f_2, f_3, \ldots$ a uniformly convergent sequence of continuous real-valued functions on $[a, b]$. Then

$$\int_a^b \left( \lim_{n\to\infty} f_n(x) \right) dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx$$

Th$^{m}$ (p 140) Let $f_1, f_2, f_3, \ldots$ be sequence of real-valued functions on open interval $U$ in $\mathbb{R}$, each continuously differentiable. Suppose that the seq. $f_1', f_2', f_3'$ converges uniformly on $U$ and that for some $a \in U$ the sequence $f_1(a), f_2(a), f_3(a) \ldots$ converges. Then $\lim_{n\to\infty} f_n$ exists, is differentiable, and

$$\left( \lim_{n\to\infty} f_n \right)' = \lim_{n\to\infty} \left( f_n' \right)$$

<u>Proof</u>: By FTC, $\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a)$ for any $x \in U$ and any $n = 1, 2, 3, \ldots$. Let $\lim_{n\to\infty} f_n' = g$.

$$\lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \int_a^x \lim_{n\to\infty} f_n'(t)\,dt = \int_a^x g(t)\,dt$$

$$\parallel$$

$$\lim_{n\to\infty} \left( f_n(x) - \underbrace{f_n(a)}_{\downarrow} \right) = \int_a^x g(t)\,dt$$

$$f(a) \Rightarrow \lim_{n\to\infty} \left( f_n(x) \right) = f(a) + \int_a^x g(t)\,dt$$

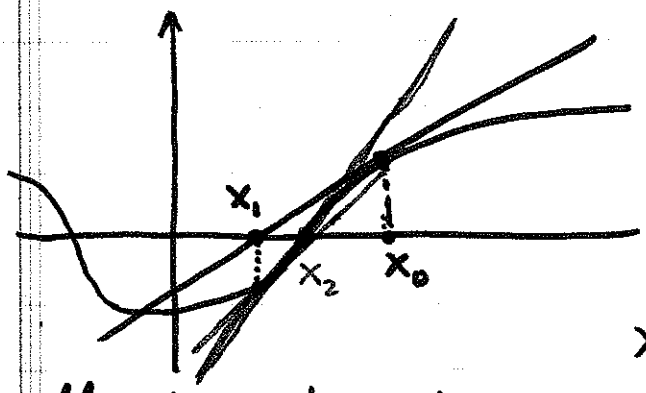
$$\Rightarrow \frac{d}{dx}\left[ \lim_{n\to\infty} f_n(x) \right] = \lim_{n\to\infty} \left[ f_n'(x) \right]$$

//

(I'm just quoting these, §2 I want to dig deeper into today.) <u>Remark</u>: decided to go to Edwards for a bit

# Newton's Method

4/14/2011




$y = f(x)$

$$L^f_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

x-intercept at $L^f_{x_0}(x) = 0$

call this intercept $x_1$ so $0 = f(x_0) + f'(x_0)(x_1 - x_0)$

Solve for $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. Evaluate $f(x_1)$

if it's nonzero then calculate $L^f_{x_1}(x)$,

$$L^f_{x_1}(x) = f(x_1) + f'(x_1)(x - x_1)$$

Say $L^f_{x_1}(x_2) = 0 = f(x_1) + f'(x_1)(x_2 - x_1)$

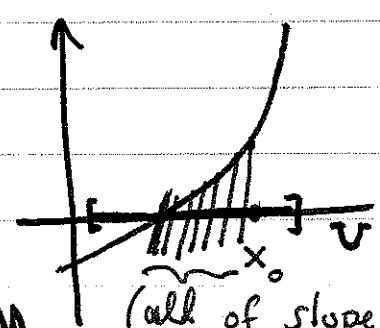
hence $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_0)}$ etc...

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ Standard Newton.

<u>Remark</u>: this method does not always work, can get stuck in an oscillation about $x_*$ where $f(x_*) = 0$ is what we're after. Various modifications possible. We do need $|f'(x)| > 0$ on some nbhd of $x_0$ for reliability of method. If $|f'(x)| \leq M$ for $x \in$ Nbhd $(x_0)$ then we can use

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{M}}$$

Modified Newton.




(all of slope $M$.)

If $f'(x) < 0$ near $x_0$ then use $M$ which has $|f'(v)| \leq M$

$M < f'(x) < 0$. We discuss contraction mappings next ☞

Th$^{m}$/ Let $\varphi : [a,b] \longrightarrow [a,b]$ be a contraction mapping with contraction constant $k$. Then $\varphi$ has a <u>unique</u> fixed point $x_*$ ; $\varphi(x_*) = x_*$. Moreover, given $x_0 \in [a,b]$, the sequence $\{x_n\}_0^\infty$ defined inductively by $x_{n+1} = \varphi(x_n)$ converges to $x_*$. In particular,

$$|x_n - x_*| \leq \frac{k^n |x_0 - x_1|}{1-k} \quad \text{for each } n = 1, 2, \ldots$$

($n=0$ also I guess)

<u>Proof</u>: To say $\varphi$ is contraction mapping on $[a,b]$ with constant of contraction $k$ means $\forall x, y \in [a,b]$ $|\varphi(x) - \varphi(y)| \leq k|x-y|$. Consider then,

$$|x_{n+1} - x_n| = |\varphi(x_n) - \varphi(x_{n-1})| \leq k|x_n - x_{n-1}|$$

& inductively $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|$. (Note for $n=0$ $|x_1 - x_0| = 1^0 |x_1 - x_0|$ so the initial step is valid) continuing with the induction,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= |\varphi(x_{n+1}) - \varphi(x_n)| \\ &\leq k|x_{n+1} - x_n| \\ &\leq k k^n |x_1 - x_0| \\ &\leq k^{n+1} |x_1 - x_0| \end{aligned}$$

Hence $|x_{n+1} - x_n| \leq k^n |x_1 - x_0| \quad \forall n = 0, 1, 2, \ldots$

Suppose $m, n > 0$ and $n < m$ wlog,

$$|x_n - x_m| \leq |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \cdots + x_{m-1} - x_m|$$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$
$$\leq k^n |x_1 - x_0| + k^{n+1} |x_1 - x_0| + \cdots + k^{m-1} |x_1 - x_0|$$
$$\leq (k^n + k^{n+1} + \cdots + k^{m-1}) |x_1 - x_0| \quad \text{sneaky.}$$
$$\leq k^n (1 + k + k^2 + \cdots) |x_1 - x_0|$$
$$\leq \frac{k^n}{1-k} |x_1 - x_0| \qquad (\text{need } k < 1\,!)$$

Therefore if $\epsilon > 0$ then we can bound $|x_n - x_m| < \epsilon$ if we choose $N$ large enough so that when $n > N$ we have $\frac{k^n}{1-k} |x_1 - x_0| < \epsilon$

Explicitly, $k^n = \frac{\epsilon(1-k)}{|x_1 - x_0|} \rightsquigarrow n \ln(k) = \ln\left(\frac{\epsilon(1-k)}{|x_1 - x_0|}\right)$

$n = \frac{1}{\ln(k)} \ln\left[\frac{\epsilon(1-k)}{|x_1 - x_0|}\right]$ then choose $N$ to be the next largest integer. Ok, well technical details aside $|x_n - x_m| \leq \frac{k^n}{1-k} |x_1 - x_0| \longrightarrow 0$ as $n, m \longrightarrow \infty$ since $k$ is a constant with $0 < k < 1$. Therefore, $\{x_n\}_0^\infty$ is Cauchy so, by completeness of $\mathbb{R}$ this sequence converges to some value $x_* \in [a, b]$. (note $[a, b]$ closed and $\{x_0, x_1, \ldots\} \subset [a, b]$ hence it must converge to pt. inside $[a, b]$ by seq. def$^{n}$ of closed set.)

We found the estimate $|x_n - x_m| \leq \frac{k^n |x_1 - x_0|}{1-k}$
Let $m \to \infty$ and find that

$$|x_n - x_*| \leq \frac{k^n |x_1 - x_0|}{1-k}$$

(we take limit $m \to \infty$ for each fixed $n$ to derive the inequality of the Contraction Mapping Th$^{\underline{m}}$.)

