\( \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\} \)

\( x \in \mathbb{R}^n \) and \( x = (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \)

\( x \) is an \( n \)-tuple.

We can add, subtract and scalar multiply:

\[
\begin{align*}
(x+y)_j &= x_j + y_j \\
(x-y)_j &= x_j - y_j \\
(cx)_j &= cx_j
\end{align*}
\]

defined for \( \mathbb{R}^n \)

\( \mathbb{R}^n \) is a vector space over \( \mathbb{R} \) and the standard basis is

\( e_1 = (1, 0, \ldots, 0) \)
\( e_2 = (0, 1, 0, \ldots, 0) \)
\( e_n = (0, 0, \ldots, 1) \)

In short, \( (e_i)_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \)

And we can write any \( v \in \mathbb{R}^n \) as a "linear combination" of the standard basis \( \{e_1, e_2, \ldots, e_n\} \)

\( v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n \)
Geometry in \( \mathbb{R}^n \) is nicely described with the help of the dot-product. If \( x, y \in \mathbb{R}^n \) then \( x \cdot y \in \mathbb{R} \) s.t.
\[
x \cdot y = \sum_{i=1}^{n} x_i y_i. \quad (\text{det}^3)
\]

We can show \( x \cdot y = \sqrt{x \cdot x} \sqrt{y \cdot y} \).

Furthermore, the length of a vector \( x \) is
\[
||x|| = \sqrt{x \cdot x} \quad (\text{det}^3)
\]

You can show that
\[
x \cdot y = ||x|| ||y|| \cos \theta
\]

Or you can define \( \theta = \cos^{-1} \left( \frac{x \cdot y}{||x|| ||y||} \right) \)

(see Dr. Mavunga's Notes from Modern Geom. For a better careful development of this).

**Einstein Notation**: (generally not used in this course until certain tasks)

- \( x \cdot y = \sum_i x_i y_i = x_i y_i \quad \text{Levi-Civita Symbol} \)

- \( a \times b = \sum_{i,j,k} \epsilon_{ijk} a_i b_j e_k = \epsilon_{ijk} a_i b_j e_k \)

\( \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \)
\( \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \)

_repeated index summed over in Einstein notation_.

\( ^2 \)
We need to think about subsets of $\mathbb{R}^n$ with special topological properties.

Set Theory

$A \subseteq B$ iff $x \in A \Rightarrow x \in B \ \forall x \in A.$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\bigcup_{x \in \Delta} A_x = \{x \mid x \in A_x \text{ for some } x \in \Delta\}$

$\bigcup_{j=1}^{\infty} A_j = \{x \mid x \in A_j \text{ for some } j \in \mathbb{N}\}$

Topology in $\mathbb{R}^n$

1.) $B_\delta(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \delta\}$

open-ball of radius $\delta$ which is centered at $x_0$.

• Note, later for limits we like to exclude the center, so we use deleted open balls $B_\delta(x_0) - \{x_0\} = (B_\delta(x_0))'$

• $B_\delta(x_0)$ these are our basic open sets

$\|x - x_0\| = |x - x_0|$ open disk $\mathbb{R}^2$
Def: \( U \subseteq \mathbb{R}^n \) is said to be open iff all its points are interior points. By interior point we mean \( x \in U \) then \( \exists \varepsilon > 0 \) such that \( B_{\varepsilon}(x) \subseteq U \).

Proposition: the union of \( A, B \) open is open.

Proof: Let \( x \in A \cup B \) then \( x \in A \) or \( x \in B \) in either case \( \exists \varepsilon > 0 \) s.t. \( B_{\varepsilon}(x) \subseteq A \) or \( B_{\varepsilon}(x) \subseteq B \) hence \( B_{\varepsilon}(x) \subseteq A \cup B \) \( \therefore x \) is an interior pt.

But \( x \) was arbitrary \( \therefore \) all pts. in \( A \cup B \) are int. Hence \( A \cup B \) is open.

Prop: the intersection of \( A, B \) open is open

Proof: \( x \in A \cap B \) then \( \exists \varepsilon_A, \varepsilon_B > 0 \) such that \( B_{\varepsilon_A}(x) \subseteq A \) and \( B_{\varepsilon_B}(x) \subseteq B \). But, we need to find \( B_{\delta}(x) \subseteq A \cap B \) to show \( x \) interior.

Let \( \delta = \min(\varepsilon_A, \varepsilon_B) \) then \( B_{\delta}(x) \subseteq B_{\varepsilon_A}(x) \subseteq A \) and \( B_{\delta}(x) \subseteq B_{\varepsilon_B}(x) \subseteq B \) \( \therefore B_{\delta}(x) \subseteq A \cap B \) \( \therefore A \cap B \) open.
Lemma: If $\delta_1 \leq \delta_2$ then $B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$

Proof: Let $y \in B_{\delta_1}(x) \Rightarrow ||y - x|| < \delta_1 \leq \delta_2$
thus $y \in B_{\delta_2}(x) \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$.

Mappings vs. functions

In Math 200, an abstract idea of function is sometimes presented $f: A \rightarrow B$ means the inputs of $f$ are taken from $A$ and result in a single value in $B$. The term "function" is used. We forsake that general idea, if I want to use it I'll say "abstract function".

- In this course a **function** has a domain which is in $\mathbb{R}$
- A mapping is an abstract function whose codomain is in $\mathbb{R}^n$.

You might say a mapping is a vector-valued function.

Mappings have component functions

\[ f(x) = (f_1(x), f_2(x), f_3(x)) \] for $x \in \mathbb{R}$.
$f: \mathbb{R} \rightarrow \mathbb{R}^3$

$f$ is a mapping with component functions $f_1, f_2, f_3$.

We write $f = (f_1, f_2, f_3)$ with this understanding.
A matrix \( A \in \mathbb{R}^{m \times n} \) is an array of \( m \) rows and \( n \) columns.

\[
A = [A_{ij}] = \begin{bmatrix} \text{col}_1(A) & \text{col}_2(A) & \cdots & \text{col}_n(A) \end{bmatrix}
\]

Notice \( \text{col}_i(A) = (A_{1i}, A_{2i}, \ldots, A_{mi}) \in \mathbb{R}^m \), whereas \( \text{row}_i(A) = [A_{i1}, A_{i2}, \ldots, A_{in}] \in \mathbb{R}^{1 \times n} \).

Moreover, \( (\text{col}_i(A))_j = A_{ij} \),

and \( (\text{row}_i(A))_j = A_{ij} \).

We can add, subtract, multiply, scalar multiply...

\[
(A + B)_{ij} = A_{ij} + B_{ij}
\]

\[
(cA)_{ij} = cA_{ij}
\]

\[
(AB)_{ij} = \sum_{k=1}^{p} A_{ik} \Theta_{kj} \quad \text{for} \quad A \in \mathbb{R}^{m \times p}, \quad B \in \mathbb{R}^{p \times n}, \quad AB \in \mathbb{R}^{m \times n}.
\]
A linear transformation is a mapping \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that
\[
L(x + y) = L(x) + L(y) \\
L(cx) = cL(x)
\]
\( \forall x, y \in \mathbb{R}^n \) and \( c \in \mathbb{R} \).

**Matrix of Linear Transformation** \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \)

\[
L(e_i) = \sum_{j=1}^{m} A_{ji} \overline{e_j}
\]

Because \( \{ \overline{e_j} \}_{j=1}^{m} \) is a basis for \( \mathbb{R}^m \) and \( L(e_i) \in \mathbb{R}^m \) \( \exists \) set of coefs. \( \{ A_{ji} \} \) such that the linear comb. above holds.

\[ \text{Det}^2 \begin{bmatrix} L \end{bmatrix} = \begin{bmatrix} A_{ji} \end{bmatrix} \]

Can calculate

\[
L(v) = L \left( \sum_{i=1}^{n} v_i e_i \right) \\
= \sum_{i=1}^{n} v_i L(e_i) \\
= \sum_{i=1}^{n} \sum_{j=1}^{m} v_i A_{ji} \overline{e_j} \\
= \sum_{j=1}^{m} \left( \sum_{i=1}^{n} A_{ji} v_i \right) \overline{e_j} = \sum_{j=1}^{m} (AV)_j \overline{e_j} = Av.
\]
Matrix Terminology

Def 1/ $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$ ← standard basis matrices.

$A \in \mathbb{R}^{m \times n}$ then $A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} E_{ij}$

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Th 2/ If $L_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n_2}$ and $L_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3}$ then $L_2 \circ L_1 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3}$ has $\begin{bmatrix} L_2 \circ L_1 \end{bmatrix} = \begin{bmatrix} L_2 \end{bmatrix} \begin{bmatrix} L_1 \end{bmatrix}$

Proof: see linear.

Useful Facts

$e_i^T A = \text{row}_i (A)$

$A e_i = \text{col}_i (A)$

$E_{ij} E_{kl} = \delta_{jk} E_{il}$

$E_{ij} = e_i e_j^T$ ← w/o qualification

$I \neq E_{ij}$ square.
Functions & Mappings, deep thoughts

\[ f(A) = \{ f(a) \mid a \in A \} \]
\[ f^{-1}(V) = \{ x \in \text{dom}(f) \mid \exists v \in V \text{ and } f(x) = v \} \]

Set of all things in the domain which map to \( V \).

**Examples:**

- **level curve in \( \mathbb{R}^2 \)**
  \[ F(x, y) = k \]
  \[ C = \{ (x, y) \in \mathbb{R}^2 \mid F(x, y) = k \} = f^{-1}(\{ k \}) \]

- **parametrised curve in \( \mathbb{R}^n \)**
  \[ C = \{ \tilde{r}(t) \mid t \in \mathbb{R} \} \]
  \[ = \{ (x_1(t), x_2(t), \ldots, x_n(t)) \mid t \in \mathbb{R} \} \]
  \[ = \tilde{r}(J) \]

---

Let \( f: U \to V \) then

* \( \text{Det}^* f \) is 1-1 or injective
  \[ \text{iff } f^{-1}\{ y \} \text{ is a singleton for each } y \in \text{range}(f) \].

\[ f(a) = f(b) \Rightarrow a = b \quad \forall a, b \in \text{dom}(f) \]

\( f \) onto \( T \subseteq V \) iff \( \exists U, \subseteq U \) s.t. \( f(U) = T \)
Special Map: the projection $\pi_j : \mathbb{R}^n \to \mathbb{R}$

$$\pi_j(x) = x_j = x \cdot e_j$$

Also, $\pi_U : \mathbb{R}^n \to U \subseteq \mathbb{R}^n$

$$\pi_U(x) = \begin{cases} x & \text{if } x \in U \\ \emptyset & \text{otherwise} \end{cases}$$

Def 3.1.9. $\pi_U$ is surjective. (Ex 3.1.10 good.)

$$\pi_V(x) = \pi_U(x_v + x_\perp) = x_v \quad \text{where } x_v \in U \quad x_\perp \in U^\perp$$

Assume $U \subseteq \mathbb{R}^n$

\[ g : \mathbb{R}^3 \to \mathbb{R}^2 \]

\[ g^{-1}\{a, b\} = \{V \subseteq \mathbb{R}^3 \mid g(V) = \{a, b\} \} \]

\[ g_1(V) = a, \quad g_2(V) = b \]
**Continuity of Mappings**

Let \( f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m \) be a mapping. We say \( f \) has limit \( b \in \mathbb{R}^m \) at limit point \( a \) of \( U \) iff for each \( \varepsilon > 0 \), \( \exists \delta > 0 \) s.t. \( x \in \mathbb{R}^n \) with \( 0 < \|x-a\| < \delta \) implies \( \|f(x)-b\| < \varepsilon \). In this case

\[
\lim_{x \to a} f(x) = b
\]

**Definition**

Let \( f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m \) be a mapping. If \( a \in U \) is a limit point of \( f \) then we say \( f \) is continuous at \( a \) iff

\[
\lim_{x \to a} f(x) = f(a).
\]

If \( a \) is an isolated point then by def. \( f \) is cont. at \( a \).

Then \( f \) is cont. on \( S \subseteq U \) iff \( f \) is cont. at each point in \( S \). If \( f \) is cont. on \( \text{dom}(f) \) then \( f \) is continuous.

**Proposition:** \( \lim_{x \to a} [f(x)] = b \) iff \( \lim_{x \to a} f_j(x) = b_j \) \( \forall j = 1, 2, \ldots, m \).

**Proof:** Edwards 7.2.
**Derivative and Differential**

Suppose that $V$ is open and $F : V \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping then we say $F$ is differentiable at $a \in V$ iff there exists a linear mapping $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{{h \to 0}} \left[ \frac{F(a+h) - F(a) - L(h)}{\|h\|} \right] = 0$$

In such a case we call $L$ the differential at $a$ and we denote $dL = dF_a$. The matrix of $dF_a$ is called the derivative of $F$ at $a$ and we denote $[dF_a] = F'(a)$.

