# Differential Equations

James S. Cook
Liberty University
Department of Mathematics

Spring 2013

# preface

## format of my notes

These notes were prepared with LaTeX. You'll notice a number of standard conventions in my notes:

1. definitions are in green.
2. remarks are in red.
3. theorems, propositions, lemmas and corollaries are in blue.
4. proofs start with a **Proof:** and are concluded with a □. However, we also use the discuss... theorem format where a calculation/discussion leads to a theorem and the formal proof is left to the reader.

The purpose of these notes is to organize a large part of the theory for this course. Your text has much additional discussion about how to calculate given problems and, more importantly, the detailed analysis of many applied problems. I include some applications in these notes, but my focus is much more on the topic of computation and, where possible, the extent of our knowledge. There are some comments in the text and in my notes which are not central to the course. To learn what is most important you should come to class every time we meet.

## sources and philosophy of my notes

I draw from a number of excellent sources to create these notes. Naturally, having taught from the text for about a dozen courses, Nagle Saff and Snider has had great influence in my thinking on DEqns, however, more recently I have been reading: the classic *Introduction to Differential Equations* by Albert Rabenstein. I reccomend that text for further reading. In particular, Rabenstein's treatment of existence and convergence is far deeper than I attempt in these notes. In addition, the required text *Differential Equations with Applications* by Ritger and Rose is a classic text with many details that are lost in the more recent generation of texts. I'm also planning to consult the texts by Rice & Strange, Zill, Edwards & Penny , Finney and Ostberg, Coddington, Zachmanoglou and Thoe, Hille, Ince, Blanchard-Devaney and Hall, Martin, Campbell as well as others I'll add to this list once these notes are more complete (some year).

Additional examples are also posted. My website has several hundred pages of solutions from problems in Nagle Saff and Snider. I hope you will read these notes and Ritger & Rose as you study differential equations this semester. My old lecture notes are sometimes useful, but I hope the theory in these notes is superior in clarity and extent. My primary goal is the **algebraic** justification of the computational essentials for differential equations. For the Spring 2013 semester I have changed to Ritger & Rose as the primary text. Nagle Saff and Snider is better in some respects, but I think $160 is a bit much and older editions are a bit tricky to find for a relatively large class. I decided to leave some comments about Nagel Saff and Snider in these notes, I'm sorry if they are a distraction, but if you are curious then you are free to take a look at my copy of Nagel Saff and Snider in office hours. You don't need to do that, but the resource is there if you want it. I hope you can tell from these notes and your homework that my thinking comes from a variety of sources and there is much more for everyone to learn. Of course we will hit all the basics in your course.

The topics which are very incomplete in this current version of my notes is:

1. the special case of Frobenius method
2. the magic formulas of Ritger and Rose on constant coefficient case
3. Laplace transforms
4. theory of orthogonal functions, Fourier techniques
5. separation of variables to solve PDEs
6. linear system analysis via Greens functions and the transfer function

I have hand-written notes on most of these topics and I will post links as the semester progresses to appropriate pdfs. More than we need is already posted on my website, but I'm trying to refine these notes in view of the presentation in Ritger and Rose hence I'll probably write up a few new sets of notes later this semester.

James Cook, January 5, 2013.

version 2.0

# Contents

# Chapter 1

# terminology and goals

## 1.1 terms and conditions

A **differential equation** (or DEqn) is simply an equation which involves derivatives. The **order** of a differential equation is the highest derivative which appears nontrivially in the DEqn. The **domain of definition** is the set of points for which the expression defining the DEqn exists. We consider real independent variables and for the most part real depedent variables, however we will have occasion to consider complex-valued objects. The complexity will occur in the range but not in the domain. We continue to use the usual notations for derivatives and integrals in this course. I will not define these here, but we should all understand the meaning of the symbols below: the following are examples of **ordinary dervatives**

$$\frac{dy}{dx} = y' \qquad \frac{d^2y}{dt^2} = y'' \qquad \frac{d^ny}{dt^n} = y^{(n)} \qquad \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

because the **dependent variables** depend on only one **independent variable**. Notation is not reserved globally in this course. Sometimes $x$ is an independent variable whereas other times it is used as a dependent variable, context is key; $\frac{dy}{dx}$ suggests $x$ is independent and $y$ is dependent whereas $\frac{dx}{dt}$ has independent variable $t$ and dependent variable $x$. A DEqn which involves only ordinary derivatives is called an **Ordinary Differential Equation** or as is often customary an "ODE". The majority of this course we focus our efforts on solving and analyzing ODEs. However, even in the most basic first order differential equations the concept of partial differentiation and functions of several variables play a key and notable role. For example, an **$n$-th order ODE** is an equation of the form $F(y^{(n)}, y^{(n-1)}, \dots, y'', y', y, x) = 0$. For example,

$$y'' + 3y' + 4y^2 = 0 \;\; (n = 2) \qquad y^{(k)}(x) - y^2 - xy = 0 \;\; (n = k)$$