<u>Uniqueness</u>? Suppose $\varphi(x_*) = x_*$ and $\varphi(x_{**}) = x_{**}$
observe $|x_* - x_{**}| = |\varphi(x_*) - \varphi(x_{**})|$
$\leq k|x_* - x_{**}|$
and as $k < 1$ it follows $|x_* - x_{**}| = 0$ hence $x_* = x_{**}$ proving uniqueness. //

This Th$^{\underline{m}}$ is at the heart of quite a few iterative analytic arguments. For example, the existence of sol$^{\underline{n}}$'s to ODEs has a contraction mapping argument. For us, we see Newton's Method shrinks $|x_1 - x_0|$ to $|x_2 - x_0|$ .... always contracting the range over which $x_*$ with $f(x_*) = 0$ resides. Let's make this explicit,

**Th$^{m}$ (1.2 p. 164 Edwards)**

Let $f:[a,b]\to\mathbb{R}$ be diff. with $f(a)<0<f(b)$ and $0<m<f'(x)\le M$ for $x\in[a,b]$. Given $x_0\in[a,b]$ the sequence $\{x_n\}_0^\infty$ defined inductively by $x_{n+1}=x_n-\frac{f(x_n)}{M}$ converges to the <u>unique</u> root $x_*\in[a,b]$ of the eq$^{n}$ $f(x)=0$. In particular,

$$|x_n-x_*|\le\left|\frac{f(x_0)}{m}\right|\left(1-\frac{m}{M}\right)^n \quad \text{for each } n=0,1,\dots.$$

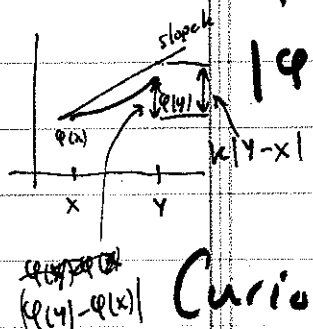
<u>Proof</u>: Let $\varphi:[a,b]\to\mathbb{R}$ be defined by $\varphi(x)=x-f(x)/M$. We argue this is a contraction mapping. Note,

$$\varphi'(x)=1-\frac{f'(x)}{M}\le 1-\frac{m}{M}\text{ "}=k\text{" } \underline{\text{since } m<f'(x)}.$$

Let $k=1-m/M$ and note $m<M$ so $\frac{m}{M}<1$ and $1-\frac{m}{M}>0$ thus $\varphi'(x)>0$ for $x\in[a,b]$. Therefore,

$$a<a-\frac{f(a)}{M}=\varphi(a)\le\varphi(x)\le\varphi(b)=b-\frac{f(b)}{M}<b$$

for all $x\in[a,b]$ where the outside $<$ & $<$ followed from the assumption $f(a)<0<f(b)$. This shows range$(\varphi)\subset[a,b]$ hence $\varphi:[a,b]\to[a,b]$. Let $x,y\in[a,b]$ and consider,




$$|\varphi(x)-\varphi(y)|=\left|x-\frac{f(x)}{M}-y-\frac{f(y)}{M}\right| \quad ?$$
$$=\left|x-y-\frac{1}{M}(f(x)-f(y))\right|\le k|x-y|$$

<u>Curious</u>: Edwards is content to say $\varphi'(x)\le k$ for $k=1-\frac{m}{M}$, why is that sufficient? (need $|\varphi(x)-\varphi(y)|\le k|x-y|$

Consider $|\varphi(x) - \varphi(x_0)| \overset{?}{\leq} k|x - x_0|$

$\Leftrightarrow \quad (\varphi(x) - \varphi(x_0))^2 \leq k^2 (x - x_0)^2$ ?

$$\frac{k^n}{1-k}|x_0 - x_1| = \frac{\cancel{M}}{m}\left(1 - \frac{m}{M}\right)^n \left|\frac{f(x_0)}{\cancel{M}}\right|$$

$$k = 1 - \frac{m}{M} \qquad x_1 = x_0 - \frac{f(x_0)}{M}$$

$$|x_0 - x_1| = \left|\frac{f(x_0)}{M}\right|$$

$$y - b = f(x) - f(a) \approx f'(a)(x - a)$$

# Examples for Inverse Mapping Th$^{m}$

- Here I find a few inverses so we can appreciate the Th$^{m}$.

$\boxed{E1}$ Let $F(x,y) = (x^2+y^2, x^2-y^2)$
Let $(a,b) = (x^2+y^2, x^2-y^2)$.
We need to solve these for $x$ & $y$,

$$a = x^2+y^2 \quad \nearrow \quad a+b = 2x^2$$
$$b = x^2-y^2 \quad \searrow \quad a-b = 2y^2$$

Thus, $x = \pm\sqrt{\frac{1}{2}(a+b)}$ & $y = \pm\sqrt{\frac{1}{2}(a-b)}$.
To specify $\pm$ we need to decide where we seek the inverse. For example,

$$F|_{U}^{-1}(a,b) = \left(\sqrt{\tfrac{1}{2}(a+b)}, -\sqrt{\tfrac{1}{2}(a-b)}\right)$$

for $U = [0,\infty) \times (-\infty, 0]$. The larger point to see here is $\nexists\, F|_{V}^{-1}$ if $V \subseteq \mathbb{R}^2$ which nontrivially overlaps the $x$ or $y$ axis. The local inverse can only work for one quadrant at a time.

**WHY?** note $F'(x,y) = \left[\frac{\partial F}{\partial x} \middle| \frac{\partial F}{\partial y}\right] = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$

thus $\det(F'(x,y)) = -4xy$. Clearly the derivative is singular along either $x=0$ ($y$-axis) or $y=0$ ($x$-axis).

$\boxed{E2}$ Let $F(x) = Ax$ where $A \in \mathbb{R}^{n\times n}$.
We know from linear algebra etc... the inverse exists iff $\det(A) \neq 0$ and in that case $F^{-1}(x) = A^{-1}x$. (or $F^{-1}(y) = A^{-1}y$ if you wish to say $F(x) = y$ and reserve notation)

<u>What about $F'$?</u> It's not hard to prove from the definition $F'(x) = A$ hence $\det(F'(x)) = \det(A)$ so clearly linear algebra and our inverse mapping Th$^{\underline{m}}$

$\boxed{E3}$ Let $\psi(r,\theta,\varphi) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$
Let $(x,y,z) = \psi(r,\theta,\varphi)$ and solve for $r,\theta,\varphi$,
You can verify $r^2 = x^2+y^2+z^2$ and we can insist $r \geq 0$ so $r = \sqrt{x^2+y^2+z^2}$
Continuing, $z = r\cos\varphi \hookrightarrow \varphi = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]$
Also, $\tan\theta = y/x \hookrightarrow \theta = \tan^{-1}[y/x]$
Now, these are local formulas defined naturally for regions with $0 < \varphi < \pi$ & $-\pi/2 < \theta < \pi/2$ and for convenience $0 \leq r < \infty$.

<u>Let's See</u>: $|\psi'(r,\theta,\varphi)| = \left|\left[\frac{\partial\psi}{\partial r}\middle|\frac{\partial\psi}{\partial\theta}\middle|\frac{\partial\psi}{\partial\phi}\right]\right|$

$$= \left|\begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & +r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}\right|$$

$$= \cos\phi\left(-r^2\sin^2\theta\sin\phi\cos\phi - r^2\cos^2\theta\sin\phi\cos\phi\right)$$
$$-r\sin\phi\left(r\cos^2\theta\sin^2\phi + r\sin^2\theta\sin^2\phi\right)$$
$$= -r^2\left(\cos^2\phi\sin\phi + \sin^3\phi\right) = \underline{-r^2\sin\phi}.$$

$\det[\psi'(r,\theta,\varphi)] = 0$ at origin and where $\sin\phi = 0 \hookrightarrow \phi = n\pi$ for $n \in \mathbb{Z}$. $\left(\begin{matrix}0<\phi<\pi \\ \text{usually} \\ \text{choice}\end{matrix}\right)$

Question: for $\boxed{E3}$ the periodicity of $\cos\theta$ and $\sin\theta$ should limit us to a $2\pi$ - length domain for $\theta$. This is apparently not seen from the examination of $\Psi'(r,\theta,\phi)$. But, this is not a deficiency of the inverse mapping Th$^{m}$. Remember, the Th$^{m}$ only says $\exists$ a <u>local</u> inverse in some nbhd about a point where the derivative mapping is non-singular. It will not reveal the trouble with $\theta$, some other idea is needed. For now we content ourselves with a case by case analysis, the case of a global inverse (E2) is special.