Note: $F'(a) \in \mathbb{R}^{m \times n}$ and $dF_a(v) = F'(a)v$ for all $v \in \mathbb{R}^n$.

Where's this from?

$$f'(a) = \lim_{{h \to 0}} \left( \frac{f(a+h) - f(a)}{h} \right) = \lim_{{h \to 0}} \left( \frac{hf'(a)}{h} \right)$$

$$\Rightarrow \lim_{{h \to 0}} \left( \frac{f(a+h) - f(a) - hf'(a)}{h} \right) = 0$$
Claim: $D_{B_a}(v) = Av$ or $D_{B_a} = \| B \|

\lim_{h \to 0} \left[ \frac{B(a + h) - B(a) - D_{B_a}(h)}{\| h \|} \right]

\lim_{h \to 0} \left[ \frac{A(a + h) - Aa - Ah}{\| h \|} \right] = \lim_{h \to 0} \left[ \frac{0}{\| h \|} \right] = 0.

$D_{B_a}$ is the best linear approx. to the change in $B$ near $a.$
Metric Spaces

The study of \((\mathbb{R}, 1-1)\) is real analysis. This is just one example of a general family of theories called metric space. 

Def. A metric space \(E\) is a set \(E\) together with a distance function \(d: E \times E \to \mathbb{R}\) such that

1. \(d(p, q) \geq 0 \quad \forall p, q \in E.\)
2. \(d(p, q) = 0 \quad \text{iff} \quad p = q.\)
3. \(d(p, q) = d(q, p) \quad \forall p, q \in E.\)
4. \(d(p, r) \leq d(p, q) + d(q, r) \quad \forall p, q, r \in E \quad (\Delta\text{-inequality})\)

Ex. \(\mathbb{R}\) with \(d(a, b) = |b - a| = \sqrt{(b-a)^2}\)

Ex. \(\mathbb{R}^2\) with \(d((v, w)) = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}\)

Ex. \(\mathbb{R}^n\) with \(d((v, w)) = \sqrt{(v_1 - w_1)(v_2 - w_2)\ldots(v_n - w_n)} = \|v - w\|^2\) squared norm

Ex. \(E_1 \subseteq \mathbb{R}\) metric space

is a subspace of \(E\). Nohiu any subset of \(E\) will do. We don't face the severe restrictions imposed by other contexts (like linear algebra)

Ex. \(\mathbb{R}^2\) with \(\|v\|_1 = |v_1| + |v_2|\) is called 1-norm.

This induces a metric \(d_1(v, w) = \|v - w\|_1\). This can also be done for \(\mathbb{R}^n\). Moreover, one studies \(\|v\|_p = \sqrt[p]{v_1^p + v_2^p + \ldots + v_n^p}\) the \(p\)-norm. The cases \(p = 2\) and \(p = 1\) are most common.
Open and Closed in Metric Space

Def. \( \mathcal{B}_r(p_0) = \{ p \in E \mid d(p, p_0) < r \} \) open ball
\( \overline{\mathcal{B}_r(p_0)} = \{ p \in E \mid d(p, p_0) \leq r \} \) closed ball

\( S \subseteq E \) is open if, for each \( p \in S \), \( \exists r > 0 \) and \( \overline{\mathcal{B}_r(p)} = S \).

We can show \( \emptyset, E \) and any union of open sets (finite or countable) is open. The intersection of finitely many open sets is open. Also, the open ball is open (see pg. 40).

Def. \( S \subseteq E \) is closed iff \( cS \) is open.

Here we denote \( cS = \{ p \in E \mid p \notin S \} \).

Closed balls are closed in the sense of the def above.
We can show \( E, \emptyset \) and the intersection of arbitrarily many closed sets is closed. However, only the finite union of closed sets is necessarily closed.

Def. \( S \subseteq E \) is bounded iff \( \exists r > 0, p_0 \in E \) such that \( S \subseteq \mathcal{B}_r(p_0) \).
Sequences

**Definition:** A sequence in \( E \) is an ordered list of elements in \( E \). Usually it is a function from \( \mathbb{N} \to E \) (although other subsets of \( \mathbb{Z} \) work provided they have a smallest element and possess the successor property). Consider \( P_1, P_2, P_3, \ldots \) in \( E \). A point \( p \in E \) is the limit of \( P_1, P_2, \ldots \) if \( \forall \varepsilon > 0 \), there exist \( N > 0 \) with \( N \in \mathbb{N} \) such that for \( n > N \) we find \( d(P_n, p) < \varepsilon \). If \( P_1, P_2, \ldots \) has a limit point \( p \) then we say it is convergent.

Comment: Usually \( N = N(\varepsilon) \). Also, it may be \( P_1, P_2, \ldots, P_n, \ldots \) converges to \( p \) in \( E \). However, \( P_1, P_2, \ldots \) in \( E' \subset E \) does not converge in \( E' \) because \( p \notin E' \). We can show the limit of a sequence is unique provided a limit exists.

**Fact:** Any subsequence of a convergent sequence of points in a metric space converges to the same point. (Prop. on pg. 46)

This fact is extremely useful for counterexamples. Also it tells us convergence really only cares about the tail \((n > 1)\) of the sequence.
Sequences continued (basics)

- Any convergent sequence \( p, p_n, \ldots \) is bounded.

\[ \epsilon > 0 \in \mathbb{N} \text{ and } \forall n > N, |p_n - p| < \epsilon, \quad r = \max \{\epsilon, d_1, d_2, \ldots\} \]

The set \( S \subseteq E \) is closed iff \( \lim_{n \to \infty} p_n \in S \) for any \( p_n \in S \) and \( \epsilon > 0 \).

Sequence in metric space also have usual limit theorems:

- \( \lim a_n = b_n = \lim a_n + \lim b_n \)
- \( \lim a_n b_n = \lim a_n \lim b_n \).

Also \( \lim a_n = a \).

There are nice, readable proofs in Rosenlicht.

Composition Test: If \( \{a_n\} \), \( \{b_n\} \) are reg. in \( E \)

such that \( a_n \leq b_n \) \( \forall n \) then \( \lim a_n \leq \lim b_n \)

Def: \( \{a_n\} \) is \( \text{increasing} \)

If \( a_n \leq a_{n+1} \) \( \forall n \) \( \Rightarrow \{a_n\} \) is \( \text{increasing} \).

If \( b_n \geq b_{n+1} \) \( \forall n \) \( \Rightarrow \{b_n\} \) is \( \text{decreasing} \).

If \( \{a_n\} \) is either inc or dec then \( \{a_n\} \) \( \text{monotonic} \).

Pr. A bounded monotonic seq. is convergent.

**Ex.** If \( |a| < 1 \) can show \( \lim_{n \to \infty} a^n = 0 \)

(see pg. 51 very cute.)
Completeness

**Def.** A sequence of points \( \{ P_n \} \) is Cauchy if for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( d(P_n, P_m) < \varepsilon \) for any \( n, m \geq N \).

Naturally \( N = N(\varepsilon) \) in a typical example.

Also it's easy to show convergent \( \Rightarrow \) Cauchy. However the converse fails to be true. Basically the problem is our given subspace or metric space could be missing some points. This to follow from pg. 52 of Rosenlicht.

- Prop: any subsequence of Cauchy is Cauchy.
- Prop: A Cauchy sequence of points in a metric space is bounded.
- Prop: Cauchy with convergent subsequences \( \Rightarrow \) convergent. (NICE)

**Def.** A metric space \( E \) is complete if every Cauchy sequence of pts. in \( E \) converges to a pt. of \( E \).

Comments: closed subsets of complete space complete, also \( \mathbb{R} \) complete and \( \mathbb{R}^n \) complete however \( \mathbb{Q} \) not complete. (see pg. 53 of Rosenlicht)

- pages 54 - 61 discuss compact/connected and how they connect with completeness for \( E \). We now essentially skip to 56 on p. 83 for our purposes.
DIGRESSION: (OR NOT) MATRIX EXPONENTIAL

\[ e^A = I + A + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \]

(\(\frac{A^n}{n!} = \frac{1}{n!} A^n\))

Obvious question: why does this series (of matrices !) converge for any \(A\)? Let's try to make this definition meaningful.

1.) \(E = \mathbb{R}^{n \times n}\) is a metric space where the distance is induced from the following norm:

\[ \|A\| = |A_{11}| + |A_{22}| + \cdots + |A_{nn}| \]

(\(|A_{ij}| = a_{ij}\))

\[ d\left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right) = \sqrt{(1-5)^2 + (2-6)^2 + (3-7)^2 + (4-8)^2} \]

2.) Why metric space? Notice \(\Psi: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}\)

\(\Psi (A) = \overline{A}\)

\(\Psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \overline{a} \\ \overline{d} \end{bmatrix}\)

Notice \(\Psi \left( B_{\varepsilon}(A_0) \right) = B_{\varepsilon}(\overline{A_0})\)

\([a \ b] \text{ with } d\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \right) < \varepsilon\)

\(\Psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \overline{a} \\ \overline{d} \end{bmatrix}\) has \(\sqrt{(a-a_0)^2 + \cdots + (d-d_0)^2} < \varepsilon\)

\(d(\overline{A}, \overline{A_0}) < \varepsilon\)
3.) If \( \{ \bar{A}_n \}_{n=1}^\infty \) is a Cauchy sequence in \( \mathbb{R}^{n \times n} \) with limit \( \bar{A} \) then \\
\( \{ \mathcal{F}^{-1}(\bar{A}_n) \}_{n=1}^\infty \) is a sequence of \\
matrices in \( \mathbb{R}^{n \times n} \) and \\
it is Cauchy with limit \( \mathcal{F}^{-1}(\bar{A}) \).

**Proof:** Assume \( \{ \bar{A}_n \}_{n=1}^\infty \) is Cauchy. Let \\
\( B_n = \mathcal{F}^{-1}(\bar{A}_n) \) for \( n \geq 1 \). Let \( \varepsilon > 0 \) \\
choose \( \bar{N} > 0 \) such that \( \forall m,n > \bar{N} \) \\
we have \( \mathcal{F} \) is continuous \( d(\bar{A}_m,\bar{A}_n) < \varepsilon \). \\
(I can choose such an \( \bar{N} \) be. \( \mathcal{F} \) is continuous) \\
Let \( m,n > \bar{N} \) then \\
\[ d(\bar{A}_m,\bar{A}_n) < \varepsilon \]

\[ \Rightarrow \] \\
\[ d(\mathcal{F}^{-1}(\bar{A}_m),\mathcal{F}^{-1}(\bar{A}_n)) < \varepsilon \]

\[ d(\bar{B}_m,\bar{B}_n) < \varepsilon \]

\[ \therefore \{ \bar{B}_n \}_{n=1}^\infty \text{ is Cauchy.} \]

Thus it follows \( \mathbb{R}^{n \times n} \) complete \( \Rightarrow \mathbb{R}^{n \times n} \) complete.
Let \( A_i \in \mathbb{R}^{n \times n} \) for all \( i \in \mathbb{N} \). Then
\[
A_1 + A_2 + A_3 + \cdots = \lim_{m \to \infty} \left( \sum_{i=1}^{m} A_i \right) = \sum_{i=1}^{\infty} A_i.
\]

(Here, \( \lim_{m \to \infty} B_m = B \) if \( \forall \epsilon > 0 \exists M \in \mathbb{N} \)
such that \( d(B_m, B) < \epsilon \) for \( m > M \).)