When $n = 1$ we say we have a **first-order ODE**, often it is convenient to write such an ODE in the form $\frac{dy}{dx} = f(x, y)$. For example,

$$\frac{dy}{dx} = x^2 + y^2 \text{ has } f(x,y) = x^2 + y^2) \qquad \frac{dr}{d\theta} = r\theta + 7 \text{ has } f(r,\theta) = r\theta + 7 + 7$$

A **system of ODEs** is a set of ODEs which share a common independent variable and a set of several dependent variables. For example, the following system has dependent variables $x, y, z$ and independent variable $t$:

$$\frac{dx}{dt} = x^2 + y + \sin(t)z, \qquad \frac{d^2y}{dt^2} = xyz + e^t, \qquad \frac{dz}{dt} = \sqrt{x^2 + y^2 + z^2}.$$

The examples given up to this point were all **nonlinear** ODEs because the dependent variable or it's derivatives appeared in a nonlinear manner. Such equations are actually quite challenging to solve and the general theory is not found in introductory textbooks. It turns out that we can solve many nonlinear first order ODEs, however, solvable higher-order nonlinear problems are for the most part beyond the reach of this course.

A **$n$-th order linear ODE in standard form** is a DEqn of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y + a_o y = g$$

where $a_n, a_{n-1}, \dots, a_1, a_o$ are the **coefficients** which are generally functions and $g$ is the **forcing function** or **inhomogenous term**. Continuing, if an $n$-th order ODE has $g = 0$ then we say it is a **homogenous** DEqn. When the coefficients are simply constants then the DEqn is said to be a **constant coefficient** DEqn. It turns out that we can solve **any** constant coefficient $n$-th order ODE. A system of ODEs for which each DEqn is linear is called a **system of linear DEqns**. For example:

$$x'' = x + y + z + t \qquad y'' = x - y + 2z, \qquad z'' = z + t^3.$$

If each linear DEqn in the system has constant coefficients then the system is also said to be a **constant coefficient** system of linear ODEs. We will see how to solve any **constant coefficient** linear system. Linear differential equations with nonconstant coefficients are not as simple to solve, however, we will solve a number of interesting problems via the series technique.

**Partial derivatives** are defined for functions or variables which depend on multiple independent variables. For example,

$$u_x = \frac{\partial u}{\partial x} \qquad T_{xy} = \frac{\partial^2 T}{\partial y \partial x} \qquad \nabla^2 \Phi = \partial_x^2 \Phi + \partial_y^2 \Phi + \partial_z^2 \Phi \qquad \nabla \bullet \vec{E} \qquad \nabla \times \vec{B}.$$

You should have studied the divergence $\nabla \bullet \vec{E}$ and curl $\nabla \times \vec{B}$ in multivariable calculus. The expression $\nabla^2 \Phi$ is called the **Laplacian** of $\Phi$. A DEqn which involves partial derivatives is called a **Partial Differential Equation** or as is often customary a "PDE". We study PDEs towards the conclusion of this course. It turns out that solving PDEs is naturally accomplished by a mixture of ODE and general series techniques.

## 1.2 philosophy and goals

What is our primary goal in this course? In a nutshell; to find the solution. Obviously this begs a question: "what is the solution to a DEqn?" I would answer that as follows: **a solution to a DEqn is a function or level set for which the given differential equation is a differential consequence of the solution.** In other words, a solution to a given DEqn is some object that satifies the DEqn when you "plug it in". For example, $y = \cos(x)$ is a solution of $y'' + y = 0$ since $y = \cos(x)$ implies $y'' = -\cos(x) = -y$ thus $y'' + y = 0$. However, I prefer the term "differential consequence" as it is more honest as to what I expect in calculations. Sometimes we cannot even solve for the solution as a function, it may be that implicit differentiation is more natural. Thinking in terms of functions alone would be a severely limiting perspective in this course. For example, if you implicitly differentiate $xy^3 + y^2 = \sin(x) + 3$ then it is easy to see the equation $xy^3 + y^2 = \sin(x) + 3$ defines a solution of $y^3 + 3xy^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \cos(x) = 0$. I would rather not find the solution as a function of $x$ in that example. That said, it is convenient to define an **explicit solution** on $I \subseteq \mathbb{R}$ for an $n$-th order ODE $F(y^{(n)}, y^{(n-1)}, \ldots, y'', y', y, x) = 0$ is a function $\phi$ such that $F(\phi^{(n)}(x), \phi^{(n-1)}(x), \ldots, \phi''(x), \phi'(x), \phi(x), x) = 0$ for all $x \in I$. In many problems we do not discuss $I$ since $I = \mathbb{R}$ and it is obvious, however, when we discuss singular points in the later portion of the course the domain of defintion plays an interesting and non-trivial role.

