Coordinate Change for Linear Transformations on $\mathbb{R}^n$

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be basis for $\mathbb{R}^n$. Recall for $x \in \mathbb{R}^n$ we say $[x]_\beta = (c_1, c_2, \ldots, c_n) \iff x = c_1 v_1 + \cdots + c_n v_n$

$$= \underbrace{[v_1 | v_2 | \cdots | v_n]}_{[\beta]} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[x]_\beta}$$

That is, $\boxed{[x]_\beta = [\beta]^{-1} x}$ and $\boxed{x = [\beta][x]_\beta}$

We also defined $[T] \in \mathbb{R}^{m \times n}$ as the <u>standard matrix</u> of $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(x) = [T]x \quad \forall x \in \mathbb{R}^n$. We found $\boxed{[T] = [T(e_1) | T(e_2) | \cdots | T(e_n)]}$ where $\{e_1, e_2, \ldots, e_n\}$ is the <u>standard basis</u> for $\mathbb{R}^n$.

Def^n/ Given linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ the <u>matrix of T</u> with respect to basis β is denoted $[T]_{\beta,\beta}$ and is <u>defined</u> by

$$[T]_{\beta,\beta} = \big[[T(v_1)]_\beta \,|\, [T(v_2)]_\beta \,|\, \cdots \,|\, [T(v_n)]_\beta \big]$$

where $\beta = \{v_1, v_2, \ldots, v_n\}$

Lemma: for $\beta = \{V_1, V_2, \dots, V_n\}$ a basis for $\mathbb{R}^n$ the coordinate map $\Phi_\beta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\Phi_\beta(X) = [X]_\beta$ has

$$\left.\begin{array}{l} 1.)\ \Phi_\beta(X+Y) = [X+Y]_\beta = [X]_\beta + [Y]_\beta \\ 2.)\ \Phi_\beta(cX) = [cX]_\beta = c[X]_\beta \end{array}\right\} \text{in other words, } \Phi_\beta \text{ is linear transformation}$$

3.) $\Phi_\beta(V_j) = e_j$

4.) $[\Phi_\beta] = [\beta]^{-1}$

Proof: if $X = c_1 V_1 + \cdots + c_n V_n$ then $[X]_\beta = (c_1, \dots, c_n)$ by Def^{n}

likewise $Y = b_1 V_1 + \cdots + b_n V_n$ then $[Y]_\beta = (b_1, \dots, b_n)$.

Notice,

$$\begin{aligned} cX + Y &= c(c_1 V_1 + \cdots + c_n V_n) + b_1 V_1 + \cdots + b_n V_n \\ &= (cc_1 + b_1)V_1 + \cdots + (cc_n + b_n)V_n \end{aligned}$$

Hence

$$\begin{aligned} [cX+Y]_\beta &= (cc_1 + b_1, \dots, cc_n + b_n) \\ &= c(c_1, \dots, c_n) + (b_1, \dots, b_n) \\ &= c[X]_\beta + [Y]_\beta \quad \text{(this proves (1.) and (2.))} \end{aligned}$$

For (3.) note $V_j = 0V_1 + \cdots + 1V_j + \cdots + 0V_n \Rightarrow [V_j]_\beta = (0, \dots, 1, \dots, 0) = e_j$.

For (4.) $X = c_1 V_1 + \cdots + c_n V_n = [V_1 | \cdots | V_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\beta][X]_\beta \Rightarrow [X]_\beta = [\beta]^{-1}X$

Therefore $\Phi_\beta(X) = [X]_\beta = [\beta]^{-1}X \Rightarrow [\Phi_\beta] = [\beta]^{-1}$. //

Proposition: $[T]_{\beta\beta}[x]_\beta = [T(x)]_\beta$

Proof:

$$\begin{aligned}
[T(x)]_\beta &= \Phi_\beta\Big(T\big(\textstyle\sum_i c_i v_i\big)\Big) && : [x]_\beta = (c_1, \dots, c_n) \\
&= \Phi_\beta\Big(\textstyle\sum_i c_i T(v_i)\Big) && : \text{linearity of } T \\
&= \textstyle\sum_i c_i \, \Phi_\beta(T(v_i)) && : \text{linearity of } \Phi_\beta \\
&= \textstyle\sum_i c_i \, [T(v_i)]_\beta \\
&= \Big[[T(v_1)]_\beta \,\Big|\, [T(v_2)]_\beta \,\Big|\, \cdots \,\Big|\, [T(v_n)]_\beta \Big] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\
&= \underline{[T]_{\beta\beta}[x]_\beta} \; //
\end{aligned}$$

Th^{m}/ $[T]_{\beta\beta} = [\beta]^{-1}[T][\beta]$ where $T: \mathbb{R}^n \to \mathbb{R}^n$ and $[T]$ is standard matrix whereas $\beta = \{v_1, \dots, v_n\}$ has $[\beta] = [v_1 | \cdots | v_n]$ and $[T]_{\beta\beta} = [\,[T(v_1)]_\beta | \cdots | [T(v_n)]_\beta\,]$

Proof: using prop. and Lemma, $\underline{[T]_{\beta\beta}[x]_\beta} = [T(x)]_\beta = \underline{[\beta]^{-1}T(x)}$

hence $\underline{[T]_{\beta\beta}[\beta]^{-1}x} = \underline{[\beta]^{-1}[T]x}$ for all $x \in \mathbb{R}^n$

thus $[T]_{\beta\beta}[\beta]^{-1} = [\beta]^{-1}[T] \Rightarrow \underline{[T]_{\beta\beta} = [\beta]^{-1}[T][\beta]}$ //