$\boxed{E4}$ Consider $F(r,\theta) = (r\cos\theta, r\sin\theta)$
then $F^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \tan^{-1}(y/x)\right)$.

<u>Note</u>: $F'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$\det(F'(r,\theta)) = r\cos^2\theta + r\sin^2\theta = r$

Again, the fact that $\cos(\theta + 2\pi k) = \cos\theta$ is not revealed by $F'(r,\theta)$. Only the degeneracy of $r = 0$ is apparent.

# Examples for Implicit Function Th$^{m}$

$\boxed{E1}$ Suppose $ax+by+cz=0$ then we can solve for

(1) $x = -\frac{b}{a}y - \frac{c}{a}z$ $\quad (x = x(y,z))$

(2) $y = -\frac{a}{b}x - \frac{c}{b}z$ $\quad (y = y(x,z))$

(3) $z = -\frac{a}{c}x - \frac{b}{c}y$ $\quad (z = z(x,y))$

HERE $F(x,y,z) = ax+by+cz = 0$ gives level surface and

(1) $\partial F/\partial x = a \neq 0 \Rightarrow$ can write level surface as graph $x = f(y,z)$.

(2) $\partial F/\partial y = b \neq 0 \Rightarrow y = g(x,z)$ gives $F^{-1}\{0\}$.

(3) $\partial F/\partial z = c \neq 0 \Rightarrow F^{-1}\{0\}$ can be written as graph of $z = h(x,y)$.

$\boxed{E2}$ Suppose $x^2+y^2+z^2 = R^2$. Here we can solve $z = \pm\sqrt{R^2-x^2-y^2}$ locally.

HERE $F(x,y,z) = x^2+y^2+z^2-R^2 = 0$ can be implicitly (locally) solved about a point with $\frac{\partial F}{\partial z} = 2z \neq 0$ for $z = f(x,y)$. The $\pm$ in the algebra reflects this reality. Local sol$^{n}$'s above or below $z=0$ exist (but not accross $z=0$).

$\boxed{E3}$ Consider the set of eq$^{n}$'s,

$$x + y + z = 3$$
$$x^2 + y^2 = 1$$

We have 3 variables and <u>two</u> seemingly independent eq$^{n}$'s, we should expect only one genuine d.o.f. survives as we simultaneously solve these eq$^{n}$'s, lets keep $x$,

$$y = \pm\sqrt{1-x^2}$$
$$z = 3 - x - y = 3 - x \mp \sqrt{1-x^2}$$

Thus $G(x) = (\pm\sqrt{1-x^2},\ 3 - x \mp \sqrt{1-x^2}) = (y, z)$ provides local sol$^{n}$ to system in the sense $H(x,y,z) = (x+y+z-3,\ x^2+y^2-1)$ has

$$H^{-1}(\{0\}) \longleftrightarrow \underbrace{(y,z) = G(x)}_{\approx\ y\ \&\ z \text{ are graph of } G.}$$

<u>Does $H'(x,y,z)$ tell us anything</u>?

$$H'(x,y,z) = \left[\frac{\partial H}{\partial x} \middle| \frac{\partial H}{\partial y} \middle| \frac{\partial H}{\partial z}\right] = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 0 \end{bmatrix}$$

see $\det\begin{bmatrix} 1 & 1 \\ 2y & 0 \end{bmatrix} = -2y \neq 0$ for $y \neq 0$ we cannot cross $y = 0$, this is reflected in the algebra $y = \pm\sqrt{1-x^2}$, we have to choose $(+)$ or $(-)$, can't have both.

note $\det\begin{bmatrix} 1 & 1 \\ 2x & 2y \end{bmatrix} = 2y - 2x = 2(y - x)$ so we cannot parametrize sol$^{n}$ by $z$ ~~along~~ over line $y = x$. Try to solve in terms of $z$, see what happens.

# Multilinearity: (suppose $V$ is vectorspace over $\mathbb{R}$ throughout) ①

- **Bilinearity:**

$b: V \times V \longrightarrow \mathbb{R}$ is bilinear iff

1.) $b(x+y, z) = b(x, z) + b(y, z) \quad \forall x, y, z \in V.$

2.) $b(x, y+z) = b(x, z) + b(x, y) \quad \forall x, y, z \in V.$

3.) $b(cx, y) = c\, b(x, y) \quad \forall x, y \in V, c \in \mathbb{R}$

4.) $b(x, cy) = c\, b(x, y) \quad \forall x, y \in V, c \in \mathbb{R}.$

Basically $b$ is a bilinear mapping on $V$ iff it's a function from $V \times V \longrightarrow \mathbb{R}$ which is linear in each "slot".

① Ex $b(x, y) = x \cdot y$ for $x, y \in \mathbb{R}^n$

② Ex $b(\vec{x}, \vec{y}) = (\vec{x} \times \vec{y}) \cdot e_1$ for $\vec{x}, \vec{y} \in \mathbb{R}^3$

$$
\begin{aligned}
b(\vec{x}_1 + \vec{x}_2, \vec{y}) &= [(\vec{x}_1 + \vec{x}_2) \times \vec{y}] \cdot e_1 \\
&= (\vec{x}_1 \times \vec{y}) \cdot e_1 + (\vec{x}_2 \times \vec{y}) \cdot e_1 \\
&= b(\vec{x}_1, \vec{y}) + b(\vec{x}_2, \vec{y}).
\end{aligned}
$$

Notice Ex ① and ② have nice prop. ②

① $b(x,y) = x \cdot y = y \cdot x = b(y,x)$

this makes $b$ <u>symmetric</u>.

② $b(\vec{x},\vec{y}) = (\vec{x} \times \vec{y}) \cdot e_1$

$= (-\vec{y} \times \vec{x}) \cdot e_1$

$= -b(\vec{y},\vec{x})$

this makes $b$ <u>antisymmetric</u>.

Notice $b(\vec{x},\vec{y}) = \vec{x} \cdot \vec{y} + (\vec{x} \times \vec{y}) \cdot e_1$ for $\vec{x},\vec{y} \in \mathbb{R}^3$

then $b$ is neither sym. or antisym.

Thm/ $T_2^0(V) = \{ b \mid b$ is bilinear form on $V \}$ forms a vector space w.r.t. the natural defn of function addition; $\forall x,y \in V$ & $\forall b_1, b_2 \in T_2^0(V)$

1.) $(b_1 + b_2)(x,y) \equiv b_1(x,y) + b_2(x,y)$

2.) $(c\, b_1)(x,y) = c\, b_1(x,y)$

<u>Proof:</u> Show $b_1, b_2 \in T_2^0(V) \Rightarrow b_1 + b_2 \in T_2^0(V)$

I'd show $(b_1+b_2)(x+y,z) = (b_1+b_2)(x,z) + (b_1+b_2)(y,z)$

# Bilinear Forms & Coordinates:

$V = \text{span}\{e_i \mid i = 1, 2, \ldots, n\} = \text{span}\,\beta$

$V$ is finite dimensional & $\dim(V) = n$

<u>Consider</u>: If $x, y \in V$ then $\exists\, x^i, y^j \in \mathbb{R}$ $\forall\, i, j \in \mathbb{N}_n = \{1, 2, \ldots, n\}$ such that

$$x = \sum_{i=1}^{n} x^i e_i \qquad [x^i] = [x]_\beta = \Phi_\beta(x).$$

$$y = \sum_{j=1}^{n} y^j e_j$$

With the above in mind,

$$b(x, y) = b\left(\sum_i x^i e_i, \sum_j y^j e_j\right)$$

$$= \sum_i x^i\, b\left(e_i, \sum_j y^j e_j\right)$$

$$= \sum_i \sum_j x^i y^j\, b(e_i, e_j)$$

Apparently the values of $b$ on basis $\beta$ determine the action of $b$ over all of $V$.

# Reminder about Dual Space $V^*$

$$V^* = \{ f : V \to \mathbb{R} \mid f \text{ is linear} \}$$

The dual space is the set of all linear functionals on $V$.