Very well, the concept of a solution is not too difficult. Let's ask a harder question: how do we find solutions? Begin with a simple problem:

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad \int \frac{dy}{dx}dx = \int 0\, dx \quad \Rightarrow \quad \boxed{y = c_o.}$$

Integrating revealed that solutions of $y' = 0$ are simply constant functions. Notice each distinct value for $c_o$ yields a distinct solution. What about $y'' = 0$?

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \int \frac{d}{dx}\left[\frac{dy}{dx}\right]dx = \int 0\, dx \quad \Rightarrow \quad \frac{dy}{dx} = c_1.$$

Integrate indefinitely once more,

$$\int \frac{dy}{dx}\, dx = \int c_1\, dx \quad \Rightarrow \quad y = c_1 x + c_o.$$

We derive $\boxed{y = c_1 x + c_o}$ is a solution for each pair of constants $c_o, c_1$. In other words, there is a whole family of solutions for $y'' = 0$ the solution set is $\{f \mid f(x) = c_1 x + c_o \text{ for } c_o, c_1 \in \mathbb{R}\}$. Each integration brings in a new integration constant. To solve $y^{(n)}(x) = 0$ we can integrate $n$-times to derive $\boxed{y = \frac{1}{(n-1)!}c_{n-1}x^{n-1} + \cdots + \frac{1}{2}c_2 x^2 + c_1 x + c_o.}$ We should know from Taylor's Theorem in second semester calculus the constants are given by $y^{(n)}(0) = c_n$ since the solution is a Taylor polynomial centered at $x = 0$. Hence we can write the solution in terms of the value of $y, y', y''$ etc... at $x = 0$: suppose $y^{(n)}(0) = y_n$ are given **initial conditions** then

$$y(x) = \frac{1}{(n-1)!}y_n x^{n-1} + \cdots + \frac{1}{2}y_2 x^2 + y_1 x + y_o$$

We see that the arbitary constants we derived allow for different initial conditions which are possible. In calculus we add $C$ to the indefinite integral to allow for all possible antiderivatives. In truth, $\int f(x)\,dx = \{F \mid F'(x) = f(x)\}$, it is a set of antiderivatives of the integrand $f$. However, almost nobody writes the set-notation because it is quite cumbersome. Likewise, in our current context we will be looking for the solution set of a DEqn, but we will call it the **general solution**. The **general solution** is usually many solutions which are indexed by a few arbitrary constants. For example, the general solution to $y''' - 4y'' + 3y' = 0$ is $y = c_1 + c_2e^t + c_3e^{3t}$. Or the general solution to $x' = -y$ and $y' = x$ is given by $x = c_1\cos(t) + c_2\sin(t)$ and $y = c_1\sin(t) - c_2\cos(t)$. To be careful, this is not always the case, there are curious DEqns which have just one solution or even none:

$$(y')^2 + y^2 = 0 \qquad (y')^2 + y^2 = -1.$$

There are other nonlinear examples where the constants index over most of the solution set, but miss a few special solutions.

We just saw that integration can sometimes solve a problem. However, can we always integrate? I mentioned that $y'' + y = 0$ has solution $y = \cos(x)$. In fact, you can show that $y = c_1\cos(x) + c_2\sin(x)$ is the general solution. Does integration reveal this directly?

$$y'' = -y \quad \Rightarrow \quad \int y''\,dx = \int -y\,dx \quad \Rightarrow \quad y' = C + \int -y\,dx$$

at this point we're stuck. In order to integrate we need to know the formula for $y$. But, that is what we are trying to find! DEqns that allow for solution by direct integration are somewhat rare. I'm not saying the solution cannot have an integral, for example, $y''(x) = g(x)$ for some continuous function $g$ has a solution which is obtained from twice integrating the DEQn: I'll find a solution in terms of the initial condtions at $x = 0$:

$$y'' = g \quad \Rightarrow \quad \int_0^x y'(t)\,dt = \int_0^x g(t)\,dt \quad \Rightarrow \quad y'(x) = y'(0) + \int_0^x g(t)\,dt$$

integrate once more, this time use $s$ as the dummy variable of integration,
note $y'(s) = y'(0) + \int_0^s g(t)\,dt$ hence

$$\int_0^x y'(s)\,ds = \int_0^x \left[y'(0) + \int_0^s g(t)\,dt\right] ds \quad \Rightarrow \quad \boxed{y(x) = y(0) + y'(0)x + \int_0^x \int_0^s g(t)\,dt\,ds.}$$

Note that the integral above does not involve $y$ itself, if we were give a nice enough function $g$ then we might be able to find an simple form of the solution in terms of elementary functions.

If direct integration is not how to solve *all* DEqns then what should we do? Well, that's what I'm going to be showing you this semester. Overall it is very similar to second semester calculus and integration. We make educated guesses then we differentiate to check if it worked. Once we find something that works then we look for ways to reformulate a broader class of problems back into those basic templates. But, the key here is guessing. Not blind guessing though. Often we make a general guess that has flexibility built-in via a few *parameters*. If the guess or *ansatz* is wise then the parameters are naturally chosen through some condition derived from the given DEqn.