$\boxed{E1}$ Suppose $T(x, y) = \left(\frac{1}{3}(-56x + 46y), \frac{1}{3}(-67x + 56y)\right)$

Let's find $[T]_{\beta,\beta}$ where $\beta = \{V_1, V_2\}$ and $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$[\beta] = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \Rightarrow [\beta]^{-1} = \frac{1}{5-8}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\begin{aligned}
[T]_{\beta\beta} &= [\beta]^{-1}[T][\beta] \\
&= \left(\frac{1}{3}\begin{bmatrix} -5 & 4 \\ 2 & -1 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} -56 & 46 \\ -67 & 56 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\right) \\
&= \frac{1}{9}\begin{bmatrix} -5 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 36 & 6 \\ 45 & 12 \end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix} 0 & 18 \\ 27 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}
\end{aligned}$$

Let $(x, y) = y_1 V_1 + y_2 V_2$

then $[(x,y)]_\beta = (y_1, y_2)$ and

$$[T]_{\beta\beta}\,[(x,y)]_\beta = [T(x,y)]_\beta$$

$$\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_2 \\ 3y_1 \end{bmatrix}$$

indicates

$$T(x,y) = (2y_2)V_1 + (3y_1)V_2$$

or,

$$\underline{T(y_1V_1 + y_2V_2) = 2y_2V_1 + 3y_1V_2}.$$

Remark:

$\det[T]_{\beta\beta} = \det[T] = -6$

$\text{trace}\,[T]_{\beta\beta} = \text{trace}\,[T] = 0$

$[T] = \frac{1}{3}\begin{bmatrix} -56 & 46 \\ -67 & 56 \end{bmatrix}$

$[T]_{\beta\beta}$ and $[T]$ are <u>similar</u> <u>matrices</u>; $[T]_{\beta\beta} = B[T]B^{-1}$.

E2

Consider $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 5 \end{bmatrix}$. Is <u>T onto or 1-1</u> ?

Let's calculate $\ker(T) = \text{Null}(A)$,

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Ax = 0 \text{ gives } x = (-x_3, -x_3, x_3)$$

so $\text{Null}(A) = \text{span}(-1,-1,1)$.

We find $\text{nullity}(T) = 1$ and $\text{rank}(T) = 2$.

Thus T is not 1-to-1 and T is not onto.

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ serve as basis for $\mathbb{R}^3$. We wish to calculate $[T]_{\beta,\beta}$

Since $[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]$ and $[T] = A$. After some calculation,

$$\text{rref}\left[\underbrace{\begin{matrix} 1 & 0 & -1 \\ 2 & 2 & -1 \\ 3 & 2 & 1 \end{matrix}}_{[\beta]} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right] = \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 4/6 & -2/6 & 2/6 \\ -5/6 & 4/6 & -1/6 \\ -2/6 & -2/6 & 2/6 \end{matrix}}_{[\beta]^{-1}} \right]$$

Hence calculate,

$$[T]_{\beta,\beta} = \frac{1}{6}\begin{bmatrix} 4 & -2 & 2 \\ -5 & 4 & -1 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & -2 & 2 \\ -5 & 4 & -1 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 0 \\ 18 & 12 & 0 \\ 22 & 14 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 8 & 4 & 0 \\ 30 & 24 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{[T]_{\beta,\beta} = \begin{bmatrix} 8/6 & 4/6 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

E2 continued

For $V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ and $V_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

If $X = aV_1 + bV_2 + cV_3$ then $[X]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

If $X = y_1 V_1 + y_2 V_2 + y_3 V_3$ then $[x]_\beta = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$T(x) = T(y_1 V_1 + y_2 V_2 + y_3 V_3)$$
$$= y_1 T(V_1) + y_2 T(V_2) + y_3 T(V_3)$$
$$= y_1 \underbrace{\left(\tfrac{8}{6} V_1 + 5 V_2\right)} + y_2 \underbrace{\left(\tfrac{4}{6} V_1 + 4 V_2\right)}$$

$$[T]_{\beta,\beta} = \begin{bmatrix} 8/6 & 4/6 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$[T]_{\beta,\beta} = \left[[T(V_1)]_\beta \,\middle|\, [T(V_2)]_\beta \,\middle|\, [T(V_3)]_\beta \right]$$

to say $[T(V_1)]_\beta = \begin{bmatrix} 8/6 \\ 5 \\ 0 \end{bmatrix}$ means $T(V_1) = \frac{8}{6} V_1 + 5 V_2$

to say $[T(V_2)]_\beta = \begin{bmatrix} 4/6 \\ 4 \\ 0 \end{bmatrix}$ means $T(V_2) = \frac{4}{6} V_1 + 4 V_2$

Thus, $\boxed{T(x) = \left(\tfrac{4}{3} y_1 + \tfrac{2}{3} y_2\right) V_1 + \left(5 y_1 + 4 y_2\right) V_2}$

Notice $T(x) = T(x_1 e_1 + x_2 e_2 + x_3 e_3) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1 + x_3) e_1 + (2x_1 + 2x_2 + 4x_3) e_2 + (3x_1 + 2x_2 + 5x_3) e_3$

$$[T]_{\beta\beta} [x]_\beta = [T(x)]_\beta \longrightarrow \begin{bmatrix} 4/3 & 2/3 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (4/3) y_1 + (2/3) y_2 \\ 5 y_1 + 4 y_2 \\ 0 \end{bmatrix}$$