Consider
\[
I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots = \sum_{m=0}^{\infty} \frac{1}{m!} A^m = S_m,
\]

for \( m > 1 \)
\[
S_m - S_k = I + A + \cdots + \frac{1}{k!} A^k + \frac{1}{(k+1)!} A^{k+1} + \cdots - \frac{1}{m!} A^m
\]
\[
= \frac{1}{(k+1)!} A^{k+1} + \cdots + \frac{1}{m!} A^m
\]

(I'm thinking \( d \left( (S_m, S_k)^2 = \| S_m - S_k \|^2 \)

\[
S_m - S_k = \sum_{k=k+1}^{m} \frac{1}{k!} A^k
\]

\[
\| S_m - S_k \| \leq \sum_{k=k+1}^{m} \frac{1}{k!} \| A^k \| \leq \sum_{k=k+1}^{m} \frac{1}{k!} \| A \|^k
\]

\[
\Delta\text{-inequality for } \mathbb{R}^{n \times n}
\]

\[
\| S_m - S_k \| \leq \sum_{k=k+1}^{m} \frac{1}{k!} \| A \|^k
\]

\[
\Delta_m = \sum_{m=1}^{\infty} \frac{1}{m!} \| A \|^m
\]

\[
\Delta\text{-inequality for } \mathbb{R}^{n \times n}
\]
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad e^{At} = ? \]

\[ A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \]

\[ A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A \]

\[ A^4 = A^2 A^2 = I \]

\[ A^5 = A^4 A = A \]

\[ A^6 = A^4 A^2 = \overline{A} (-I) = -A I \]

\[ A^7 = A^4 A^3 = A^3 = -A \]

\[ A^{4k} = I, \quad A^{4k+1} = A, \quad A^{4k+2} = -I \]

\[ A^{4k+3} = -A, \quad \forall k \in \mathbb{N} \cup \{0\} \]

\[ e^{At} = \sum_{y=0}^{\infty} \frac{(tA)^y}{y!} = \sum_{k=0}^{\infty} \frac{t^k}{(4k)!} \]

\[ + \sum_{k=0}^{\infty} \frac{t^{4k+1}}{(4k+1)!} A + \sum_{k=0}^{\infty} \frac{t^{4k+2}}{(4k+2)!} (-I) + \sum_{k=0}^{\infty} \frac{t^{4k+3}}{9k!} (-A) \]

\[ = \left( \sum_{k=0}^{\infty} \frac{t^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{t^{4k+2}}{(4k+2)!} \right) I \]

\[ + \left( \sum_{k=0}^{\infty} \frac{t^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{t^{4k+3}}{(4k+3)!} \right) A \]

\[ = \left( 1 + \frac{t^4}{4!} + \frac{t^8}{8!} + \ldots - \frac{t^2}{2!} - \frac{t^6}{6!} - \frac{t^{10}}{10!} + \ldots \right) I + \left( \cos t - \sin t \right) A \]
An indirect method to calculate $e^{At}$:

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (\lambda + 2)\lambda + 4$

$= \lambda^2 + 2\lambda + 4$

$= (\lambda + 1)^2 + 3$

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \Rightarrow \det(A - \lambda I) = -\lambda(-2-\lambda) + 1$

$= \lambda^2 + 2\lambda + 1$

$= (\lambda + 1)^2$

$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(A+I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(A+I)\vec{u}_2 = \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u + v = 1 \Rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e^{At} = e^t (I + t(A+I) + \frac{t^2}{2!}(A+I)^2)$

$e^{At} \vec{u}_1 = e^t \vec{u}_1$

$e^{At} \vec{u}_2 = e^t (\vec{u}_2 + t\vec{u}_1)$

$\frac{d}{dt} = AX$

$x(t) = e^{At} \vec{u}_1 + \ldots + c e^{At} \vec{u}_n$

$e^{At} [\vec{u}_1, \vec{u}_2] = \begin{bmatrix} e^{t} \vec{u}_1 \\ e^t(\vec{u}_2 + t\vec{u}_1) \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{t} \vec{u}_1 \\ e^t(\vec{u}_2 + t\vec{u}_1) \end{bmatrix} [\vec{u}_1, \vec{u}_2]^{-1}$
This shows \( e^{At} \) is fundamental \( \text{sole matrix} \) for \( \text{system of ODES} \) \( \frac{dx}{dt} = Ax \). In other words, one independent \( \text{variable} \) ordinary \( \text{differential equation} \).
A couple of examples on \( e^{At} \)’s utility

**Ex:** \( y'' + 2y' + y = 0 \)

2nd order ODE with constant coefficients.

Use reduction of order to convert to system of 2 1st order ODE’s:

\[
\begin{align*}
    x_1 &= y \
    x_2 &= y'
\end{align*}
\]

Hence

\[
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

a.k.a. \( \frac{dx}{dt} = AX \)

\( A \) is the “complementary” matrix to \( y'' + 2y' + y = 0 \)

\( \text{no accident!} \)

\[
\det \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} = \lambda (\lambda + 2) + 1 = \lambda^2 + 2\lambda + 1 = 0
\]

Hence \( (\lambda + 1)^2 = 0 \) \( \Rightarrow \lambda_1 = \lambda_2 = -1 \).

E-vector: \( (A + I)\overline{u}_1 = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \overline{u}_1 = 0 \Rightarrow \overline{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) will do.

generalized: \( (A + I)\overline{u}_2 = \overline{u}_1 \Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \overline{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) e-vector order 2 and solve \( \overline{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) works.

Then \( x(t) = c_1 e^{-t} \overline{u}_1 + c_2 e^{-t}(\overline{u}_2 + t\overline{u}_1) \)

\[
= \begin{pmatrix} c_1 e^{-t} + c_2 (e^{-t} + te^{-t}) \\ -c_1 e^{-t} + c_2 te^{-t} \end{pmatrix} = y
\]

\( y = (c_1 + c_2) e^{-t} + c_2 te^{-t} = c_1 e^{-t} + \overline{c}_2 te^{-t} \) (double root)

\( y' = -(c_1 + c_2) e^{-t} + c_2(e^{-t} - te^{-t}) = -c_1 e^{-t} - \overline{c}_2 te^{-t} \)

(One way to see need for \( t \) in double root case)
Jordan Bloch matrix has standard basis as generalized e-vectors.
we can read e-values -1, -1, -1, 3, 3, 2, 2 off the diagonal since A is triangular.
More over, it's easy to verify that
\[
\begin{align*}
(A + I)e_1 &= 0 \\
(A + I)e_2 &= e_1 \\
(A + I)e_3 &= e_2 \\
(A + I)e_4 &= 0 \\
(A - 3I)e_5 &= e_4 \\
(A - 3I)e_6 &= (A - 3I)e_7 = 0
\end{align*}
\]
(\text{diagonal Blocks $\sim$ e-vectors})
non-diag. Blocks $\sim$ e-vector & g.e.-vectors.

Th3! For any matrix $B$ with real e-values $\in \mathbb{P}eGL(n)$ such that $A = p^{-1} BP$ is Jordan Block.
The matrix $P$ is comprised of gen. e-vectors concatenated to gather.

\[
\frac{dx}{dt} = Bx \quad \text{coord. change} \quad \frac{dv}{dt} = Ay \quad \text{solve via } e^{At} y \text{ change to } \frac{d}{dt}.
\]
Sequences of Functions

- Note: §2 and §3 of Rosenlicht we’re already covered for us in my notes or Edwards. He has some abstraction to metric spaces but the essential arguments is same. Use composition, product, sum, projection to build continuity for polynomials etc....

- §4.8 S he discusses compactness, uniform continuity etc... we’ll add det’s as we need. But let’s start on §6.

**Def**

Let $E, E'$ be metric spaces for $n = 1, 2, 3, \ldots$ let $f_n : E_n \to E'$ be a function. If $p \in E$ we say sequence $f_1, f_2, f_3, \ldots$ conv. at $p$ if the seq. $f_1(p), f_2(p), f_3(p), \ldots$ converges in $E'$. We say $f_1, f_2, \ldots$ converge on $E$ (or converge) if the sequence converges at each $p \in E$. We write

$$f(p) = \lim_{n \to \infty} f_n(p)$$

for all $p \in E$. Or $\lim f_n = f$.

Note to say $\{f_n(p)\}_{n=1}^{\infty}$ converges in $E'$ means for each $\varepsilon > 0 \exists N \in \mathbb{N}$ s.t. for all $n > N$ we have $d'(f_n(p), f(p)) < \varepsilon$.

At the moment, the necessity for the metric space structure of $E$ is unclear.
\[ \text{Ex} \]
\[ f_n : [0,1] \rightarrow \mathbb{R} \quad \text{with} \quad f_n(x) = x - \frac{1}{n}x \]
\[ f_n \quad \text{id as } n \to \infty, \quad f_n(x) \to x \quad \text{as } n \to \infty. \]
\[ \text{id}(x) = x \]

\[ \text{Ex} \]
\[ f_n(x) = x^n \quad \text{for } f_n : [0,1] \rightarrow \mathbb{R} \]
\[ \lim_{n \to \infty} (x^n) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \]
\[ \left\{ f, f_1, f_2, f_3, \ldots \right\} \quad \text{all continuous functions} \]

\[ \text{Def}^n \quad \text{Let } E, E' \text{ be metric spaces for } n=1,2,3,\ldots \]
\[ \text{let } f_n : E \to E' \text{ and } f : E \to E' \text{ another function.} \]
\[ \text{Then the seq } f, f_1, f_2, \ldots \text{ is said to converge uniformly to } f \text{ if, for any } \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t.} \]
\[ d' \left( f(p), f_n(p) \right) < \epsilon \text{ whenever } n > N \text{ and } \forall p \in E. \]

\[ \text{Ex} \]
\[ \{ x - \frac{1}{n}x \}_{n=1}^{\infty} \quad \text{where domain is } [0,1] \quad f_n(x) = x - \frac{1}{n}x \]
\[ d \left( x - \frac{1}{n}x, x \right) = |x - (x - \frac{1}{n}x)| = \frac{1}{n} \leq \frac{\epsilon}{n} < \epsilon \]
\[ \text{Let } \epsilon > 0 \text{ choose } N = \frac{1}{\epsilon}. \]
Prop: Let $E, E'$ be metric spaces, with $E'$ complete. Then for $n = 1, 2, 3, \ldots$ the seq of functions $f_n, f_2, \ldots$ is uniformly convergent iff for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n > N \implies d'(f_n(p), f_m(p)) < \varepsilon$ for all $p \in E$.

Aka: A sequence of functions is uniformly convergent iff it is a Cauchy sequence of functions.

Thm: Let $E, E'$ be metric spaces and $f_1, f_2, \ldots$ be uniformly conv. seq of continuous functions from $E \to E'$. Then lim $f_n$ is continuous function from $E \to E'$.

Comment about generalizing Thm (from §7):

\[ f_n(x) = \begin{cases} -x + \frac{x}{n} & -1 \leq x \leq 0 \\ x^n & 0 \leq x \leq 1 \end{cases} \]

$\{f_n\}$ conv. uniformly on $(-1, 0)$ but no $(0, 1)$. Next, Chapter 4.
Concerning the metric structure of function space (p. 87-90)

Lemma: Let $E$ and $E'$ be metric spaces and let $f, g : E \to E'$ be continuous functions. Then the function $P \mapsto \sqrt{d'(f(P), g(P))}$ is continuous on $E$. \(\lim_{P \to P_0} h(P) = h(P_0)\)

Proof: Let $\varepsilon > 0$, we need to find $\delta > 0$ s.t.
whenever $d(P, P_0) < \delta \Rightarrow \sqrt{d'(f(P), g(P))} < \varepsilon$.

Note: $f, g$ continuous $\Rightarrow \exists \delta_1, \delta_2 > 0$ such that

\[d(P, P_0) < \delta_1 \Rightarrow d'(f(P), f(P_0)) < \varepsilon/2\]

\[d(P, P_0) < \delta_2 \Rightarrow d'(g(P), g(P_0)) < \varepsilon/2\]

Let $\delta = \min(\delta_1, \delta_2)$ and consider

\[
\frac{1}{2} \sqrt{d'(f(P), g(P)) - d'(f(P), g(P_0))}
\leq \frac{1}{2} \sqrt{d'(f(P), g(P)) - d'(f(P), g(P_0))}
\leq \frac{1}{2} \sqrt{d'(f(P), f(P_0)) + d'(g(P), g(P_0))}
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\leq \varepsilon \quad \forall P \in E \quad \text{s.t.} \quad d(P, P_0) < \delta.
\]

Remark: in case $E = E' = \mathbb{R}$ this lemma simply states $P \mapsto |f(P) - g(P)|$ is continuous f.m. on $\mathbb{R}$.

Def: \(\mathcal{D}(E, E') = \sigma f = \text{set of all continuous functions on } E\).

- next we explain how $E'$ compact gives us
  a nice idea of distance in $\mathcal{F}$.
Let \( f : E \rightarrow E' \) be continuous. We define \( D(f, g) = \max \{ d'(f(p), g(p)) \mid p \in E \} \).

**Claim:** \( D \) so defined gives \( J \) a metric space structure.

To begin note \( d \circ (f \times g) \) is continuous (by the Lemma) function from \( E \times E \) into \( \mathbb{R} \) hence \( \exists \max \) for \( d \circ (f \times g) \) for each \( f, g \in J \) hence \( D : J \times J \rightarrow \mathbb{R} \)

is indeed a binary operator.

1. \( D(f, f) = \max \{ d'(f(p), f(p)) \mid p \in E \} = \max \{ 0 \mid p \in E \} = 0 \)

2. \( D(f, g) = \max \{ d'(f(p), g(p)) \mid p \in E \} = 0 \) since \( d' \) is a metric.

3. \( D(g, f) = \max \{ d'(g(p), f(p)) \mid p \in E \} = \max \{ 0 \mid p \in E \} = 0 \) since \( d' \) is symmetric.

For all \( D(f, g) = \max \{ d'(f(p), g(p)) \mid p \in E \} \),

\[ d'(f(p), g(p)) \] for some \( p \in E \), by \( \max \) value obtained by the function on compact space as \( d' \) is metric.