If we make a guess then how do we know we didn't miss some possibility? I suppose we don't know. Unless we discuss some of the theory of differential equations. Fortunately there are deep and broad existence theorems which not only say the problems we are trying to solve are solvable, even more, the theory tells us how many *linearly independent* solutions we must find. The theory has the most to say about the linear case. However, as you can see from the nonlinear examples $(y')^2 + y^2 = 0$ and $(y')^2 + y^2 = -1$, there is not much we can easily say in general about the structure of solutions for nonlinear ODEs.

We say a set of conditions are **initial conditions (IC)** if they are all given at the same value of an independent variable. In contrast, **boundary conditions** or **BCs** are given at two or more values of the independent variables. If pair a DEqn with a set of initial conditions then the problem of solving the DEqn subject to the intial conditions is called an **initial value problem** or **IVP**. If pair a DEqn with a set of boundary conditions then the problem of solving the DEqn subject to the boundary conditions is called a **boundary value problem** or **BVP**. For example,

$$y'' + y = 0 \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 1$$

is an IVP. The **unique** solution is simply $y(x) = \sin(x)$. On the other hand,

$$y'' + y = 0 \qquad \text{with } y(0) = 0 \text{ and } y(\pi) = 0$$

is a BVP which has a family of solutions $y(x) = \sin(nx)$ indexed by $n \in \mathbb{Z}_{\geq 0}$. Other BVPs may have no solutions at all. We study BVPs in our analysis of PDEs towards the end of this course. Given a linear ODE with continuous coefficient and forcing functions on $I \subseteq \mathbb{R}$ the IVP has a unique solution which extends to all of $I$. In particular, the constant coefficient linear ODE has solutions on $\mathbb{R}$. This is a very nice result which is physically natural; given a DEqn which models some phenomenon we find that the same thing happens every time we start the system with a particular initial condition. In constrast, nonlinear DEqns sometime allow for the same initial condition to yield infinitely many possible solutions[1]

The majority of our efforts are placed on finding functions or equations which give solutions to DEqns. These are **quantitative** results. There is also much that can be said **qualitatively** or even **graphically**. In particular, we can study **autonomous** systems of the form $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$ by plotting the **direction field** of the system. The solutions can be seen by tracing out curves which line-up with the arrows in the direction field. Software[2] will plot direction fields for autonomous systems and you can easily see what types of behaviour are possible. All of this is possible when explicit quantitative solutions are intractable.

[1] I don't mean to say nonlinear DEqns are unphysical, however, the linear case is easier to understand.
[2] such as pplane of **Matlab** which is built-in to an applet linked on the course page

Numerical solutions to DEqns is one topic these notes neglect. If you are interested in numerical methods then you should try to take the numerical methods course. That course very useful to those who go on to business and industry, probably linear algebra is the only other course we offer which has as wide an applicability. Sadly most students avoid this course due to its supposed difficulty. The other topic which is neglected in these notes is rigor. For one thing, I use the concept of a differential in this course. Recall that if $F$ is a function of $x_1, x_2, \dots, x_n$ then we defined

$$dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \cdots + \frac{\partial F}{\partial x_n}dx_n = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}dx_i.$$

When the symbol $d$ acts on an equation it is understood we are taking the total differential. I assume that is reasonable to either write

$$\frac{dy}{dx} = f(x,y) \qquad \text{or} \qquad dy = f(x,y)dx \qquad \text{or} \qquad f(x,y)dx - dy = 0.$$

Some mathematicians will say the expressions on the right are not so meaningful. I reject that. They are meaningful and I explain in great detail in the advanced calculus course how the method of differentials enjoys its sucess on the back of the implicit and inverse function theorems. However, this is not advanced calculus so I will not prove or deeply discuss those things here. I'm just going to use them, *formally* if you wish. More serious is our lack of focus on existence and convergence, those analytical discussions tend to beg questions from real analysis and are a bit beyond the level of these notes and this course.

## 1.3 a short overview of differential equations in basic physics

I'll speak to what I know a little about. These comments are for the reductionists in the audience.

1. **Newtonian Mechanics** is based on Newton's Second Law which is stated in terms of a time derivative of three functions. We use vector notation to say it succinctly as

$$\boxed{\frac{d\vec{P}}{dt} = \vec{F}_{net}}$$

where $\vec{P}$ is the momentum and $\vec{F}_{net}$ is the force applied.

2. **Lagrangian Mechanics** is the proper way of stating Newtonian mechanics. It centers its focus on energy and conserved quantities. It is mathematically equivalent to Newtonian Mechanics for some systems. The fundamental equations are called the Euler Lagrange equations they follow from Hamilton's principle of least action $\delta S = \delta \int L \; dt = 0$,

$$\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] = \frac{\partial L}{\partial y}.}$$

Lagrangian mechanics allows you to derive equations of physics in all sorts of curvy geometries. Geometric constraints are easily implemented by Lagrange multipliers. In any event, the mathematics here is integration, differentiation and to see the big picture variational calculus (I sometimes cover variational calculus in the Advanced Calculus course Math 332)

3. **Electricity and Magnetism** boils down to solving Maxwell's equations subject to various boundary conditions:

$$\boxed{\nabla \cdot \vec{B} = 0, \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o}, \qquad \nabla \times \vec{B} = \mu_o \vec{J} - \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.}$$

   Again, the mathematics here is calculus of several variables and vector notations. In other words, the mathematics of electromagnetism is vector calculus.