Ex) $\alpha(v) = v \cdot w$ for $v \in \mathbb{R}^n$ and $w$ some fixed vector which makes $\alpha$ what it is.

$$\alpha(cv_1 + v_2) = (cv_1 + v_2) \cdot w$$
$$= cv_1 \cdot w + v_2 \cdot w = c\alpha(v_1) + \alpha(v_2)$$

In fact <u>any</u> linear functional on $\mathbb{R}^n$ can be expressed as a dot-product

$$\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$$

(stand basis this time)

$$\alpha(x) = \alpha\left(\sum_{i=1}^{n} x^i e_i\right) = \sum_{i=1}^{n} x^i \alpha(e_i)$$

$$= x \cdot w = w^T x$$

where $(w)_i = \alpha(e_i)$. Some people take the viewpoint $(\mathbb{R}^{n \times 1})^* = \mathbb{R}^{1 \times n}$

$$\alpha \longleftrightarrow [\alpha(e_i)] = w^T$$

## Continuing on the dual space $V^* = T^0_1(V)$

**Def$^{\underline{n}}$/** If $V = \text{span}\{e_i\}_{i=1}^n$ then $V^* = \text{span}\{e^j\}_{j=1}^n$ where $e^j(e_i) = \delta^j_i$ $\forall\, i,j \in \mathbb{N}_n$

**Why is this reasonable?**

1.) $e^j : V \longrightarrow \mathbb{R}$ has $e^j(e_i) = \delta^j_i$

we've defined a fnct. on a basis of $V$ so we extend linearly to obtain a linear function meaning $e^j \in V^*$.

$$\underline{e^j(x)} \overset{\text{def}^n}{=} \sum_{i=1}^n x^i \underbrace{e^j(e_i)}_{\delta^j_i} = \underline{x^j} \; *$$

2.) Why is $V^*$ spanned by $\beta^* = \{e^j\}_{j=1}^n$

Let $\alpha \in V^*$, $x \in V$

$$\begin{aligned} \alpha(x) &= \alpha\left(\sum_{i=1}^n x^i e_i\right) \\ &= \sum_{i=1}^n x^i \alpha(e_i) \\ &= \sum_{i=1}^n \alpha(e_i)\, e^i(x) \quad * \quad \underline{\underline{\forall x}} \end{aligned}$$

$$\therefore \; \alpha = \sum_{i=1}^n \alpha(e_i)\, e^i \in \text{span}\{e^i\}_{i=1}^n$$

# Return to bilinear forms

**Question:** what is the natural basis for the vector space $T_2^0(V)$?

**Answer:** tensor product of dual-basis

**Def$^{n}$/** Let $\alpha, \gamma \in V^*$ then $\underline{\alpha \otimes \gamma \in T_2^0(V)}$ ? defined by: $\forall x, y \in V$

$$(\alpha \otimes \gamma)(x, y) = \alpha(x)\,\gamma(y)$$

**Proof:**

$$\begin{aligned}(\alpha \otimes \gamma)(x_1 + c x_2, y) &= \alpha(x_1 + c x_2)\,\gamma(y) \\ &= (\alpha(x_1) + c\,\alpha(x_2))\,\gamma(y) \\ &= \alpha(x_1)\,\gamma(y) + c\,\alpha(x_2)\,\gamma(y) \\ &= (\alpha \otimes \gamma)(x_1, y) + c(\alpha \otimes \gamma)(x_2, y)\end{aligned}$$

**Prop:**

$$(\alpha_1 + \alpha_2) \otimes \gamma = \alpha_1 \otimes \gamma + \alpha_2 \otimes \gamma$$

$$\alpha \otimes (\gamma_1 + \gamma_2) = \alpha \otimes \gamma_1 + \alpha \otimes \gamma_2$$

Likewise constants factor out

$$\alpha \otimes (c\gamma) = c\,\alpha \otimes \gamma.$$

**CAUTION:** usually $\alpha \otimes \gamma \neq \gamma \otimes \alpha$.

Continuing with $\otimes$ for $T_2^0(V)$

Roughly speaking $T_1^0(V) \otimes T_1^0(V) = T_2^0(V)$

**Thm/** $\operatorname{span}\{e^i \otimes e^j \mid i,j \in \mathbb{N}_n\} = T_2^0(V)$.

<u>Proof</u>) Let $b \in T_2^0(V)$. On pg. 3 we found

$$b(x,y) = \sum_i \sum_j b(e_i, e_j)\, x^i y^j$$

$$= \sum_i \sum_j b(e_i, e_j)\, e^i(x)\, e^j(y)$$

$$= \sum_i \sum_j b(e_i, e_j)\, (e^i \otimes e^j)(x,y)$$

$$= \left( \sum_i \sum_j b(e_i, e_j)\, e^i \otimes e^j \right)(x,y)$$

Therefore, $\underline{b = \sum_{i,j} b(e_i, e_j)\, e^i \otimes e^j}$.

<u>Consequence</u>: $\dim(T_2^0(V)) = n^2 = \dim(\mathbb{R}^{n \times n})$

So $\exists$ some isomorphism between bilinear forms and $n \times n$ matrices. (see $\circlearrowright$)

# Reminder about Isomorphism

$$\begin{tikzcd}
V_1 \arrow[rr, "\Phi_2^{-1}\circ\Phi_1"] \arrow[d, "\Phi_1"'] & & V_2 \arrow[d, "\Phi_2"] \\
\mathbb{R}^{\dim(V_1)} \arrow[rr, "Id_n"] & & \mathbb{R}^{\dim(V_2)}
\end{tikzcd}$$

provided $\dim V_1 = \dim V_2 = n$.

$\Phi_2^{-1} \circ \Phi_1 : V_1 \longrightarrow V_2$ provides an isomorphism

<u>Prop</u>: $\Phi : T_2^0(V) \longrightarrow \mathbb{R}^{n\times n}$ where $\dim V = n$
defined by $(\Phi(b))_{ij} = b(e_i, e_j)$.
Usually, we say $(\Phi(b))_{ij} = [b]_\beta$

<u>Ex</u>] $b(x,y) = x \cdot y$ for $x, y \in \mathbb{R}^n$

$$= x^T y$$

$$= x^T I y \qquad [b] = I_{n\times n}$$

**Proposition:** the matrix of a symmetric (antisymmetric) bilinear form is symmetric (antisymmetric)

**Proof)** $(\Phi(b))_{ij} = b(e_i, e_j)$
$= b(e_j, e_i)$ symmetric
$= (\Phi(b))_{ji}$

$\therefore [b]^T = [b]$ provided $b$ symmetric

Likewise $[b]^T = -[b]$ if $b$ antisymm.

direct sum.

**Th^m/** $T_2^0(V) = S_2^0(V) \oplus AS_2^0(V)$
although I like to write $AS_2^0 = \Lambda_2(V)$

**Proof)** to say $W = W_1 \oplus W_2$ means
$W_1 \leq W$ and $W_2 \leq W$ and $W_1 \cap W_2 = \{0\}$.
and $\forall w \in W \ \exists w_1 \in W_1, w_2 \in W_2$ s.t. $w = w_1 + w_2$.

Suppose $b \in S_2^0(V)$ <u>and</u> $b \in AS_2^0(V)$ then

$b(x,y) = b(y,x)$ <u>and</u> $b(x,y) = -b(y,x)$.

$\forall x, y \in V$ so in particular $x = e_i$ & $y = e_j$

$\Rightarrow b(e_i, e_j) = -b(e_j, e_j) \ \forall ij$

$\therefore b(e_i, e_j) = 0 \ \forall i, j$.

Proof Continued:

$$b(x,y) = \underbrace{\tfrac{1}{2}\big(b(x,y) - b(y,x)\big)}_{\text{antisymmetric}} + \underbrace{\tfrac{1}{2}\big(b(x,y) + b(y,x)\big)}_{\text{symmetric}}$$

$$= b_1(x,y) + b_0(x,y)$$ ← not standard notation this $b_0$ & $b_1$ business.