4. \( D(f, g) = \max \{ d'(f(p), g(p)) \mid p \in E \} \)

\[ = \max \{ d'(f(p), h(p)) + d'(g(p), h(p)) \mid p \in E \} \]

\[ = \max \{ d'(f(p), h(p)) \mid p \in E \} + \max \{ d'(g(p), h(p)) \mid p \in E \} \]

\[ = D(f, h) + D(h, g) \]

\[ \therefore (J, D) \text{ is an abstract metric space.} \]

Here the prints are functions.
Oh, so \((Y(E, E'), D)\) where \(E\) is compact is a metric space of functions... and what?

Let \(f_1, f_2, f_3, \ldots\) be a sequence of functions in \(Y\), then we say \(f_n \to f\) as \(n \to \infty\) for some \(f \in Y\) iff
\[
\lim_{n \to \infty} D(f_n, f) = 0
\]

To unfold this, iff for each \(\varepsilon > 0\), \(\exists N \in \mathbb{N}\) s.t.
whenever \(n \geq N\) we have \(D(f, f_n) < \varepsilon\).
Which means \(\max_{p \in E} d'(f(p), f_n(p)) < \varepsilon\),
which says \(d'(f_n(p), f(p)) < \varepsilon\) \(\forall p \in E\) and \(n > N\).

Thus (1) The sequence \(f_1, f_2, f_3, \ldots \to f\) in \(Y\) iff
the sequence of functions \(f_1, f_2, f_3, \ldots\) on \(E\) converges uniformly to \(f\).

Sequential convergence in \(Y \iff \) uniform convergence for functions on \(E\).

max-metric \(D\) in terms of \(d, d'\) on \(E \neq E'\)

Continuing, \(f_1, f_2, \ldots : E \to E'\) complete \(d, d'\) on \(E \neq E'\)
\(f_1, f_2, f_3, \ldots\) is a Cauchy sequence in \(Y\) then
for all \(\varepsilon > 0\), \(\exists M \in \mathbb{N}\) such that whenever \(m, n > M\)
\(D(f_n, f_m) < \varepsilon\), hence \(\max_{p \in E} d(f_n(p), f_m(p)) < \varepsilon\)
hence \(d'(f_n(p), f_m(p)) < \varepsilon\) \(\forall p \in E\) \& \(n, m > M\).
Which makes \(f_1, f_2, f_3, \ldots\) uniformly convergent to \(f\).

By prop. on p. 86 of Kolmogorov. Hence
\[ f_n \to f \quad f : E \to E' \quad \text{as} \quad n \to \infty \quad \text{and} \quad \text{as the limit of uniformly convergent seq. of continuous functions is continuous it follows} \ f \in Y. \text{ Therefore} \]
\(\text{cauchy seq. in } Y \text{ converge in } Y \iff Y \text{ is complete.}\)
Theorem: Let $E, E'$ be metric spaces with $E$ compact and $E'$ complete, then $\mathcal{F} = \{ f: E \to E' \mid \text{continuous} \}$ is a complete metric space with distance
\[ D(f, g) = \max \{d(f(p), g(p)) \mid p \in E\}. \]
Moreover, a sequence of points in $\mathcal{F}$ is convergent iff it is a uniformly convergent sequence of functions.

Definition: Let $E$ be compact, then $C(E) = \{ f: E \to \mathbb{R} \mid \text{continuous} \}$ is a complete metric space of functions on $E$.

Remark: Many interesting exercises worthy of study on pgs. 90-95.

Now we move on to Chapter VII where we should find careful proofs of many central Theorems from calculus.

To summarize uniform continuity, convergence vs. continuity of limits:

\[
\lim_{p \to p_0} \left( \lim_{n \to \infty} f_n(p) \right) = \left( \lim_{n \to \infty} f_n \right)(p_0) \quad \text{continuity of limiting function.}
\]

\[
= \lim_{n \to \infty} \left( f_n(p_0) \right)
\]

Uniform convergence of $f_1, f_2, \ldots$ (each continuous) implies $f_n \to f$ as $n \to \infty$ and thus $\lim f$ is likewise continuous ($\lim_{p \to p_0} f(p) = f(p_0)$).
Uniform Continuity Thm's for Calculus

Thm (p. 138 Rosenlicht) Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( f_1, f_2, f_3, \ldots \) a uniformly convergent sequence of continuous real-valued functions on \([a, b]\). Then
\[
\int_a^b \left( \lim_{n \to \infty} f_n(x) \right) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx
\]

Thm (p. 140) Let \( f_1, f_2, f_3, \ldots \) be sequence of real-valued functions on open interval \( U \) in \( \mathbb{R} \), each continuously differentiable. Suppose that the seq. \( f'_1, f'_2, f'_3, \ldots \) converges uniformly on \( U \) and that for some \( a \in U \) the sequence \( f_1(a), f_2(a), f_3(a), \ldots \) converges. Then \( \lim_{n \to \infty} f_n \) exists, is differentiable, and
\[
\left( \lim_{n \to \infty} f_n \right)' = \lim_{n \to \infty} (f'_n)
\]

Proof: By FTC,
\[
\int_a^x f_n'(t) \, dt = f_n(x) - f_n(a)
\]
for any \( x \in U \) and any \( n = 1, 2, 3, \ldots \). Let \( \lim_{n \to \infty} f_n' = g \).
\[
\lim_{n \to \infty} \int_a^x f_n'(t) \, dt = \int_a^x \lim_{n \to \infty} f_n'(t) \, dt = \int_a^x g(t) \, dt
\]

\[
\lim_{n \to \infty} (f_n(x) - f_n(a)) = \int_a^x g(t) \, dt
\]

\[
f(a) = \lim_{n \to \infty} f_n(x) = f(a) + \int_a^x g(t) \, dt
\]

\[
\frac{d}{dx} \left[ \lim_{n \to \infty} f_n(x) \right] = \lim_{n \to \infty} \frac{d}{dx} f_n(x)
\]

(I'm just guessing here, so I want to dig deeper into today.) Remarks: decided to go to Edwards for a bit??
Newton's Method

\[ y = f(x) \]
\[ L_{x_0}^f(x) = f(x_0) + f'(x_0)(x-x_0) \]

x-intercept \( \Phi L_{x_0}^f(x) = 0 \)

call this intercept \( x_1 \), so \( 0 = f(x_0) + f'(x_0)(x_1-x_0) \)

Solve for \( x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \). Evaluate \( f(x_1) \)

if it's non-zero then consider \( L_{x_1}^f(x) \),

\[ L_{x_1}^f(x) = f(x_1) + f'(x_1)(x-x_1) \]

Say \( L_{x_1}^f(x_1) = 0 = f(x_1) + f'(x_1)(x_2-x_1) \)

hence \( x_2 = x_1 - \frac{f(x_1)}{f'(x_0)} \). etc...

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] Standard Newton.

Remark: this method does not always work, can get stuck in an oscillation about \( x_n \) where \( f(x_n) = 0 \) is what we're after. Various modifications possible. We do need \( |f'(x)| > 0 \) on some nbhd of \( x_0 \) for reliability of method.

If \( |f'(x)| \leq M \) for \( x \in \text{Nbhd}(x_0) \) then we can use

\[ x_{n+1} = x_n - \frac{f(x_n)}{M} \]

Modified Newton.

If \( f'(x) < 0 \) near \( x_0 \) then use \( M \) which has \( |f'(v)| \leq M \)

\( M < f'(x) < 0 \). We discuss contraction mappings next 2
Theorem: Let $\Phi: [a, b] \to [a, b]$ be a contraction mapping with contraction constant $k$. Then $\Phi$ has a unique fixed point $x_*$; $\Phi(x_*) = x_*$. Moreover, given $x_0 \in [a, b]$, the sequence \( \{x_n\} \) defined inductively by $x_{n+1} = \Phi(x_n)$ converges to $x_*$. In particular,

$$|x_n - x_*| \leq \frac{k^n |x_0 - x_1|}{1-k} \quad \text{for each} \quad n=0, 1, 2, \ldots$$

Proof: To say $\Phi$ is contraction mapping on $[a, b]$ with constant of contraction $k$ means $\forall x, y \in [a, b]$

$$|\Phi(x) - \Phi(y)| \leq k |x - y|.$$ Consider then,

$$|x_{n+1} - x_n| = |\Phi(x_n) - \Phi(x_{n-1})| \leq k |x_n - x_{n-1}|$$

Inductively $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|$. (Note for $n = 0$ $|x_1 - x_0| = 1 \cdot |x_1 - x_0|$ so the initial step is valid) Continuing with the induction,

$$|x_{n+2} - x_{n+1}| = |\Phi(x_{n+1}) - \Phi(x_n)|$$

$$\leq k |x_{n+1} - x_n|$$

$$\leq k \cdot k^n |x_1 - x_0|$$

$$\leq k^{n+1} |x_1 - x_0|$$

Hence $|x_{n+1} - x_n| \leq k^n |x_1 - x_0| \quad \forall n=0, 1, 2, \ldots$
Suppose $m, n \geq 0$ and $n < m$ wlog,

\[
|X_n - X_m| = \frac{|X_n - X_{n+1} + X_{n+1} - X_{n+2} + X_{n+2} - X_m|}{X_{m-1} - X_m}.
\]

\[
|X_n - X_{n+1}| + |X_{n+1} - X_{n+2}| + \cdots + |X_{m-1} - X_m|
\]

\[
h^n |X_i - x_0| + h^{n+1} |X_i - x_0| + \cdots + h^{m-1} |X_i - x_0|
\]

\[
(k^n + h^{n+1} + \cdots + h^{m-1}) |X_i - x_0| \geq \text{sneaky}.
\]

\[
k^n (1 + h + h^2 + \cdots) |X_i - x_0| \leq \frac{k^n}{1-k} |X_i - x_0| \quad \text{(need } k < 1\text{!)}
\]

Therefore if $\varepsilon > 0$ then we can bound

\[
|X_n - X_m| < \varepsilon \quad \text{if we choose } N \text{ large enough}
\]

so that when $n > N$ we have $\frac{k^n}{1-k} |X_i - x_0| < \varepsilon$.

Explicitly, $k^n = \frac{\varepsilon (1-k)}{|X_i - x_0|}$ and $n = \frac{1}{\ln(k)} \ln \left( \frac{\varepsilon (1-k)}{|X_i - x_0|} \right)$

Then choose $N$ to be the next largest integer. Oh, well technical details aside, \( |X_n - X_m| \leq \frac{k^n}{1-k} |X_i - x_0| \rightarrow 0 \)

as $n, m \rightarrow \infty$ since $k$ is a constant with $0 < k < 1$.

Therefore, \( \{X_n\} \) is Cauchy so, by completeness of $\mathbb{R}$ this sequence converges to some value $X_* \in [a, b]$. (note $[a, b]$ closed and

\[
\{x_0, x_1, \ldots, \} \subset [a, b] \text{ hence}
\]

it must converge to pt. inside $[a, b]$ by seq. def of closed set.)
We found the estimate \( |x_n - x_m| \leq \frac{k^n |x_1 - x_0|}{1-k} \).

Let \( m \to \infty \) and find that

\[
|x_n - x_*| \leq \frac{k^n |x_1 - x_0|}{1-k}
\]

(we take limit \( m \to \infty \) for each fixed \( n \) to derive the inequality of the contraction mapping Th.\(^2\)).

Uniqueness? Suppose \( \varphi(x_*) = x_* \) and \( \varphi(x_{**}) = x_{**} \).