4. **Special Relativity** also uses vector calculus. However, linear algebra is really needed to properly understand the general structure of Lorentz transformations. Mathematically this is actually not so far removed from electromagnetism. In fact, electromagnetism as discovered by Maxwell around 1860 naturally included Einstein's special relativity. In relativitic coordinate free differential form language Maxwell's equations are simply stated as

$$\boxed{dF = 0, \qquad d * F = *J.}$$

   Newtonian mechanics is inconsistent with these equations thus Einstein's theory was inevitable.

5. **General Relativity** uses calculus on manifolds. A manifold is a curved surface which allows for calculus in local coordinates. The geometry of the manifold encodes the influence of gravity and conversely the presence of mass curves space and time.

6. **Quantum Mechanics** based on Schrodinger's equation which is a partial differential equation (much like Maxwell's equations) governing a complex wave function. Alternatively, quantum mechanics can be formulated through the path integral formalism as championed by Richard Feynman.

7. **Quantum Field Theory** is used to frame modern physics. The mathematics is not entirely understood. However, Lie groups, Lie algebras, supermanifolds, jet-bundles, algebraic geometry are likely to be part of the correct mathematical context. Physicists will say this is done, but mathematicians do not in general agree. To understand quantum field theory one needs to master calculus, differential equations and more generally develop an ability to conquer very long calculations.

In fact, all modern technical fields in one way or another have differential equations at their core. This is why you are expected to take this course.

Differential equations are also used to model phenomenon which are not basic; population models, radioactive decay, chemical reactions, mixing tank problems, heating and cooling, financial markets, fluid flow, a snowball which gathers snow as it falls, a bus stopping as it rolls through a giant vat of peanut butter, a rope falling off a table etc... the list is endless. If you think about the course you took in physics you'll realize that you were asked about specific times and events, but there is also the question of how the objects move once the forces start to act. The step-by-step continuous picture of the motion is going to be the solution to the differential equation called Newton's Second Law. Beyond the specific examples we look at in this course, it is my hope you gain a more general appreciation of the method. In a nutshell, the leap in concept is to use derivatives to model things.

## 1.4 course overview

1. Chapter 1: you've almost read the whole thing. By now you should realize it is to be read once now and once again at the end of the semester.

2. Chapter 2: we study first order ODEs. One way or another it usually comes back to some sort of integration.

3. Chapter 3: we study $n$-th order linear ODEs. I'll lecture on a method presented in Ritger and Rose which appears magical, however, we don't just want answers. We want understanding and this is brought to us from a powerful new way of thinking called the *operator method*. We'll see how many nontrivial problems are reduced to algebra. Variation of parameters takes care of the rest.

4. Chapter 4: some problems are too tricky for the method of Chapter 3, we are forces to resort to power series techniques. Moreover, some problems escape power series as well. The method of Frobenius helps us capture behaviour near regular singular points. The functions discovered here have tremendous application across the sciences.

5. Chapter 5: we study systems of linear ODEs, we'll need matrices and vectors to properly treat this topic. The concept of eigenvectors and eigenvalues plays an important role, however the operator method shines bright once more here.

6. Chapter 6: the method of Laplace transforms is shown to solve problems with discontinuous, even infinite, forcing functions with ease.

7. Chapter 7: energy analysis and the phase plane approach. In other chapters our goal has almost always to find a solution , but here we study properties of the solution without actually finding it. This qualitative approach can reveal much without too much effort. When paired with the convenient pplane software we can ascertain many things with a minimum of effort.

8. Chapter 8: we pause to study series of functions. The concept of orthogonal functions is discussed. Fourier series and power series are important examples, however what else can we say?

9. Chapter 9: some are PDEs solved. We study heat, wave and Laplace's equations. Fourier techniques from the previous chapter play a central role.

In the Spring 2013 semester your Test 1 will focus on Chapters 1-3. Then Chapters 4-7 are covered by Test 2. Chapters 8 and 9 are likely covered on the (take-home) Test 3. The final focuses on Tests 1 and 2.