Show $b_0 \in S^0_2(V)$ and $b_1 \in \Lambda_2(V)$ that completes the proof.

$$b_0(e_i, e_j) = \tfrac{1}{2}\big(b(e_i, e_j) + b(e_j, e_i)\big)$$
$$= \tfrac{1}{2}(b_{ij} + b_{ji})$$

---

$$b_1(e_j, e_i) = \tfrac{1}{2}(b_{ij} - b_{ji})$$

$$b = \sum_{i,j} b_{ij}\, e^i \otimes e^j = \sum_{i,j} \tfrac{1}{2}(b_{ij} + b_{ji})\, e^i \otimes e^j + \sum_{i,j} \tfrac{1}{2}(b_{ij} - b_{ji})\, e^i \otimes e^j$$

Let's expand on the calculation in 10 over $\mathbb{R}^3$

$$
\begin{aligned}
b = \; & b_{11}\, e^1\otimes e^1 + b_{12}\, e^1\otimes e^2 + b_{13}\, e^1\otimes e^3 \\
& b_{21}\, e^2\otimes e^1 + b_{22}\, e^2\otimes e^2 + b_{23}\, e^2\otimes e^3 \\
& b_{31}\, e^3\otimes e^1 + b_{32}\, e^3\otimes e^2 + b_{33}\, e^3\otimes e^3
\end{aligned}
$$

$$
\begin{aligned}
= \; & b_{11}\, e^1\otimes e^1 + b_{22}\, e^2\otimes e^2 + b_{33}\, e^3\otimes e^3 \\
& + \tfrac{1}{2}(b_{12}+b_{21})\, e^1\otimes e^2 + \tfrac{1}{2}(b_{21}+b_{12})\, e^2\otimes e^1 \\
& + \tfrac{1}{2}(b_{13}+b_{13})\, e^1\otimes e^3 + \tfrac{1}{2}(b_{31}+b_{13})\, e^3\otimes e^1 \\
& + \tfrac{1}{2}(b_{23}+b_{32})\, e^2\otimes e^3 + \tfrac{1}{2}(b_{32}+b_{23})\, e^3\otimes e^2 \\
& + \tfrac{1}{2}(b_{12}-b_{21})\, e^1\otimes e^2 + \tfrac{1}{2}(b_{21}-b_{12})\, e^2\otimes e^1 \\
& + \tfrac{1}{2}(b_{13}-b_{13})\, e^1\otimes e^3 + \tfrac{1}{2}(b_{31}-b_{13})\, e^3\otimes e^1 \\
& + \tfrac{1}{2}(b_{23}+b_{32})\, e^2\otimes e^3 + \tfrac{1}{2}(b_{32}-b_{23})\, e^3\otimes e^2
\end{aligned}
$$

$$b_{11} = \tfrac{1}{2}(b_{11}+b_{11})$$

---

$$
\begin{aligned}
= \; & b_{11}\, e^1\otimes e^1 + b_{22}\, e^2\otimes e^2 + b_{33}\, e^3\otimes e^3 \\
& + \tfrac{1}{2}(b_{12}+b_{21})\left[e^1\otimes e^2 + e^2\otimes e^1\right] \\
& + \tfrac{1}{2}(b_{13}+b_{31})\left[e^1\otimes e^3 + e^3\otimes e^1\right] \\
& + \tfrac{1}{2}(b_{23}+b_{32})\left[e^2\otimes e^3 + e^3\otimes e^2\right] \\
& + \tfrac{1}{2}(b_{12}-b_{21})\left[e^1\otimes e^2 - e^2\otimes e^1\right] + \tfrac{1}{2}(b_{13}-b_{31})\left[e^1\otimes e^3 - e^3\otimes e^1\right] \\
& + \tfrac{1}{2}(b_{23}-b_{32})\left[e^2\otimes e^3 - e^3\otimes e^2\right].
\end{aligned}
$$

6 indep. basis elements spanned.

<u>In matrix notation:</u>

$$\mathbb{R}^{3\times 3} = \operatorname{span}\{E_{ii} \mid i=1,2,3\} \cup \{E_{ij}+E_{ji} \mid i<j,\ i,j\in\mathbb{N}_3\}$$

$$\oplus \operatorname{span}\{E_{ij}-E_{ji} \mid i<j,\ i,j\in\mathbb{N}_3\}.$$

<u>Goal</u>: $\sum b_{ij}\, e^i\otimes e^j = \sum \quad e^i\wedge e^j$

$$\boxed{e^i\wedge e^j \underset{\substack{\uparrow \\ \text{def}^{\underline{n}} \text{ of} \\ \text{wedge product} \\ \text{of two dual-vectors}}}{=} \frac{1}{2}\left(e^i\otimes e^j - e^j\otimes e^i\right)}$$

From (11) if $b(x,y) = -b(y,x)$ then

$$b = (b_{12}-b_{21})\, e^1\wedge e^2 + (b_{13}-b_{31})\, e^1\wedge e^3$$

$$+(b_{23}-b_{32})\, e^2\wedge e^3. \qquad b_{ij} = -b_{ji}$$

$$= 2b_{12}\, e^1\wedge e^2 + 2b_{13}\, e^1\wedge e^3 + 2b_{23}\, e^2\wedge e^3$$

$$= \sum_{i<j} 2\, b_{ij}\, e^i\wedge e^j.$$

<u>Physicists don't like bases:</u>

$\{e^1\wedge e^2, e^1\wedge e^3, e^2\wedge e^3\}$ is basis for $\Lambda_2(\mathbb{R}^3)$

$$\rightarrow b = \sum_{i,j} b_{ij}\, e^i\wedge e^j \qquad \text{(what? see 2)}$$

assume antisymmetric.

$$= \sum_{i<j} b_{ij}\, e^i\wedge e^j + \sum_{i=j} \underbrace{b_{ii}}_{0}\, e^i\wedge e^i + \sum_{j<i} b_{ij}\, e^i\wedge e^j$$

$$= \sum_{i<j} b_{ij}\, e^i\wedge e^j + \sum_{j<i} (-b_{ji})\, e^i\wedge e^j$$

$$= \sum_{i<j} b_{ij}\, e^i\wedge e^j + \sum_{j<i} (-b_{ji})(-e^j\wedge e^i)$$

$$= \sum_{k<l} b_{kl}\, e^k\wedge e^l + \sum_{k<l} b_{kl}\, e^k\wedge e^l$$

$$= \underline{\sum_{i<j} 2 b_{ij}\, e^i\wedge e^j}.$$

we're back to 12.

Physicists like to write $\underline{b = b_{ij}\, e^i\wedge e^j}$.

Wedge Product For forms extend, linearly from our previous definition.

$$\alpha \wedge \beta = \left(\sum_i \alpha_i e^i\right) \wedge \left(\sum_j \beta_j e^j\right)$$

$$= \sum_{i,j} \alpha_i \beta_j \, e^i \wedge e^j$$

def$^{\underline{n}}$.

<u>Ex</u>) $\alpha = 3e^1 + 2e^2 \qquad \beta = 4e^3 + e^1$

$$\alpha \wedge \beta = (3e^1 + 2e^2) \wedge (4e^3 + e^1)$$

$$= 3e^1 \wedge 4e^3 + \cancel{3e^1 \wedge e^1}^{\,0} + 2e^2 \wedge 4e^3 + 2e^2 \wedge e^1$$

$$= 12 e^1 \wedge e^3 + 8 e^2 \wedge e^3 + 2 e^2 \wedge e^1$$

$$= 8 e^2 \wedge e^3 - 12 e^3 \wedge e^1 - 2 e^1 \wedge e^2$$

$$= \Phi_{\langle 8, -12, -2 \rangle}$$ ← flux form corresponding to $\langle 8, -12, -2 \rangle$.