Observe \( |x_n - x_{**}| = |\varphi(x_*) - \varphi(x_{**})| \)

\[
\leq k |x_* - x_{**}|
\]

and as \( k < 1 \) it follows \( |x_n - x_{**}| = 0 \) hence \( x_* = x_{**} \) proving uniqueness. \( \Box \)

This Th.\(^2\) is at the heart of quite a few iterative analytic arguments. For example, the existence of solutions to ODEs has a contraction mapping argument. For us, we see Newton's Method shrinks \( |x_1 - x_0| \)

to \( |x_2 - x_0| \ldots \) always contracting the range over which \( x_* \) with \( f(x_*) = 0 \) resides.
Let's make this explicit,
Theorem 1.2 p. 164 Edwards

Let \( f : (a, b) \to \mathbb{R} \) be differentiable with \( f(a) < 0 < f(b) \)
and \( 0 < m < f'(x) \leq M \) for \( x \in (a, b) \). Given \( x_0 \in [a, b] \),
the sequence \( \{x_n\} \) defined inductively by
\[
    x_{n+1} = x_n - \frac{f(x_n)}{M}
\]
converges to the unique root \( x_* \in [a, b] \) of
the equation \( f(x) = 0 \). In particular,
\[
    |x_n - x_*| \leq \left| \frac{f(x_0)}{m} \right| (1 - \frac{m}{M})^n \quad \text{for each } n = 0, 1, \ldots
\]

Proof: Let \( \varphi : [a, b] \to \mathbb{R} \) be defined by
\[
    \varphi(x) = x - \frac{f(x)}{M}.
\]
We argue this is a contraction mapping. Note,
\[
    \varphi'(x) = 1 - \frac{f'(x)}{M} \leq 1 - \frac{m}{M} = \frac{M - m}{M} = k \quad \text{since } m < f'(x).
\]
Let \( k = 1 - \frac{m}{M} \) and note \( m < M \) so \( \frac{m}{M} < 1 \) and \( 1 - \frac{m}{M} > 0 \) thus \( \varphi'(x) > 0 \) for \( x \in (a, b) \).
Therefore,
\[
    a < a - \frac{f(a)}{M} = \varphi(a) \leq \varphi(x) \leq \varphi(b) = b - \frac{f(b)}{M} < b
\]
for all \( x \in (a, b) \) where the outside \( a < x \) followed from the assumption \( f(a) < 0 < f(b) \).
This shows \( \text{range}(\varphi) \subseteq [a, b] \) hence \( \varphi : [a, b] \to [a, b] \).
Let \( x, y \in (a, b) \) and consider,
\[
    |\varphi(x) - \varphi(y)| = |x - \frac{f(x)}{M} - y - \frac{f(y)}{M}|
\]
\[
    = |x - y - \frac{1}{M}(f(x) - f(y))| < k|x - y|
\]

Curious: Edwards is content to say \( \varphi(x) \leq k \)
for \( k = 1 - \frac{m}{M} \), why is that sufficient? (need \( |\varphi(x) - \varphi(y)| \leq k|x - y| \)
Consider \( |f(x) - f(x_0)| \leq k |x - x_0| \)
\[ \Leftrightarrow \frac{h^n |x_0 - x_1|}{1-h} = \frac{n}{m} \left(1 - \frac{m}{M}\right)^n \left|\frac{f(x_0)}{m}\right| \]

\[ h = 1 - \frac{m}{M} \quad x_1 = x_0 - \frac{f(x_0)}{m} \]

\[ |x_0 - x_1| = \frac{f(x_0)}{M} \]

\[ y - b = f(x) - f(a) \approx f'(a) (x - a) \]
Examples for Inverse Mapping Theorem

- Here I find a few inverses so we can appreciate the Th.2.

1) Let $F(x, y) = (x^2 + y^2, x^2 - y^2)$

Let $(a, b) = (x^2 + y^2, x^2 - y^2)$.

We need to solve these for $x$ and $y$,

\[ a = x^2 + y^2 \Rightarrow a + b = 2x^2 \]
\[ b = x^2 - y^2 \Rightarrow a - b = 2y^2 \]

Thus, $x = \pm \sqrt{\frac{1}{2}(a + b)}$ & $y = \pm \sqrt{\frac{1}{2}(a - b)}$.

To specify we need to decide where we seek the inverse. For example,

\[ F^{-1}_V(a, b) = (\sqrt{\frac{1}{2}(a + b)}, -\sqrt{\frac{1}{2}(a - b)}) \]

for $V = [0, \infty) \times (-\infty, 0]$. The larger point to see here is $F^{-1}_V$ if $V \subseteq \mathbb{R}^2$ which nontrivially overlaps the $x$ or $y$ axis. The local inverse can only work for one quadrant at a time.

Why? note $F'(x, y) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$

thus $\det(F'(x, y)) = -4xy$. Clearly the derivative is singular along either $x = 0$ (y-axis) or $y = 0$ (x-axis).
Let \( F(x) = Ax \) where \( A \in \mathbb{R}^{n \times n} \). We know from linear algebra etc. the inverse exists iff \( \det(A) \neq 0 \) and in that case \( F^{-1}(x) = A^{-1}x \). (or \( F^{-1}(y) = A^{-1}y \) if you wish to say \( F(x) = y \) and reserve notation.)

What about \( F' \)? It's not hard to prove from the definition \( F'(x) = A \) hence
\[
\det(F'(x)) = \det(A) \text{ so clearly linear algebra and our inverse mapping \( \Theta = \)}
\[
\begin{align*}
\text{Let } \Psi(r, \theta, \varphi) &= (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi) \\
\text{Let } (x, y, z) &= \Psi(r, \theta, \varphi) \text{ and solve for } r, \theta, \varphi.
\end{align*}
\]
You can verify \( r^2 = x^2 + y^2 + z^2 \) and we can insist \( r \geq 0 \) so \( r = \sqrt{x^2 + y^2 + z^2} \).
Continuing, \( z = r \cos \varphi \Rightarrow \varphi = \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \)
Also, \( \tan \theta = y/x \Leftrightarrow \theta = \tan^{-1} \left[ \frac{y}{x} \right] \)
Now, these are local formulas defined naturally for regions where \( -\pi < \varphi < \pi \) and \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \) and for convenience \( 0 \leq r < \infty \).

Let's see: \( \Psi'(r, \theta, \varphi) = \\
\begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}
\end{vmatrix} = \\
\begin{vmatrix}
\cos \theta \sin \varphi & -r \sin \theta \sin \varphi & r \cos \theta \cos \varphi \\
\sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \cos \theta \cos \varphi \\
\cos \varphi & 0 & -r \sin \varphi
\end{vmatrix} = \\
\begin{vmatrix}
\cos \varphi (r^2 \sin^2 \theta \sin \varphi \cos \varphi + r^2 \cos \theta \sin \varphi) & -r \sin \varphi (r \cos \theta \sin \varphi + r \sin \theta \sin \varphi) \\
- \sin \varphi (r \cos \theta \sin \varphi + r \sin \theta \sin \varphi) & - r^2 (r \cos \theta \sin^2 \varphi + r \sin \theta \sin \varphi) \\
\end{vmatrix} = \\
- r^2 (\cos^2 \varphi \sin \varphi + \sin^2 \varphi) = -r^2 \sin \varphi.
Question: for $E_3$, the periodicity of \( \cos \theta \) and \( \sin \theta \) should limit us to a \( 2\pi \)-length domain for \( \theta \). This is apparently not seen from the examination of \( \psi'(r, \theta, \phi) \). But, this is not a deficiency of the inverse mapping \( \Theta^{-1} \). Remember, the \( \Theta^{-1} \) only says \( \exists \) a local inverse in some nbhd about a point where the derivative mapping is non-singular. It will not reveal the trouble with \( \Theta \), some other idea is needed. For now we content ourselves with a case by case analysis, the case of a global inverse \( (E_2) \) is special.

**E4** Consider \( F(r, \theta) = (r \cos \theta, r \sin \theta) \) then \( F^{-1}(x, y) = \left( \sqrt{x^2 + y^2}, \tan^{-1}(y/x) \right) \).

Note: \( F'(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \)

\[ \det \mathbf{(F'(r, \theta))} = r \cos^2 \theta + r \sin^2 \theta = r \]

Again, the fact that \( \cos(\theta + 2\pi k) = \cos \theta \) is not revealed by \( F'(r, \theta) \). Only the degeneracy of \( r = 0 \) is apparent.
Examples for Implicit Function Thm

### E1

Suppose \( ax + by + cz = 0 \) then we can solve for:

1. \( x = -\frac{b}{a} y - \frac{d}{a} z \) \( (x = x(y, z)) \)
2. \( y = -\frac{a}{b} x - \frac{d}{b} z \) \( (y = y(x, z)) \)
3. \( z = -\frac{a}{c} x - \frac{b}{c} y \) \( (z = z(x, y)) \)

Here \( F(x, y, z) = ax + by + cz = 0 \) gives level surfaces and

1. \( \frac{\partial F}{\partial x} = a \neq 0 \Rightarrow \) can write level surface as graph \( x = f(y, z) \).
2. \( \frac{\partial F}{\partial y} = b \neq 0 \Rightarrow y = g(x, z) \) gives \( F^{-1}\{0\} \).
3. \( \frac{\partial F}{\partial z} = c \neq 0 \Rightarrow F^{-1}\{0\} \) can be written as graph of \( z = h(x, y) \).

### E2

Suppose \( x^2 + y^2 + z^2 = R^2 \), Here we can solve \( z = \pm \sqrt{R^2 - x^2 - y^2} \) locally.

Here \( F(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0 \) can be implicitly (locally) solved about a point with \( \frac{\partial F}{\partial z} = 2z \neq 0 \) for \( z = f(x, y) \). The \( \pm \) in the algebra reflects this reality. Local solutions above or below \( z = 0 \) exist (but not across \( z = 0 \)).
Consider the set of eqs,
\[ \begin{align*}
  x + y + z &= 3 \\
  x^2 + y^2 &= 1 
\end{align*} \]

We have 3 variables and two seemingly independent eqs, we should expect only one genuine solution to survive as we simultaneously solve these eqs. Let's keep \( x \),
\[ y = \pm \sqrt{1-x^2} \]
\[ z = 3 - x - y = 3 - x \pm \sqrt{1-x^2} \]
Thus \( G(x) = (\pm \sqrt{1-x^2}, 3-x \pm \sqrt{1-x^2}) = (y, z) \)
provides local soln to system in the sense \( H(x, y, z) = (x+y+z-3, x^2+y^2-1) \) has \( H^{-1}(\{0\}) \leftrightarrow (y, z) = G(x) \)
\( \Rightarrow y \neq 0 \) are graph of G.

**Does \( H'(x, y, z) \) tell us anything?**
\[ H'(x, y, z) = \begin{bmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 0 \end{bmatrix} \]
see \( \det \begin{bmatrix} 1 & 1 \\ 2y & 0 \end{bmatrix} = -2y \neq 0 \) for \( y \neq 0 \)
we cannot cross \( y=0 \) this is reflected in the algebra \( y = \pm \sqrt{1-x^2} \) we have to choose (+) or (-), can't have both.

Note \( \det \begin{bmatrix} 2x & 2y' \\ 2y & 0 \end{bmatrix} = 2y - 2x = 2(y-x) \)
so we cannot parametrize soln by \( z \) along any line \( y = x \). Try to solve in terms of \( z \), see what happens.
Multilinearity: (suppose $V$ is vector space over $\mathbb{R}$ throughout)

**Bilinearity:**

$b : V \times V \rightarrow \mathbb{R}$ is bilinear iff

1. $b(x+y, z) = b(x, z) + b(y, z)$ $\forall x, y, z \in V.$
2. $b(x, y+z) = b(x, z) + b(x, y)$ $\forall x, y, z \in V.$
3. $b(cx, y) = c \cdot b(x, y)$ $\forall x, y \in V$, $c \in \mathbb{R}$
4. $b(x, cy) = c \cdot b(x, y)$ $\forall x, y \in V$, $c \in \mathbb{R}$.

Basically $b$ is a bilinear mapping on $V$ iff it's a function from $V \times V \rightarrow \mathbb{R}$ which is linear in each "slot".

**Example 1:** $b(x, y) = x \cdot y$ for $x, y \in \mathbb{R}^n$

**Example 2:** $b(\vec{x}, \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{e}_i$ for $\vec{x}, \vec{y} \in \mathbb{R}^3$

\[
b(\vec{x}_1 + \vec{x}_2, \vec{y}) = \left[(\vec{x}_1 + \vec{x}_2) \times \vec{y}\right] \cdot \vec{e}_i
\]

\[
= (\vec{x}_1 \times \vec{y}) \cdot \vec{e}_i + (\vec{x}_2 \times \vec{y}) \cdot \vec{e}_i
\]

\[
= b(\vec{x}_1, \vec{y}) + b(\vec{x}_2, \vec{y}).
\]
Notice $\text{Ex 1} \text{ and 2} \text{ have nice props.}$

1. $b(x, y) = x \cdot y = y \cdot x = b(y, x)$
   
   This makes $b$ **symmetric**.

2. $b(x, y) = (\vec{x} \times \vec{y}) \cdot e$, 
   
   $= (-\vec{y} \times \vec{x}) \cdot e$, 
   
   $= -b(\vec{y}, \vec{x})$

   This makes $b$ **antisymmetric**.

Notice $b(x, y) = x \cdot y + (\vec{x} \times \vec{y}) \cdot e$, for $x, y \in \mathbb{R}^3$

Then $b$ is neither sym. or antisym.