# Chapter 2

# ordinary first order problem

We wish to solve problems of the form $\frac{dy}{dx} = f(x,y)$. An **explicit solution** is a function $\phi : I \subset \mathbb{R} \to \mathbb{R}$ such that $\frac{d\phi}{dx} = f(x, \phi(x))$ for all $x \in I$. There are several common techniques to solve such problems, although, in general the solution may be impossible to find in terms of elementary functions. You should already anticipate this fact from second semester calculus. Performing an indefinite integration is equivalent to solving a differential equation; observe that

$$\int e^{x^2}\, dx = y \qquad \Leftrightarrow \qquad \frac{dy}{dx} = e^{x^2}.$$

you may recall that the integration above is not amenable to elementary techniques[1]. However, it is simple enough to solve the problem with series techniques. Using term-by-term integration,

$$\int e^{x^2}\, dx = \int \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} x^{2n+1} + c.$$

This simple calculation shows that $y = x + \frac{1}{6}x^3 + \frac{1}{10}x^5 + \cdots$ will solve $\frac{dy}{dx} = e^{x^2}$. We will return to the application of series techniques to find analytic solutions later in this course. For this chapter, we wish to discuss those techniques which allow us to solve first order problems via algebra and integrals of elementary functions. There are really three[2] main techniques:

1. separation of variables
2. integrating factor method
3. identification of problem as an exact equation

Beyond that we study substitutions which bring the problem back to one of the three problems above in a new set of variables. The methods of this chapter are by no means complete or algorithmic. Solving arbitrary first order problems is an art, not unlike the problem of parametrizing a level curve. That said, it is not a hidden art, it is one we all must master.

[1] the proof of that is not elementary!

[2] you could divide these differently, it is true that the integrating factor technique is just a special substitution

## 2.1 separation of variables

Suppose you are faced with the problem $\frac{dy}{dx} = f(x,y)$. If it happens that $f$ can be factored into a product of functions $f(x,y) = g(x)h(y)$ then the problem is said to be **separable**. Proceed formally for now, suppose $h(y) \neq 0$,

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{dy}{h(y)} = g(x)\,dx \quad \Rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Ideally, we can perform the integrations above and solve for $y$ to find an explicit solution. However, it may even be preferrable to not solve for $y$ and capture the solution in an **implicit** form. Let me provide a couple examples before I prove the method at the end of this section.

**Example 2.1.1. Problem:** *Solve* $\frac{dy}{dx} = 2xy$.

**Solution:** *Separate variables to find* $\int \frac{dy}{y} = \int 2x\,dx$ *hence* $\ln|y| = x^2 + c$. *Exponentiate to obtain* $|y| = e^{x^2+c} = e^c e^{x^2}$. *The constant* $e^c \neq 0$ *however, the absolute value allows for either* $\pm$. *Moreover, we can also observe directly that* $y = 0$ *solves the problem. We find* $\boxed{y = ke^{x^2}}$ *is the general solution to the problem.*

An explicit solution of the differential equation is like an antiderivative of a given integrand. The general solution is like the indefinite integral of a given integrand. The general solution and the indefinite integral are not functions, instead, they are a family of functions of which each is an explicit solution or an antiderivative. Notice that for the problem of indefinite integration the constant can always just be thoughtlessly tacked on at the end and that will nicely index over all the possible antiderivatives. On the other hand, for a differential equation the constant could appear in many other ways.

**Example 2.1.2. Problem:** *Solve* $\frac{dy}{dx} = \frac{-2x}{2y}$.

**Solution:** *separate variables and find* $\int 2y\,dy = -\int 2x\,dx$ *hence* $y^2 = -x^2 + c$. *We find* $x^2 + y^2 = c$. *It is clear that* $c < 0$ *give no interesting solutions. Therefore, without loss of generality, we assume* $c \geq 0$ *and denote* $c = R^2$ *where* $R \geq 0$. *Altogether we find* $\boxed{x^2 + y^2 = R^2}$ *is the general* **implicit** *solution to the problem. To find an explicit solution we need to focus our efforts, there are two cases:*

1. *if* $(a,b)$ *is a point on the solution and* $b > 0$ *then* $y = \sqrt{a^2 + b^2 - x^2}$.

2. *if* $(a,b)$ *is a point on the solution and* $b < 0$ *then* $y = -\sqrt{a^2 + b^2 - x^2}$.

Notice here the constant appeared inside the square-root. I find the implicit formulation of the solution the most natural for the example above, it is obvious we have circles of radius $R$. To capture a single circle we need two function graphs. Generally, given an implicit solution we can solve for an explicit solution locally. The implicit function theorems of advanced calculus give explicit conditions on when this is possible.

**Example 2.1.3. Problem:** *Solve* $\frac{dy}{dx} = e^{x-2\ln|y|}$.

**Solution:** *recall* $e^{x-\ln|y|^2} = e^x e^{\ln|y|^2} = e^x|y|^2 = e^x y^2$. *Separate variables in view of this algebra:*

$$\frac{dy}{y^2} = e^x\,dx \quad \Rightarrow \quad \frac{-1}{y} = e^x + C \quad \Rightarrow \quad \boxed{y = \frac{-1}{e^x + C}.}$$

When I began this section I mentioned the justification was *formal*. I meant that to indicate the calculation seems plausible, but it is not justified. We now show that the method is in fact justified. In short, I show that the notation works.