$\alpha = \omega_{\langle 3,2,0 \rangle} \qquad \beta = \omega_{\langle 1,0,4 \rangle}$

$$\alpha \wedge \beta = \boxed{\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A} \times \vec{B}}}$$

$$(3\hat{i} + 2\hat{j}) \times (4\hat{k} + \hat{i}) = 8\hat{i} - 12\hat{j} - 2\hat{k}$$

In $\mathbb{R}^3$ we have two ways to represent a given vector $\vec{A} = \langle a, b, c \rangle$

$$\omega_{\vec{A}} = a e^1 + b e^2 + c e^3$$

$$\Phi_{\vec{A}} = a e^2 \wedge e^3 + b e^3 \wedge e^1 + c e^1 \wedge e^2$$

Generally true $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A} \times \vec{B}}$

$$\binom{3}{2} = \dim \Lambda^2(\mathbb{R}^3) = \dim(\Lambda^1(\mathbb{R}^3)) = \binom{3}{1}$$

In $\mathbb{R}^4$ $(e^1, e^2, e^3, e^4)$

$$\binom{4}{2} \quad \left.\begin{matrix} e^1 \wedge e^2, & e^1 \wedge e^3, & e^1 \wedge e^4 \\ e^2 \wedge e^3, & e^2 \wedge e^4, & e^3 \wedge e^4 \end{matrix}\right\} \text{ 6-dim'l}$$

$$\binom{4}{1} \quad \left. e^1, e^2, e^3, e^4 \right\} \text{ 4-dim'l.}$$

$$\binom{4}{3} \quad \begin{matrix} e^1 \wedge e^2 \wedge e^3, & e^1 \wedge e^2 \wedge e^4 \\ e^2 \wedge e^3 \wedge e^4, & e^1 \wedge e^3 \wedge e^4 \end{matrix}$$

it's natural to match up 1-form with a vector. (good for $\mathbb{R}^n$)

$$T^r_s(V) = \{ T : \overbrace{V \times V \times \cdots \times V}^{s} \times \overbrace{V^* \times \cdots \times V^*}^{r} \to \mathbb{R} \}$$

where $T$ is linear in each slot.

$T$ is called a <u>tensor</u>. rank $\binom{r}{s}$

<u>Ex</u>) $T^1_1(V) = \{ \sum_{i,j} b^i_j \, e^j \otimes e_i \mid b^i_j \in \mathbb{R}$ for $i,j \in \mathbb{N}_n \}$

$$b \in T^1_1(V) \Rightarrow b(v, \alpha) = \left( \sum_{i,j} b^i_j \, e^j \otimes e_i \right)(v, \alpha)$$

$$(e^j \otimes e_i)(v, \alpha) \underset{\text{def}^{\underline{n}}}{=} e^j(v) \, \alpha(e_i)$$

$$b = \sum_{i,j} b(e_j, e^i) \, e^j \otimes e_i$$

---

w/o some additional concepts it seems $T^0_s(V)$ & $T^r_0(V)$ are logical places to ask about symmetric or antisymmetric.

<u>Question</u>: $T : V \times V \times V \longrightarrow \mathbb{R}$

can we write $T$ as a sum of a symmetric & antisymmetric part?

antisymmetric

$$\boxed{T(V_1, V_2, V_3) = \cancel{\operatorname{sgn}(\sigma)}\, T(V_{\sigma(1)}, V_{\sigma(2)}, V_{\sigma(3)})}$$

(symmetric)

where $\sigma : \mathbb{N}_3 \longrightarrow \mathbb{N}_3$ is a <u>permutation</u>.

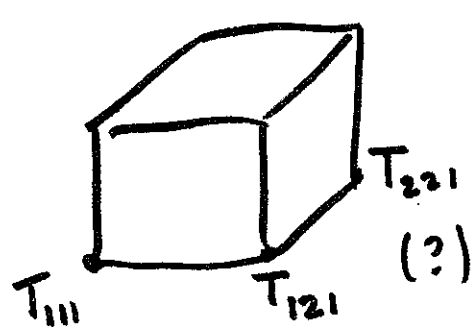
$$T(V_1, V_2, V_3) = -\,T(V_2, V_1, V_3)$$

$$\operatorname{sgn}(\sigma) = (-1)^{\#\text{ cycles comprising } \sigma}$$

<u>Answer</u>: no.

$$T = T_{ijk}\, e^i \otimes e^j \otimes e^k \quad \text{(Einstein notation)}$$

no longer a matrix.



# Wedge Product:

Suppose $V$ is $n$-dim'l with basis $e_1, e_2, \ldots, e_n$
$V^*$ is $n$-dim'l with dual basis $e^1, e^2, \ldots, e^n$

The <u>exterior</u> <u>algebra</u> is:

$$\Lambda(V) = \mathbb{R} \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

Where $\Lambda^0(V) = \mathbb{R}$ dim 1

$\Lambda^1(V) = V^* = \text{span}\{e^1, e^2, \ldots, e^n\}$ dim $n$

$\Lambda^2(V) = \text{span}\{e^i \wedge e^j\}_{\substack{i,j=1 \\ i<j}}^{n}$ $\rightarrow$ dim is $\binom{n}{2}$

$\Lambda^3(V) = \text{span}\{e^i \wedge e^j \wedge e^k\}_{i<j<k}$ $\binom{n}{3}$

$\vdots$

$\Lambda^{n-1}(V) = \text{span}\{\underbrace{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge}_{e^I}{}^{i_{n-1}}\}_{i_1 < i_2 < \cdots < i_{n-1}}$ dim $n$

(multi-index notation)

$\Lambda^n(V) = \text{span}\{e^{i_1} \wedge \cdots \wedge e^{i_n}\}_{i_1 < \cdots < i_n}$
$= \text{span}\{e^1 \wedge e^2 \wedge \cdots \wedge e^n\}$ dim 1

$$\dim(\Lambda(V)) = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{1} + \binom{n}{0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^{k} = (1+1)^n$$

$$= (2)^n$$

Binomial Th (Recall): $\displaystyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$

<u>Properties of $\Lambda(V)$</u>

Given $\alpha \in \Lambda^p(V)$ and $\beta \in \Lambda^q(V)$
then $\alpha \wedge \beta \in \Lambda^{p+q}(V)$.

$$(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta$$

$$(c\alpha) \wedge \beta = \alpha \wedge (c\beta) = c(\alpha \wedge \beta)$$

Can define $\wedge$-Product via Determinant

$$\alpha \wedge \beta = (-1)^{p+q} \beta \wedge \alpha$$

Similarly $\alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2$.

$$\left\{ \begin{array}{l} (A_1 \wedge A_2 \wedge \cdots \wedge A_n)(e_1, e_2, \ldots, e_n) = \det[A] \\ \text{where } \mathrm{col}_j(A) = [A_j(e_1), A_j(e_2), \ldots, A_j(e_n)]^T, \end{array} \right.$$

# Why $\alpha^p \wedge \beta^q = (-1)^{pq} \beta^q \wedge \alpha^p$.

2.5 or 9

Let $\alpha^p = \sum_{\substack{i_1, i_2 \cdots i_p \\ i_1 < \cdots < i_p}} \alpha_{i_1 i_2 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$

Likewise $\beta^q = \sum_{\substack{j_1 j_2 \cdots j_q \\ j_1 < j_2 < \cdots < j_q}} \beta_{j_1 j_2 \cdots j_q} dx^{j_1} \wedge \cdots \wedge dx^{j_q}$

Try to reassemble $\beta^q \wedge \alpha^p$ and see what happens.

$$\alpha^p \wedge \beta^q = \sum_I \sum_J \alpha_I \beta_J \, dx^I \wedge dx^J$$

linearity of $\wedge$.

flip just #'s.

$$= \sum_{\substack{i_1 < i_2 < \cdots < i_p \\ j_1 < j_2 < \cdots j_q}} \alpha_{i_1 i_2 \cdots i_p} \beta_{j_1 j_2 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

to move $dx^j$ up front.

$$= \sum_{I,J} \alpha_I \beta_J (-1)^{\cancel{i}p} \, dx^{j_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q}$$

$$= \sum_{IJ} \alpha_J \beta_J \underbrace{(-1)^p (-1)^p \cdots (-1)^p}_{q} \, dx^J \wedge dx^I$$

$$= \underline{(-1)^{pq} \beta^q \wedge \alpha^p}.$$

Example:

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (A_1 \wedge A_2)(e_1, e_2)$$

$$= \Big(\underbrace{[ae^1 + ce^2]}_{[(ae^1+ce^2)(e_1),\ (ae^1+ce^2)(e_2)]^T = [a,c]^T} \wedge [be^1 + de^2]\Big)(e_1, e_2)$$

$$= (ade^1 \wedge e^2 + cbe^2 \wedge e^1 + \underbrace{abe^1 \wedge e^1}_{0} + \underbrace{cde^2 \wedge e^2}_{0})(e_1, e_2)$$

$$= (ad - bc)(e^1 \wedge e^2)(e_1, e_2)$$

$$= (ad-bc)\left[(e^1 \otimes e^2 - e^2 \otimes e^1)(e_1, e_2)\right]$$

$$= (ad-bc)\left(e^1(e_1)\, e^2(e_2) - \underbrace{e^2(e_1)}_{0}\, \underbrace{e^1(e_2)}_{0}\right)$$

$$= ad - bc.$$

This holds $\forall e_1, e_2$ thus,

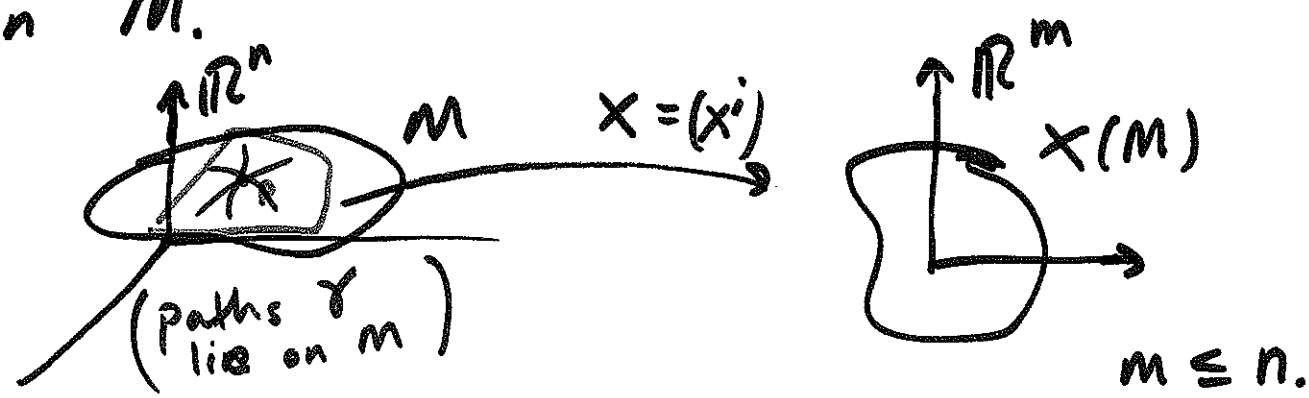
$$\cancel{e_1 A e_2} \Rightarrow (Ae_1)^* \wedge (Ae_2)^* = \det(A)\, e^1 \wedge e^2$$

(not the Hodge Dual)

# Tangent Space and the Cotangent Space

Everything we've calculated thus far is for a single isolated vector space. But, differential forms perhaps should be called "form fields" because they're the assignment of a form at each point along some curve or more commonly over some manifold. (we'll stick to subspaces of $\mathbb{R}^n$)

Consider $M \subset \mathbb{R}^n$ and $(x^i)$ coordinates on $M$.




$$T_p M = \{\text{all tangent vectors to } M \text{ at } p\}$$

$$= \{\gamma'(0) \mid \gamma \text{ is path with } \gamma(0) = p \text{ and } \gamma : I \subseteq \mathbb{R} \longrightarrow M \subseteq \mathbb{R}^n\}$$

$$= \operatorname{span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}.$$

<u>Claim:</u> $\underbrace{\text{span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}}_{\substack{\text{set of smooth} \\ \text{derivations on functs} \\ \text{on } M \text{ at } p}} \cong T_pM.$

vector space isomorphism

$$\gamma'(0) = \vec{V} = V_1 e_1 + V_2 e_2 + \cdots + V_m e_m + \cdots + V_n e_n$$

To construct $\cong$ we need a mapping to convert $\gamma'(0)$ to a corresponding derivation ( $\mathcal{D}er_p(M) = \{ X : C_p(M) \to C_p(M) \,/\, X$ Linear, Leibniz

$$X(fg) = X(f)\cdot g + f\,X(g) \quad \text{Leibniz}$$

$$\left.\begin{aligned} X(cf) &= c\,X(f) \\ X(f+g) &= X(f) + X(g) \end{aligned}\right\} \text{Linear}$$

$$\frac{\partial}{\partial x^i}(f) = \frac{\partial}{\partial u^i}\left[\underbrace{(f \circ x^{-1})}\right](x(p))$$

funct of $u_1, u_2, \ldots, u_m$ coord. down on $\mathbb{R}^m$.

$$\begin{tikzcd} f : M \arrow[r, "f"] \arrow[d, "x"'] & \mathbb{R} \\ \mathbb{R}^m \arrow[ur, "f \circ x^{-1}"'] & \end{tikzcd}$$

$$\boxed{X : M \subseteq \mathbb{R}^n \to \mathbb{R}^m}$$

$$\vec{V}_p = \langle 1, 3 \rangle \longleftrightarrow V = 1\frac{\partial}{\partial x}\Big|_p + 3\frac{\partial}{\partial y}\Big|_p$$

<u>To summarize</u>

$$T_p M = \text{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p, \dots \frac{\partial}{\partial x^m}\Big|_p \right\}$$

$$(T_p M)^* = T_p M^* = \text{span}\{ d_p x^1, d_p x^2, \dots, d_p x^m \}$$

$$(d_p x^j)\left(\frac{\partial}{\partial x^k}\Big|_p\right) = \delta^j_k$$

$e^j(e_k) = \delta^j_k$ this <u>defines</u> $e^j$ since we extend linearly

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Def$^{\underline{n}}$/ $\underbrace{\alpha = df}$ and $\underbrace{X \text{ is a vector field.}}$

$$\alpha = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n \qquad p \longmapsto X_p = X_p^1 \frac{\partial}{\partial x^1}\Big|_p + \dots + X_p^m \frac{\partial}{\partial x^m}\Big|_p$$

$$\alpha(X)(p) = df(X)(p) = X_p(f)$$

$$\boxed{df(X) = X(f)}$$

$$dx^j(X) = X(x^j)$$

$$e^j(v^i e_i) = v^i e^j(e_i) = v^i \delta^j_i = v^j$$

$$= \sum_{k=1}^m X^k \frac{\partial}{\partial x^k}(x^j) = \sum_{k=1}^m X^k \delta^j_k = X^j$$

**Example** $\mathbb{R}^2 = M$ "$u_j = x^j$"

$$T_p\mathbb{R}^2 = \text{span}\left\{ \frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p \right\} = \text{span}\left\{ \frac{\partial}{\partial r}\Big|_p, \frac{\partial}{\partial \theta}\Big|_p \right\}$$

$$(T_p\mathbb{R}^2)^* = \text{span}\{ d_p x, d_p y \} = \text{span}\{ d_p r, d_p \theta \}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$X : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$dx : \underset{\substack{\uparrow \\ \text{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}}{T_p\mathbb{R}^2} \longrightarrow \underset{\substack{\uparrow \\ \text{span}\left(\frac{\partial}{\partial x}\right)}}{T\mathbb{R}} \qquad \left( \underset{\substack{\| \\ \text{span}(e_1, e_2)}}{\mathbb{R}^2} \longrightarrow \underset{\substack{\| \\ \text{span}(1)}}{\mathbb{R}} \right)$$

$$\lim_{h \to 0} \frac{\| X(p+h) - X(p) - (d_p X)(h) \|}{\| h \|} = 0$$

How are $\partial/\partial x$, $\partial/\partial y$ & $\partial/\partial r$ & $\partial/\partial\theta$ related:

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \qquad r = \sqrt{x^2+y^2}$$

$$= \frac{x}{r} dx + \frac{y}{r} dy \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \cos\theta\, dx + \sin\theta\, dy \qquad \theta = \tan^{-1}(y/x)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = \frac{-y}{r^2} dx + \frac{x}{r^2} dy$$

Continuing:

$$dr = \cos\theta\, dx + \sin\theta\, dy$$
$$d\theta = -\frac{1}{r}\sin\theta\, dx + \frac{1}{r}\cos\theta\, dy$$

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = r \begin{bmatrix} \cos\theta/r & -\sin\theta \\ \sin\theta/r & \cos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\, dr - r\sin\theta\, d\theta \\ \sin\theta\, dr + r\cos\theta\, d\theta \end{bmatrix}$$

$$\boxed{\begin{aligned} dx &= \cos\theta\, dr - r\sin\theta\, d\theta \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta \end{aligned}}$$

$$\mathrm{Vol}_2 = dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$
$$= r\, dr \wedge d\theta = \underbrace{\mathrm{Vol}_2}_{\text{volume element in } \mathbb{R}^2.}$$