**Theorem 1**

$T_2^0(V) = \{ b \mid b \text{ is bilinear form on } V \}$

forms a vector space w.r.t. the natural
def. of function addition; $\forall x, y \in V, \forall b, b_1, b_2 \in T_2^0(V)$

1. $(b_1 + b_2)(x, y) \equiv b_1(x, y) + b_2(x, y)$

2. $(cb)(x, y) = c \cdot b(x, y)$

**Proof:** Show $b_1, b_2 \in T_2^0(V) \Rightarrow b_1 + b_2 \in T_2^0(V)$

I'd show $(b_1 + b_2)(x + y, z) = (b_1 + b_2)(x, z) + (b_1 + b_2)(y, z)$
Bilinear Forms & Coordinates:

\[ V = \text{span}\{ e_i \mid i = 1, 2, \ldots, n \} = \text{span}\beta \]

\( V \) is finite dimensional & \( \dim(V) = n \)

Consider: If \( x, y \in V \) then \( \exists x^i, y^j \in \mathbb{R} \)

\( \forall i, j \in \{1, 2, \ldots, n\} \) such that

\[ x = \sum_{i=1}^{n} x^i e_i \quad [x^i] = [x]_\beta = \mathcal{F}_\beta(x) \]

\[ y = \sum_{j=1}^{n} y^j e_j \]

With the above in mind,

\[ b(x, y) = b\left( \sum_{i} x^i e_i, \sum_{j} y^j e_j \right) \]

\[ = \sum_{i} x^i b(e_i, \sum_{j} y^j e_j) \]

\[ = \sum_{i} \sum_{j} x^i y^j b(e_i, e_j) \]

Apparently the values of \( b \) on bases \( \beta \)

determine the action of \( b \) over all of \( V \).
Reminder about Dual Space $V^*$

$V^* = \{ f : V \to \mathbb{R} \mid f \text{ is linear} \}$

The dual space is the set of all linear functionals on $V$.

Example: $\alpha(V) = V \cdot W$ for $V \in \mathbb{R}^n$ and $W$ some fixed vector which makes $\alpha$ what 578. it is.

\[ \alpha(cV_1 + V_2) = (cV_1 + V_2) \cdot W = cV_1 \cdot W + V_2 \cdot W = c\alpha(V_1) + \alpha(V_2) \]

In fact any linear functional on $\mathbb{R}^n$ can be expressed as a dot product:

$\alpha : \mathbb{R}^n \to \mathbb{R}$ (stand basis this time)

$\alpha(x) = \alpha\left( \sum_{i=1}^{n} x_i e_i \right) = \sum_{i=1}^{n} x_i \alpha(e_i)$

$= x \cdot W = W^T x$

where $(W^T)_i = \alpha(e_i)$. Some people take the viewpoint $(\mathbb{R}^{n \times 1})^* = \mathbb{R}^{1 \times n}$

$\alpha \leftrightarrow [\alpha(e_i)] : e = W^T$
Continuity on the dual space $V^* = T^0(V)$

Def. If $V = \text{span}\{e_i\}_{i=1}^n$, then
\[ V^* = \text{span}\{e^i\}_{i=1}^n, \text{ where } e^i(e_j) = \delta^i_j \]
\[ \forall i, j \in \mathbb{N}_n \]

Why is this reasonable?

1.) $e^i : V \to \mathbb{R}$ has $e^i(e_i) = \delta^i_i$
we've defined a function on a basis of $V$ so we extend linearly to obtain a linear function meaning $e^i \in V^*$.
\[ e^i(x) \overset{\text{def}}{=} \sum_{i=1}^n x^i e^i(e_i) = x^i e^i \]

2.) Why is $V^*$ spanned by $\beta^* = \{e^i\}_{i=1}^n$?
Let $\alpha \in V^*$, $x \in V$
\[ \alpha(x) = \alpha\left(\sum_{i=1}^n x^i e_i\right) \]
\[ = \sum_{i=1}^n x^i \alpha(e_i) \]
\[ = \sum_{i=1}^n x^i \cdot e^i(x) \]
\[ \alpha \in \text{span}\{e^i\}_{i=1}^n \]
\[ \forall x \]
\[ \therefore \alpha = \sum_{i=1}^n \alpha(e_i) e^i \in \text{span}\{e^i\}_{i=1}^n \]
Return to bilinear forms

Question: what is the natural basis for the vector space $T^0_2(V)$?

Answer: tensor product of dual-basis

Define: Let $\alpha, \gamma \in V^*$, then $\alpha \otimes \gamma \in T^0_2(V)$ defined by: $\forall x, y \in V$

$(\alpha \otimes \gamma)(x, y) = \alpha(x) \gamma(y)$

Proof: $(\alpha \otimes \gamma)(x_1 + cx_2, y) = \alpha(x_1 + cx_2) \gamma(y)$

$= (\alpha(x_1) + c\alpha(x_2)) \gamma(y)$

$= \alpha(x_1) \gamma(y) + c\alpha(x_2) \gamma(y)$

$= (\alpha \otimes \gamma)(x_1, y) + c(\alpha \otimes \gamma)(x_2, y)$

Prop: $(\alpha_1 + \alpha_2) \otimes \gamma = \alpha_1 \otimes \gamma + \alpha_2 \otimes \gamma$

$\alpha \otimes (\gamma_1 + \gamma_2) = \alpha \otimes \gamma_1 + \alpha \otimes \gamma_2$

Likewise constants factor out

$\alpha \otimes (c \gamma) = c \alpha \otimes \gamma$

Caution: usually $\alpha \otimes \gamma \neq \gamma \otimes \alpha$. 
Continuing with $\otimes$ for $T_2^0(V)$

Roughly speaking, $T_1^0(V) \otimes T_1^0(V) = T_2^0(V)$

$Th =$ span $\{ e^i \otimes e^j \mid i, j \in \mathbb{N}_n \} = T_2^0(V)$.

**Proof:** Let $b \in T_2^0(V)$. On pg. 3 we found

$$b (x, y) = \sum_i \sum_j b(e^i, e^j) x^i y^j$$

$$= \sum_i \sum_j b(e^i, e^j) e^i(x) e^j(y)$$

$$= \sum_i \sum_j b(e^i, e^j) (e^i \otimes e^j)(x, y)$$

$$= \left( \sum_i \sum_j b(e^i, e^j) e^i \otimes e^j \right)(x, y)$$

Therefore, $b = \sum_i \sum_j b(e^i, e^j) e^i \otimes e^j$.

**Consequence:** $\dim(T_2^0(V)) = n^2 = \dim(\mathbb{R}^{nxn})$

So there is some isomorphism between bilinear forms and $nxn$ matrices. (see 2)
Reminder about Isomorphism

\[
\begin{align*}
&\Phi_1 &\Phi_2 \\
&V_1 &V_2 \\
&\mathbb{R}^{\dim(V_1)} &\mathbb{R}^{\dim(V_2)} \\
&\Phi_2 \circ \Phi_1 &\text{Id}_n
\end{align*}
\]

provided \( \dim V_1 = \dim V_2 = n \).

\( \Phi_2 \circ \Phi_1 : V_1 \rightarrow V_2 \) provides an isomorphism

**Prop:** \( \Phi : T^\circ_2(V) \rightarrow \mathbb{R}^{n \times n} \) where \( \dim V = n \)

defined by \( (\Phi(b))_{ij} = b(e_i, e_j) \).

Usually, we say \( (\Phi(b))_{ij} = [b]_{ij} \)

\(\text{Ex.} \quad b(x, y) = x \cdot y \quad \text{for } x, y \in \mathbb{R}^n \)

\[= x^T y\]

\[= x^T I y \quad [b] = I_{n \times n}\]
**Proposition:** the matrix of a symmetric (antisymmetric) bilinear form is symmetric (antisymmetric)

**Proof:**
\[
(\Phi(b))_{ij} = b(e_i, e_j) = b(e_j, e_i) \text{ symmetric} \\
= (\Phi(b))_{ji}
\]
\[
\therefore \ [b]^T = [b] \quad \text{provided } b \text{ symmetric}
\]
Likewise \([b]^T = -[b]\) if \(b\) antisymmetric.

**Theorem:** \(T^o_2(V) = S^o_2(V) \oplus \Lambda^o_2(V)\)

**Proof:** to say \(W = W_1 \oplus W_2\) means 
\(W_1 \subseteq W\) and \(W_2 \subseteq W\) and \(W_1 \cap W_2 = \{0\}\) and 
\(\forall w \in W \exists w_1 \in W_1, w_2 \in W_2\) s.t. \(w = w_1 + w_2\).

Suppose \(b \in S^o_2(V)\) and \(b \in \Lambda^o_2(V)\) then
\[
b(x, y) = b(y, x) \quad \text{and} \quad b(x, y) = -b(y, x).
\]
\(\forall x, y \in V\) so in particular \(x = e_i\) and \(y = e_j\) 
\[
\Rightarrow b(e_i, e_j) = -b(e_j, e_i) \quad \forall ij \\
= b(e_i, e_j) = 0 \quad \forall ij.
\]
Proof Continued:

\[ b(x, y) = \frac{1}{2} (b(x, y) - b(y, x)) + \frac{1}{2} (b(x, y) + b(y, x)) \]

 antisymmetric Symmetric

\[ = b_0(x, y) + b_1(x, y) \]

Show \( b_0 \in S^0_2(V) \) and \( b_1 \in \Lambda^1_2(V) \)

Thus completes the proof.

\[ b_0(e_i, e_j) = \frac{1}{2} (b(e_i, e_j) + b(e_i, e_j)) \]

\[ = \frac{1}{2} (b_{ij} + b_{ji}) \]

\[ b_1(e_i, e_i) = \frac{1}{2} (b_{ij} - b_{ji}) \]

\[ b = \sum_{i, j} b_{ij} e^i \otimes e^j = \sum_{i, j} \frac{1}{2} (b_{ij} + b_{ji}) e^i \otimes e^j + \]

\[ + \sum_{i, j} \frac{1}{2} (b_{ij} - b_{ji}) e^i \otimes e^j \]
Let's expand on the calculation in $\mathbb{R}^3$

\[ b = b_{11} e^1 \otimes e^1 + b_{12} e^1 \otimes e^2 + b_{13} e^1 \otimes e^3 \]
\[ b_{21} e^2 \otimes e^1 + b_{22} e^2 \otimes e^2 + b_{23} e^2 \otimes e^3 \]
\[ b_{31} e^3 \otimes e^1 + b_{32} e^3 \otimes e^2 + b_{33} e^3 \otimes e^3 \]

\[ = b_{11} e^1 \otimes e^1 + b_{22} e^2 \otimes e^2 + b_{33} e^3 \otimes e^3 \]
\[ + \frac{1}{2} (b_{12} + b_{21}) e^1 \otimes e^2 + \frac{1}{2} (b_{21} + b_{12}) e^2 \otimes e^1 \]
\[ + \frac{1}{2} (b_{13} + b_{31}) e^1 \otimes e^3 + \frac{1}{2} (b_{31} + b_{13}) e^3 \otimes e^1 \]
\[ + \frac{1}{2} (b_{23} + b_{32}) e^2 \otimes e^3 + \frac{1}{2} (b_{32} + b_{23}) e^3 \otimes e^2 \]
\[ + \frac{1}{2} (b_{12} - b_{21}) e^1 \otimes e^2 + \frac{1}{2} (b_{21} - b_{12}) e^2 \otimes e^1 \]
\[ + \frac{1}{2} (b_{13} - b_{31}) e^1 \otimes e^3 + \frac{1}{2} (b_{31} - b_{13}) e^3 \otimes e^1 \]
\[ + \frac{1}{2} (b_{23} - b_{32}) e^2 \otimes e^3 + \frac{1}{2} (b_{32} - b_{23}) e^3 \otimes e^2 \]

\[ = b_{11} e^1 \otimes e^1 + b_{22} e^2 \otimes e^2 + b_{33} e^3 \otimes e^3 \]
\[ + \frac{1}{2} (b_{12} + b_{21}) [e^1 \otimes e^2 + e^2 \otimes e^1] \]
\[ + \frac{1}{2} (b_{13} + b_{31}) [e^1 \otimes e^3 + e^3 \otimes e^1] \]
\[ + \frac{1}{2} (b_{23} + b_{32}) [e^2 \otimes e^3 + e^3 \otimes e^2] \]
\[ + \frac{1}{2} (b_{12} - b_{21}) [e^1 \otimes e^2 - e^2 \otimes e^1] + \frac{1}{2} (b_{21} - b_{12}) [e^2 \otimes e^1] \]
\[ + \frac{1}{2} (b_{13} - b_{31}) [e^1 \otimes e^3 - e^3 \otimes e^1] . \]
In matrix notation:

\[ \mathbb{R}^{3 \times 3} = \text{span} \left\{ E_{ii} \mid i = 1, 2, 3 \right\} \cup \left\{ E_{ij} + E_{ji} \mid i \neq j, \ i, j \in \mathbb{N}_3 \right\} \]

\[ \oplus \text{span} \left\{ E_{ij} - E_{ji} \mid i \neq j, \ i, j \in \mathbb{N}_3 \right\} \]

**Goal:** \[ \sum b_{ij} e^i \otimes e^j = \sum e^i \otimes e^j \]

\[ e^i \otimes e^j = \frac{1}{2} (e^i \otimes e^j - e^j \otimes e^i) \]

**Definition of wedge product of two dual-vectors**

From (10) if \( b(x, y) = -b(y, x) \) then

\[ b = (b_{12} - b_{21}) e^1 \otimes e^2 + (b_{13} - b_{31}) e^1 \otimes e^3 \]

\[ + (b_{23} - b_{32}) e^2 \otimes e^3 \]

\[ b_{ij} = -b_{ji} \]

\[ = 2b_{12} e^1 \otimes e^2 + 2b_{13} e^1 \otimes e^3 + 2b_{23} e^2 \otimes e^3 \]

\[ = \sum_{i < j} 2b_{ij} e^i \otimes e^j \]
Physicists don't like bases:
\[
\{ e^i n^e, e^i n^e, e^i n^e \} \text{ is basis for } \Lambda^2 (\mathbb{R}^9)
\]
Assume antisymmetric.
\[
b = \sum_{i < j} b_{ij} e^i n^e + \sum_{i = j} b_{ii} e^i + \sum_{i < j} b_{ji} e^j n^i
\]
\[
= \sum_{i < j} b_{ij} e^i n^e + \sum_{i < j} (-b_{ji}) e^j n^i
\]
\[
= \sum_{i < j} b_{ij} e^i n^e + \sum_{i < j} (-b_{ji}) (-e^i n^j)
\]
\[
= \sum_{k < l} b_{kl} e^k n^l + \sum_{k < l} b_{kl} e^k n^l
\]
\[
= \sum_{i < j} 2 b_{ij} e^i n^e
\]

we're back to (13).