**Proposition 2.1.4.** *separation of variables:*

The differential equation $\frac{dy}{dx} = g(x)h(y)$ has an implicit solution given by

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

for $(x, y)$ such that $h(y) \neq 0$.

**Proof:** to say the integrals above are an implicit solution to $\frac{dy}{dx} = g(x)h(y)$ means that the differential equation is a differential consequence of the integral equation. In other words, if we differentiate the integral equation we should hope to recover the given DEqn. Let's see how this happens, differentiate implicitly,

$$\frac{d}{dx}\int \frac{dy}{h(y)} = \frac{d}{dx}\int g(x)\,dx \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x) \quad \Rightarrow \quad \frac{dy}{dx} = h(y)g(x). \qquad \square$$

**Remark 2.1.5.**

Technically, there is a gap in the proof above. How did I know implicit differentiation was possible? Is it clear that the integral equation defines $y$ as a function of $x$ at least locally? We could use the implicit function theorem on the level curve $F(x, y) = \int \frac{dy}{h(y)} - \int g(x)\,dx = 0$. Observe that $\frac{\partial F}{\partial y} = \frac{1}{h(y)} \neq 0$ hence the implicit function theorem provides the existence of a function $\phi$ which has $F(x, \phi(x)) = 0$ at points near the given point with $h(y) \neq 0$. This comment comes to you from the advanced calculus course.

## 2.2 integrating factor method

Let $p$ and $q$ be continuous functions. The following differential equation is called a **linear differential equation** in standard form:

$$\boxed{\frac{dy}{dx} + py = q \qquad (\star)}$$

Our goal in this section is to solve equations of this type. Fortunately, linear differential equations are very nice and the solution exists and is not too hard to find in general, well, at least up-to a few integrations.

Notice, we cannot directly separate variables because of the $py$ term. A natural thing to notice is that it sort of looks like a product, maybe if we multiplied by some new function I then we could separate and integrate: multiply $\star$ by $I$,

$$I\frac{dy}{dx} + pIy = qI$$

Now, if we choose $I$ such that $\frac{dI}{dx} = pI$ then the equation above separates by the product rule:

$$\frac{dI}{dx} = pI \quad\Rightarrow\quad I\frac{dy}{dx} + \frac{dI}{dx}y = qI \quad\Rightarrow\quad \frac{d}{dx}\big[Iy\big] = qI \quad\Rightarrow\quad Iy = \int qI\,dx \quad\Rightarrow\quad \boxed{y = \frac{1}{I}\int qI\,dx.}$$

Very well, but, is it possible to find such a function $I$? Can we solve $\frac{dI}{dx} = pI$? Yes. Separate variables,

$$\frac{dI}{dx} = pI \quad\Rightarrow\quad \frac{dI}{I} = p\,dx \quad\Rightarrow\quad \ln(I) = \int p\,dx \quad\Rightarrow\quad \boxed{I = e^{\int p\,dx}.}$$

**Proposition 2.2.1.** *integrating factor method:*

Suppose $p, q$ are continuous functions which define the linear differential equation $\frac{dy}{dx} + py = q$ (label this $\star$). We can solve $\star$ by the following algorithm:

1. define $I = exp(\int p\,dx)$,
2. multiply $\star$ by $I$,
3. apply the product rule to write $I\star$ as $\frac{d}{dx}\big[Iy\big] = Iq$.
4. integrate both sides,
5. find general solution $y = \frac{1}{I}\int Iq\,dx$.

**Proof:** Define $I = e^{\int p\,dx}$, note that $p$ is continuous thus the antiderivative of $p$ exists by the FTC. Calculate,

$$\frac{dI}{dx} = \frac{d}{dx}e^{\int p\,dx} = e^{\int p\,dx}\frac{d}{dx}\int p\,dx = pe^{\int p\,dx} = pI.$$

Multiply $\star$ by $I$, use calculation above, and apply the product rule:

$$I\frac{dy}{dx} + Ipy = Iq \quad \Rightarrow \quad I\frac{dy}{dx} + \frac{dI}{dx}y = Iq \quad \Rightarrow \quad \frac{d}{dx}\big[Iy\big] = Iq.$$

Integrate both sides,

$$\int \frac{d}{dx}\big[Iy\big]\,dx = \int Iq\,dx \quad \Rightarrow \quad Iy = \int Iq\,dx \quad \Rightarrow \quad y = \frac{1}{I}\int Iq\,dx. \qquad \square$$

The integration in $y = \frac{1}{I}\int Iq\,dx$ is indefinite. It follows that we could write $y = \frac{C}{I} + \frac{1}{I}\int Iq\,dx$. Note once more that the constant is not simply added to the solution.