# Exterior Derivative

Let $\alpha \in \Lambda^p(M)$ then $d\alpha \in \Lambda^{p+1}(M)$

where $\alpha = \sum_I \underbrace{\alpha_I(p)}_{\text{component functions}} d_p x^I$ is

defined by

$$\boxed{d_p\alpha = \sum_I d_p\alpha_I \wedge d_p x^I}$$

Ex) $\alpha = x\, dy \wedge dz + e^{xyz}\, dx \wedge dy$

$$d\alpha = dx \wedge dy \wedge dz + \underbrace{d(e^{xyz})} \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz + \left(xye^{xyz} dz + xze^{xyz} dy + yze^{xyz} dx\right) \wedge dx \wedge dy$$

$$= \underline{\left[1 + xye^{xyz}\right] dx \wedge dy \wedge dz}$$

Ex) $\omega_{\vec{F}} = a\,dx + b\,dy + c\,dz$

(where $\vec{F}(P) = \langle a(P), b(P), c(P) \rangle$ this is a vector field.)

$$d\omega_{\vec{F}} = da \wedge dx + db \wedge dy + dc \wedge dz$$

$$= \left(\frac{\partial a}{\partial x} dx + \frac{\partial a}{\partial y} dy + \frac{\partial a}{\partial z} dz\right) \wedge dx$$

$$+ \left(b_x dx + b_y dy + b_z dz\right) \wedge dy$$

$$+ \left(c_x dx + c_y dy + c_z dz\right) \wedge dz$$

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$$= \left(a_y - b_x\right) dy \wedge dx$$

$$+ \left(a_z - c_x\right) dz \wedge dx$$

$$+ \left(b_z - c_y\right) dz \wedge dy + 0 + 0 + 0.$$

---

$$= \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz$$

$$\boxed{d\omega_{\vec{F}} = \Phi_{\nabla \times \vec{F}}}$$

$$\Phi_{(a,b,c)} = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$$

Ex) $\Phi_{\vec{G}}$ $\quad \vec{G} = \langle G_1, G_2, G_3 \rangle$

$$d\Phi_{\vec{G}} = d\left(G_1\, dy\wedge dz + G_2\, dz\wedge dx + G_3\, dx\wedge dy\right)$$

$$= dG_1\wedge dy\wedge dz + dG_2\wedge dz\wedge dx + dG_3\wedge dx\wedge dy$$

$$= \frac{\partial G_1}{\partial x}\, dx\wedge dy\wedge dz + \frac{\partial G_2}{\partial y}\, dy\wedge dz\wedge dx + \frac{\partial G_3}{\partial z}\, dz\wedge dx\wedge dy$$

$$= \underbrace{\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right)}\, dx\wedge dy\wedge dz$$

$$\boxed{d\Phi_{\vec{G}} = (\nabla\cdot\vec{G})\, dx\wedge dy\wedge dz}$$

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge d\beta$$

Proof) $d(\alpha\wedge\beta) = \sum_{I,J} d(\alpha_I\beta_J)\, dx^I\wedge dx^J$

$$= \sum_{k,I,J} \underbrace{\frac{\partial}{\partial x^k}(\alpha_I\beta_J)}\, dx^k\wedge dx^I\wedge dx^J$$

$$= \sum_{k,I,J} \underbrace{\frac{\partial\alpha_I}{\partial x^k}\beta_J}\, dx^k\wedge dx^I\wedge dx^J + \alpha_I\frac{\partial\beta_J}{\partial x^k}\, dx^k\wedge dx^I\wedge dx^J$$

$$= d\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge d\beta.$$

$$\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$$

$$d(\Phi_{\vec{A}\times\vec{B}}) = d(\omega_{\vec{A}} \wedge \omega_{\vec{B}})$$

$$= d\omega_{\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge d\omega_{\vec{B}}$$

$$= \Phi_{\nabla\times\vec{A}} \wedge \omega_{\vec{B}} - \omega_{A} \wedge \Phi_{\nabla\times\vec{B}}$$

$$(\nabla\cdot(\vec{A}\times\vec{B}))\, dx \wedge dy \wedge dz = \Phi_{\nabla\times\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge \Phi_{\nabla\times\vec{B}}$$

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## Generalized Stokes' Th$^{\underline{m}}$

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha} \longrightarrow \iint_M d\omega_F = \int_{\partial M} \omega_{\vec{F}}$$

$$\iint_M d\omega_F = \iint_M (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial M} \vec{F}\cdot d\vec{l}$$

$$\iiint_M d\Phi_G = \iint_{\partial M} \Phi_G$$

$$\iiint_M (\nabla\cdot\vec{G})\, dV = \iint_{\partial M} G\cdot d\vec{S}$$

# Hodge Duality

Consider $V$ a vector space over $\mathbb{R}$ with dimension $n$. Then we can construct the exterior algebra $\Lambda(V) = \mathbb{R} \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V)$

$$m = \binom{n}{k} = \binom{n}{n-k} \longrightarrow \Lambda^k(V) \cong \Lambda^{n-k}(V)$$

$$\begin{tikzcd} \Lambda^k(V) \arrow[d, "\Phi_k"] & \Lambda^{n-k}(V) \arrow[dl, "\Phi_{n-k}"] \\ \mathbb{R}^m & \end{tikzcd}$$

Ex) $V = \mathbb{R}^3$

$$*(dx) = dy \wedge dz \quad , \quad *(dy) = dz \wedge dx$$

$$*(dy \wedge dz) = dx$$

$$*(\omega_{\vec{F}}) = \Phi_{\vec{F}}$$

$$*(*\alpha) = \alpha$$

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$$\alpha = \sum_I \alpha_{i_1 i_2 \cdots i_k} \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$*\alpha = \sum_{I,J} \alpha^{i_1 i_2 \cdots i_k} \epsilon_{i_1 i_2 \cdots i_k j_1 j_2 \cdots j_{n-k}} \, dx^{j_1} \wedge \cdots dx^{j_2} \wedge \cdots dx^{j_{n-k}}$$

$$\alpha^{i_1 i_2 \cdots i_k} = \underbrace{g^{i_1 k_1} g^{i_2 k_2} \cdots g^{i_k k_k}}_{\text{inverse metric}} \alpha_{k_1 k_2 \cdots k_k}$$

# Metrics (PHYSICSY MATH)

$$g : V \times V \longrightarrow \mathbb{R}$$

- $g(v,w) = g(w,v)$ symmetric
- nondegenerate $\rightarrow$ $g(v,w) = 0 \quad \forall w \Rightarrow v = 0$
- $g$ bilinear

Mathematicians usually insist $g(v,v) \geq 0$. and $g(v,v) = 0$ iff $v = 0$. (positive definite)

Ex) $\mathbb{R}^4$, $\eta = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

$g(v,w) = v^T \eta w$ Minkowski Metric.

Note $\eta^T = \eta$

$$\begin{aligned} g(w,v) &= w^T \eta v \\ &= (w^T \eta v)^T \\ &= v^T \eta^T w^{T^T} \\ &= v^T \eta w = g(v,w). \end{aligned}$$

$g(e_i, e_j) = [g]_{ij} = \eta_{ij} \qquad g^{ki} g_{jl} = \delta^k_l$

# Metric isomorphisms

$$T : V \times V \longrightarrow \mathbb{R} \quad \rightarrow T(e_i, e_j) = T_{ij}$$

$$\tilde{T} : V^* \times V \longrightarrow \mathbb{R}$$ via the metric

$$\tilde{T}(\alpha, v) = \tilde{T}(\alpha_i e^i, v^j e_j) \quad \left(\sum_{\text{implicit}}\right)$$

$$= \alpha_i v^j \tilde{T}(e^i, e_j)$$

$$= \alpha_i v^j \tilde{T}^i{}_j$$

$$= \alpha_i v^j g^{ik} T_{kj}$$

$$= g^{ik} \alpha_i v^j T_{kj}$$

$$= T(\tilde{\alpha}, v)$$

vector which is $g$-dual to $\alpha$.

$$\alpha = \alpha_i e^i \longrightarrow \tilde{\alpha} = \underbrace{g^{ik} \alpha_i}_{\#'s} \underbrace{e_k}_{\text{vector}}$$

Continuing

$$\tilde{\tilde{T}} : V^* \times V^* \longrightarrow \mathbb{R}$$

$$\tilde{\tilde{T}}(e^i, e^j) = g^{ik} g^{jl} T_{kl} = T^{ij}$$