Physicists like to write \( b = b_{ij} e^i n^e \).
Wedge Product For forms extend linearly from our previous definition.

\[ \alpha \wedge \beta = (\sum \alpha_i e^i) \wedge (\sum \beta_j e^j) \]

\[ = \sum_{i,j} \alpha_i \beta_j e^i \wedge e^j \]

\[ \text{det}^2. \]

Example:

\[ \alpha = 3e^1 + 2e^2 \quad \beta = 4e^3 + e^1 \]

\[ \alpha \wedge \beta = (3e^1 + 2e^2) \wedge (4e^3 + e^1) \]

\[ = 3e^1 \wedge 4e^3 + 3e^1 \wedge 4e^3 \]

\[ + 2e^2 \wedge 4e^3 + 2e^2 \wedge 4e^3 \]

\[ + 2e^1 \wedge e^3 \]

\[ = 12e^1 \wedge e^3 + 8e^2 \wedge e^3 + 2e^3 \wedge e^1 \]

\[ = 8e^3 \wedge e^1 - 2e^3 \wedge e^1 \]

\[ = 12e^3 \wedge e^1 - 2e^3 \wedge e^1 \]

\[ = 12e^3 \wedge e^1 \]

\[ \equiv <8, -12, 2> \quad \text{flux form corresponding} \]

\[ \alpha = \omega <3, 2, 0> \quad \beta = \omega <1, 0, 4> \rightarrow <8, -12, -2> \]

\[ \alpha \wedge \beta = (\omega_A \wedge \omega_B) = \omega_A \times \omega_B \]

\[ (3\hat{i} + 2\hat{j}) \times (4\hat{k} + \hat{i}) = 8\hat{i} - 12\hat{j} - 2\hat{k} \]
In $\mathbb{R}^3$ we have two ways to represent a given vector $\vec{A} = (a, b, c)$

$$\omega_A = ae^1 + be^2 + ce^3$$

$$\varphi_A = ae^2\delta^3 + be^3\delta^1 + ce^1\delta^2$$

Generally true $\omega_A \wedge \omega_B = \varphi_A \times B$

$\dim \Lambda^3(\mathbb{R}^3) = \dim (\Lambda^1(\mathbb{R}^3)) = 3$

In $\mathbb{R}^4 (e^1, e^2, e^3, e^4)$

$\{ e^1 e^2, e^1 e^3, e^1 e^4 \} \ \text{6-dim'd}$

$\{ e^2 e^3, e^2 e^4, e^3 e^4 \} \ \text{5-dim'd}$

$\{ e^1, e^2, e^3, e^4 \} \ \text{4-dim'd}$

$\{ e^1 e^2 e^3, e^1 e^2 e^4 \}$

$\{ e^2 e^3 e^4, e^1 e^3 e^4 \}$

It's natural to match up 1-form with a vector. (good for $\mathbb{R}^n$)
\[ T^r_s(V) = \left\{ T : V \times V \times \cdots \times V \times V^* \times \cdots \times V^* \rightarrow \mathbb{R} \right\} \]

where \( T \) is linear in each slot.

\( T \) is called a tensor, rank \((r)\)

\[ \exists! \ T^i_j(V) = \left\{ \sum_{i,j} b_{ij} e_i \otimes e_j : b_{ij} \in \mathbb{R} \quad \text{for} \quad i,j \in \mathbb{N}_n \right\} \]

\( b \in T^i_j(V) \Rightarrow b(V, \alpha) = \left( \sum_{i,j} b_{ij} e_i \otimes e_j \right)(V, \alpha) \)

\[ (e_i \otimes e_i)(V, \alpha) \triangleq e_i^2(V) \alpha(e_i) \]

\[ b = \sum_{i,j} b(e_i, e_j) e_i \otimes e_j \]

w/o some additional concepts it seems \( T^0_0(V) \) & \( T^r_0(V) \)

are logical places to ask about symmetric or antisymmetric.
Question: \( T: V \times V \times V \to \mathbb{R} \)

can we write \( T \) as a sum of a symmetric & antisymmetric part?

\[
T( V_1, V_2, V_3 ) = \text{sgn}(\sigma) T( V_{\sigma(1)}, V_{\sigma(2)}, V_{\sigma(3)})
\]  

\( \text{antisymmetric} \)

where \( \sigma: \mathbb{N}_3 \to \mathbb{N}_3 \) is a permutation.

\[
T( V_1, V_2, V_3 ) = - T( V_2, V_1, V_3 )
\]

\( \text{sgn}(\sigma) = (-1)^{\# \text{cycles comprising } \sigma} \)

Answer: no.

\[
T = \sum_{ijkl} T_{ijkl} e^{i\phi} e^{j\theta} e^{k\phi} \quad (\text{Einstein notation})
\]

\[ T_{1111} \quad T_{121} \quad (?) \]

\( \text{no longer a matrix.} \)
Wedge Product:

Suppose $V$ is $n$-dim'l with basis $e_1, e_2, \ldots, e_n$.
$V^*$ is $n$-dim'l with dual basis $e^1, e^2, \ldots, e^n$.

The exterior algebra is:

$$\Lambda(V) = \mathbb{R} \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

Where:

$$\Lambda^0(V) = \mathbb{R} \quad \text{dim 1}$$

$$\Lambda^1(V) = V^* = \text{span} \{ e^1, e^2, \ldots, e^n \} \quad \text{dim } n$$

$$\Lambda^2(V) = \text{span} \{ e^i e^j | i < j \} \quad \rightarrow \text{dim is } \binom{n}{2}$$

$$\Lambda^3(V) = \text{span} \{ e^i e^j e^k | i < j < k \} \quad \binom{n}{3}$$

$$\vdots$$

$$\Lambda^{n-1}(V) = \text{span} \{ e^{i_1} e^{i_2} \cdots e^{i_{n-1}} | i_1 < i_2 \cdots < i_{n-1}, \text{ multi-index notation } \} \quad \text{dim } n$$

$$\Omega^1 =$$

$$\Lambda^n(V) = \text{span} \{ e^{i_1} e^{i_2} \cdots e^{i_n} | i_1 < \cdots < i_n \} = \text{span} \{ e^1 e^2 \cdots e^n \} \quad \text{dim 1}$$
\[
\dim(\Lambda(V)) = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \ldots + \binom{n}{1} + \binom{n}{0}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k = (1+1)^n
\]
\[
= 2^n
\]

Binomial Theorem
\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n
\]

Properties of \( \Lambda(V) \)

Given \( \alpha \in \Lambda^p(V) \) and \( \beta \in \Lambda^q(V) \),

- Then \( \alpha \wedge \beta \in \Lambda^{p+q}(V) \).

\[
(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta
\]
\[
(c \alpha) \wedge \beta = \alpha \wedge (c \beta) = c(\alpha \wedge \beta)
\]
\[
\alpha \wedge \beta = (-1)^{p+q} \beta \wedge \alpha
\]

Can define \( \wedge \)-product via determinants.

Similarly \( \alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2 \).

\[
\{(A, \wedge A_2 \wedge \ldots \wedge A_n)(e_1, e_2, \ldots, e_n) = \det[A]
\]

where \( \text{col}_j(A) = [A_j(e_1), A_j(e_2), \ldots, A_j(e_n)]^T \).
Why \( \alpha^p \land \beta^q = (-1)^{pq} \beta^q \land \alpha^p \) 

Let \( \alpha^p = \sum_{i_1, i_2, \ldots, i_p} \alpha_{i_1, i_2, \ldots, i_p} dx^{i_1} \land dx^{i_2} \land \ldots \land dx^{i_p} \).

Linearize \( \beta^q = \sum_{j_1, j_2, \ldots, j_q} \beta_{j_1, j_2, \ldots, j_q} dx^{j_1} \land dx^{j_2} \land \ldots \land dx^{j_q} \).

Try to reassemble \( \beta^q \land \alpha^p \) and see what happens.

\[ \alpha^p \beta^q = \sum \sum \alpha_{i_1, i_2, \ldots, i_p} \beta_{j_1, j_2, \ldots, j_q} dx^{i_1} \land dx^{i_2} \land \ldots \land dx^{i_p} \land dx^{j_1} \land dx^{j_2} \land \ldots \land dx^{j_q} \]

Flip just #s.

\[ = \sum \sum \alpha_{i_1, i_2, \ldots, i_p} (-1)^{i_1} \beta_{j_1, j_2, \ldots, j_q} dx^{i_1} \land dx^{i_2} \land \ldots \land dx^{i_p} \land dx^{j_1} \land dx^{j_2} \land \ldots \land dx^{j_q} \]

To move \( dx^q \) up front.

\[ = \sum (-1)^{q} \alpha_{i_1, i_2, \ldots, i_p} \beta_{j_1, j_2, \ldots, j_q} dx^{j_1} \land dx^{j_2} \land \ldots \land dx^{j_q} \land dx^{i_1} \land dx^{i_2} \land \ldots \land dx^{i_p} \]

\[ = (-1)^{pq} \beta^q \land \alpha^p. \]
Example:

\[
\det \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} = (A_1 \cap A_2) (e_1, e_2)
\]

\[
= \left( \left[ ae' + ce^2 \right] \cap \left[ be' + de^2 \right] \right) (e_1, e_2)
\]

\[
= \left[ \left( ae' + ce^2 \right)(e_1), \left( ae' + ce^2 \right)(e_2) \right]^T = [a, c]^T
\]

\[
= \frac{ade^ne^2 + cbe^2ne'}{s + a e'n e' + c d e^n e^2} (e_1, e_2)
\]

\[
= (ad - bc)(e^1 e^2) (e_1, e_2)
\]

\[
= (ad - bc) [ (e'^1 e e^2 - e^2 e e' ) (e_1, e_2) ]
\]

\[
= (ad - bc) \left( e'^1 (e_1) e^2 (e_2) - e^2 (e_1) e'^1 (e_2) \right)
\]

\[
= ad - bc.
\]

This holds \( \forall e_1, e_2 \) thus,

\[
e_{e1} \wedge e_{e2} = (A e_{e1})^* \wedge (A e_{e2})^* = \det(A) e' ne^2
\]

(not the Hodge dual)
Tangent Space and the Cotangent Space

Everything we've calculated thus far is for a single isolated vector space. But, differential forms perhaps should be called "form fields" because they're the assignment of a form at each point along some curve or more commonly, over some manifold. (we'll stick to subspaces of $\mathbb{R}^n$)

Consider $M \subset \mathbb{R}^n$ and $(x^i)$ coordinates on $M$.

\[ T_p M = \{ \text{all tangent vectors to } M \text{ at } p \} \]
\[ = \{ \gamma'(0) \mid \gamma \text{ is path with } \gamma(0) = p \text{ and } \gamma: I \subset \mathbb{R} \rightarrow M \subset \mathbb{R}^n \} \]
\[ = \text{span} \left\{ \frac{\partial}{\partial x^i} \bigg|_p \right\}. \]
Claim: \( \text{span } \left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^{n} = T_p M \)

Set of smooth derivations at fact on \( M \) at \( p \)

\[ \gamma'(0) = \sqrt{v_1 e_1 + v_2 e_2 + \ldots + v_m e_m + \ldots + v_n e_n} \]

To construct \( \eta \), we need a mapping to convert \( \gamma'(0) \) to a corresponding derivation:

\[ \text{Der}_p(M) = \left\{ \xi: C_p(M) \to C_p(M) \right\} \text{ s.t. } \]

\[ \xi(fg) = \xi(f) \cdot g + f \xi(g) \quad \text{Leibniz} \]

\[ \xi(cf) = c \xi(f) \quad \text{Linear} \]

\[ \xi(f + g) = \xi(f) + \xi(g) \]

\[ \frac{\partial}{\partial x_i}(f) = \frac{\partial}{\partial u} \left[ (f \circ x^{-1}) \right] (x(p)) \]

\[ f: M \to \mathbb{R} \]

\[ x: M \subseteq \mathbb{R}^n \to \mathbb{R}^m \]

fct of \( U_1, U_2, \ldots, U_m \)

coord. down on \( T_p M \).
\( \hat{V} = \langle 1, 3 \rangle \iff \nabla = \frac{\partial}{\partial x} \langle p \rangle + 3 \frac{\partial}{\partial y} \langle p \rangle \)

To summarize,

\[
T_p M = \text{span} \left\{ \frac{\partial}{\partial x^1} \langle p \rangle, \frac{\partial}{\partial x^2} \langle p \rangle, \ldots, \frac{\partial}{\partial x^m} \langle p \rangle \right\}
\]

\[
(T_p M)^* = T_p M^* = \text{span} \left\{ dx^1, dx^2, \ldots, dx^m \right\}
\]

\[
(d_p x^i) \left( \frac{\partial}{\partial x^k} \right) \langle p \rangle = \delta^i_k
\]

\[
e^\circ (e_k) = \delta^j_k
\]

This defines \( e^* \) such we extend linearly.

\[
\text{Det} f / \quad \alpha = df \quad \text{and} \quad \mathbf{X} \quad \text{is a vector field.}
\]

\[
\alpha = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n
\]

\[
\alpha(\mathbf{X})(p) = df(\mathbf{X})(p) = \mathbf{X}_p(f)
\]

\[
df(\mathbf{X}) = \mathbf{X}(f)
\]

\[
e^\circ (V^i e_i) = e^\circ (V^i) e_i = e^\circ (e_i) = e^\circ (e^\circ (e_i)) = e^\circ (e^\circ (V^i)) e_i = e^\circ (V^i)
\]

\[
\sum_{k=1}^m x^k \frac{\partial}{\partial x^k} (x^i) = \sum_{k=1}^m x^k \delta^i_k = x^i
\]
Example \( R^2 = M \) \( u_i = x^i \)

\[
\text{Example} \quad R^2 = M \quad \text{"} u_i = x^i \text{"} \\

T_p R^2 = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \\
(T_p R^2)^* = \text{span} \left\{ dx, dy \right\} = \text{span} \left\{ dx, dy \right\} \\
\text{span} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \text{span} \left( e_1, e_2 \right) \\
X = r \cos \theta \\
Y = r \sin \theta \\
X : R^2 \rightarrow R \\
dx : T_p R^2 \rightarrow TR \\
\lim_{h \rightarrow 0} \frac{\|X(p+h) - X(p) - (d_p x)(h)\|}{\|h\|} = 0 \\
\text{How are } \partial/\partial x, \partial/\partial y \text{ and } \partial/\partial r, \partial/\partial \theta \text{ related?} \\
\frac{dr}{dx} = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
= \frac{x}{r} dx + \frac{y}{r} dy \\
= \cos \theta dx + \sin \theta dy \\
\frac{d\theta}{dx} = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \\
= \frac{-y}{r^2} dx + \frac{x}{r^2} dy \\
\frac{2r}{\partial x} = \frac{x}{r} \\
\theta = \tan^{-1} \left( \frac{y}{x} \right) \\
\frac{2\theta}{\partial x} = \frac{1}{1 + (\frac{y}{x})^2} \left( \frac{y}{x} \right)
Continuing:

\[ dr = \cos \theta \, dx + \sin \theta \, dy \]
\[ d\theta = -\frac{1}{r} \sin \theta \, dx + \frac{1}{r} \cos \theta \, dy \]

\[
\begin{bmatrix}
    dr \\
    d\theta
\end{bmatrix} =
\begin{bmatrix}
    \cos \theta & \sin \theta \\
    -\frac{\sin \theta}{r} & \frac{\cos \theta}{r}
\end{bmatrix}
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix}
\]

\[
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix} = r
\begin{bmatrix}
    \frac{\cos \theta}{r} & -\sin \theta \\
    \frac{\sin \theta}{r} & \frac{\cos \theta}{r}
\end{bmatrix}
\begin{bmatrix}
    dr \\
    d\theta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    \cos \theta \, dr - r \sin \theta \, d\theta \\
    \sin \theta \, dr + r \cos \theta \, d\theta
\end{bmatrix}
\]

\[
dx = \cos \theta \, dr - r \sin \theta \, d\theta
\]
\[
dy = \sin \theta \, dr + r \cos \theta \, d\theta
\]

\[\text{Vol}_e = dx \wedge dy = (\cos \theta \, dr - r \sin \theta \, d\theta) \wedge (\sin \theta \, dr + r \cos \theta \, d\theta)
\]
\[= r \cos^2 \theta \, dr \wedge d\theta - r \sin^2 \theta \, d\theta \wedge dr
\]
\[= r \, dr \wedge d\theta = \text{Vol}_e
\]

Volume element in \( \mathbb{R}^2 \).
Exterior Derivative

Let $\alpha \in \Lambda^p(M)$ then $d\alpha \in \Lambda^{p+1}(M)$

where $\alpha = \sum \alpha_i^p \, dx^i$ is

component function,

defined by

$$d\alpha = \sum_{I=1} \delta_i \alpha_i^p \wedge d\alpha^i$$

\[\text{(10)}\]

\[\text{Ex.}\]

$\alpha = x \, dy \wedge dz + e^{xyz} \, dx \wedge dy$

$d\alpha = dx \wedge dy \wedge dz + d(e^{xyz}) \wedge dx \wedge dy$

$= dx \wedge dy \wedge dz + (x \, ye^{xyz} \, dz + xze^{xyz} \, dy + yze^{xyz} \, dx) \wedge dx \wedge dy$

$= \left[ 1 + xye^{xyz} \right] dx \wedge dy \wedge dz$. 
\[ \omega = a \, dx + b \, dy + c \, dz \]

(when \( \vec{F}(P) = \langle a(P), b(P), c(P) \rangle \))

This is a vector field.

\[ d\omega = da \wedge dx + db \wedge dy + dc \wedge dz \]

\[ = \left( \frac{\partial a}{\partial x} \, dx + \frac{\partial a}{\partial y} \, dy + \frac{\partial a}{\partial z} \, dz \right) \wedge dx \]

\[ + \left( b_x \, dx + b_y \, dy + b_z \, dz \right) \wedge dy \]

\[ + \left( c_x \, dx + c_y \, dy + c_z \, dz \right) \wedge dz \]

\[ = \left( \frac{\partial a_y}{\partial y} - b_x \right) \, dy \wedge dx \]

\[ + \left( a_z - c_x \right) \, dz \wedge dx \]

\[ + \left( b_z - c_y \right) \, dz \wedge dy \]

\[ + o + o + o. \]

\[ = \left( \frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y} \right) dx \wedge dy + \left( \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial x} \right) dz \wedge dx \]

\[ + \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy \wedge dz \]

\[ d\omega = \nabla \times \vec{F} \]

\[ \nabla \times \vec{F} = \langle a, b, c \rangle = a \, dy \wedge dz + b \, dz \wedge dx + c \, dx \wedge dy \]
\[ \mathcal{G} = \langle G_1, G_2, G_3 \rangle \]

\[
\begin{aligned}
\mathcal{G} &= \mathcal{G} \left( G_1 \, dxdyndz + G_2 \, dxdzndx + G_3 \, dxndydy \right) \\
&= \partial G_1 \, dxdyndz + \partial G_2 \, ndzndx + \partial G_3 \, ndxndy \\
&= \left( \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right) dxdyndz
\end{aligned}
\]

\[
\mathcal{G} = (\nabla \cdot \mathcal{G}) \, dxdyndz
\]

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta
\]

**Proof**: \[ d(\alpha \wedge \beta) = \sum_{I, J} d(\alpha_I \beta_J) \, dx^I \wedge dx^J \]

\[
= \sum_{k, I, J} \left( \frac{\partial}{\partial x^k} (\alpha_I \beta_J) \right) d\alpha^k \wedge dx^I \wedge dx^J + \alpha_I \frac{\partial \beta_J}{\partial x^k} \, dx^k \wedge d\alpha^I \wedge dx^J
\]

\[
= d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta.
\]
\[ \omega_A \wedge \omega_B = \overrightarrow{A} \wedge \overrightarrow{B} \]

\[
\begin{align*}
d(\overrightarrow{A} \times \overrightarrow{B}) &= d(\omega_A \wedge \omega_B) \\
&= d\omega_A \wedge \omega_B - \omega_A \wedge d\omega_B \\
&= \overrightarrow{V} \times \overrightarrow{A} \wedge \omega_B - \omega_A \wedge \overrightarrow{V} \times \overrightarrow{B}
\end{align*}
\]

\[
\left(\nabla \cdot (A \times B)\right) dV d\eta d\xi d\zeta = \overrightarrow{V} \times \overrightarrow{A} \wedge \omega_B - \omega_A \wedge \overrightarrow{V} \times \overrightarrow{B}
\]

---

**Generalized Stokes' Thm**

\[
\int_{m} d\alpha = \int_{m} \alpha \quad \Rightarrow \quad \int_{m} d\omega_f = \int_{m} \omega_f
\]

\[
\int_{m} \oint d\varphi_{m_{a_{m}}} = \int_{m} \varphi_{a_{m}} \quad \Rightarrow \quad \int_{m} \oint (\overrightarrow{\nabla} \cdot \overrightarrow{F}) \cdot d\overrightarrow{s} = \int_{m} \overrightarrow{F} \cdot d\overrightarrow{l}
\]

\[
\int_{m} \mathbf{E} \cdot d\mathbf{s} = \int_{m} \mathbf{G} \cdot d\mathbf{s}
\]
Hodge Duality

Consider $V$ a vector space over $\mathbb{R}$ with dimension $n$. Then we can construct the exterior algebra $\Lambda(V) = \mathbb{R} \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V)$.

$m = \binom{n}{k} = \binom{n}{n-k} \rightarrow \Lambda^k(V) \cong \Lambda^{n-k}(V)$

$\exists x \bigg| V = \mathbb{R}^3$

$*(dx) = dy \wedge dz$, $*(dy) = dz \wedge dx$

$*(dy \wedge dz) = dx$

$*(\omega^\perp) = \perp \omega^\perp$

$*(*\alpha) = \alpha$

$\alpha = \sum_{i=1}^{n} \alpha_{i_1, i_2, \ldots, i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}$

$* \alpha = \sum_{i,j} \alpha^{'i_1, i_2, \ldots, i_k} \varepsilon_{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_m} \cdot dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$

$\alpha_{i_1 i_2 \cdots i_n} = g_{i_1 k_1} g_{i_2 k_2} \cdots g_{i_n k_n} \delta_{k_1 k_2 \cdots k_n}$

inverse metric
*Metrics* (Physics, Math)

\[ g : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R} \]

- \[ g(v, w) = g(w, v) \] symmetric
- \[ g(v, w) = 0 \quad \forall w \implies v = 0 \] nondegenerate

Mathematicians usually insist \( g(v, v) \geq 0 \). and \( g(v, v) = 0 \) if \( v = 0 \). (positive definite)

\[ \exists ! \mathbb{R}^n, \eta = (1, 0) \]

\[ g(v, w) = v^T \eta w \quad \text{Minkowski Metric.} \]

Note \( \eta^T = \eta \)

\[ g(w, v) = w^T \eta^T = (w^T \eta v)^T = v^T \eta w^T = v^T \eta w = g(v, w) \]

\[ g(e_i, e_j) = [g]_{ij} = \eta_{ij} \quad \delta_{ij} g_{ij} = \delta^i \]

Metric isomorphisms

\[ T : V \times V \to \mathbb{R} \to T(e_i, e_j) = T_{ij} \]

\[ \bar{T} : V^* \times V \to \mathbb{R} \text{ via the metric} \]

\[ \bar{T}(\alpha, \nu) = \bar{T}(\alpha_i e^i, \nu^j e_j) \quad (\Sigma \text{ implicit}) \]

\[ = \alpha_i \nu^j \bar{T}(e^i, e_j) \]

\[ = \alpha_i \nu^j \bar{T}_i^j \]

\[ = \alpha_i \nu^j g^{ik} T_{kj} \]

\[ = g^{ik} \alpha_i \nu^j T_{kj} \]

\[ = T(\alpha^i, \nu) \]

\[ \text{vector which is} \quad g^- \text{-dual to } \alpha. \]

\[ \alpha = \alpha_i e^i \quad \to \quad \alpha = g^{ik} \alpha_i e_k \quad \#'s \text{ vector} \]
Continuing

\[ \hat{T}: V^* \times V^* \rightarrow \mathbb{R} \]

\[ \hat{T}(e^i, e^j) = g^{ik} g^{jl} T_{kl} = T_{ij} \]