**Example 2.2.2. Problem:** *find the general solution of* $\frac{dy}{dx} + \frac{2}{x}y = 3$

**Solution:** *Identify that* $p = 2/x$ *for this linear DE. Calculate, for* $x \neq 0$,

$$I = exp\bigg(\int \frac{2dx}{x}\bigg) = exp(2\ln|x|) = exp(\ln|x|^2) = |x|^2 = x^2$$

*Multiply the DEqn by* $I = x^2$ *and then apply the reverse product rule;*

$$x^2\frac{dy}{dx} + 2xy = 3x^2 \quad \Rightarrow \quad \frac{d}{dx}\big[\,x^2y\,\big] = 3x^2$$

*Integrate both sides to obtain* $x^2y = x^3 + c$ *therefore* $\boxed{y = x + c/x^2}$.

We could also write $y(x) = x + c/x^2$ to emphasize that we have determined $y$ as a function of $x$.

**Example 2.2.3. Problem:** *let* $r$ *be a real constant and suppose* $g$ *is a continuous function, find the general solution of* $\frac{dy}{dt} - ry = g$

**Solution:** *Identify that* $p = r$ *for this linear DE with independent variable* $t$. *Calculate,*

$$I = exp\bigg(\int r\,dt\bigg) = e^{rt}$$

*Multiply the DEqn by* $I = e^{rt}$ *and then apply the reverse product rule;*

$$e^{rt}\frac{dy}{dt} + re^{rt}y = ge^{rt} \quad \Rightarrow \quad \frac{d}{dt}\big[\,e^{rt}y\,\big] = ge^{rt}$$

*Integrate both sides to obtain* $e^{rt}y = \int g(t)e^{rt}\,dt + c$ *therefore* $\boxed{y(t) = ce^{-rt} + e^{-rt}\displaystyle\int g(t)e^{rt}\,dt}$. *Now that we worked this in general it's fun to look at a few special cases:*

1. *if* $g = 0$ *then* $\boxed{y(t) = ce^{-rt}}$.

2. *if* $g(t) = e^{-rt}$ *then* $y(t) = ce^{-rt} + e^{-rt}\int e^{-rt}e^{rt}\,dt$ *hence* $\boxed{y(t) = ce^{-rt} + te^{-rt}}$.

3. *if* $r \neq s$ *and* $g(t) = e^{-st}$ *then* $y(t) = ce^{-rt} + e^{-rt}\int e^{-st}e^{rt}\,dt = ce^{-rt} + e^{-rt}\int e^{(r-s)t}\,dt$ *consqeuently we find that* $y(t) = ce^{-rt} + \frac{1}{r-s}e^{-rt}e^{(r-s)t}$ *and thus* $\boxed{y(t) = ce^{-rt} + \dfrac{1}{r-s}e^{-st}}$.

## 2.3 exact equations

Before we discuss the theory I need to introduce some new notation:

**Definition 2.3.1.** *Pffafian form of a differential equation*

Let $M, N$ be functions of $x, y$ then $Mdx + Ndy = 0$ is a differential equation in **Pfaffian form**.

For example, if $\frac{dy}{dx} = f(x, y)$ then $dy - f(x, y)dx = 0$ is the differential equation in its Pfaffian form. One advantage of the Pfaffian form is that it puts $x, y$ on an equal footing. There is no artificial requirement that $y$ be a function of $x$ implicit within the set-up, instead $x$ and $y$ appear in the same way. The natural solution to a differential equation in Pfaffian form is a level curve.

**Example 2.3.2.** *Consider the circle* $x^2 + y^2 = R^2$ *note that* $2xdx + 2ydy = 0$ *hence the circle is a solution curve of* $2xdx + 2ydy = 0$

Recall the total differential[3] of a function $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ was defined by:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Let $k$ be a constant and observe that $F(x, y) = k$ has $dF = 0dx + 0dy = 0$. Conversely, if we are given $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$ then we find natural solutions of the form $F(x, y) = k$ for appropriate constants $k$. Let us summarize the technique:

**Proposition 2.3.3.** *exact equations:*

If differential equation $Mdx + Ndy = 0$ has $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$ for some differentiable function $F$ then the solutions to the differential equation are given by the level-curves of $F$.

A **level-curve** of $F$ is simply the collection of points $(x, y)$ which solve $F(x, y) = k$ for a constant $k$. You could also call the solution set of $F(x, y) = k$ the $k$-level curve of $F$ or the fiber $F^{-1}\{k\}$.

**Example 2.3.4. Problem:** *find the solutions of* $y^2dx + 2xydy = 0$.

**Solution:** *we wish to find* $F$ *such that*

$$\frac{\partial F}{\partial x} = y^2 \qquad \& \qquad \frac{\partial F}{\partial y} = 2xy$$

*You can integrate these equations holding the non-integrated variable fixed,*

$$\frac{\partial F}{\partial x} = y^2 \quad \Rightarrow \quad F(x, y) = \int y^2\, dx = xy^2 + C_1(y)$$

[3] You might wonder what precisely $dx$ and $dy$ mean in such a context. If you want to really know then take advanced calculus. For our purposes here is suffices to inform you that you can multiply and divide by differentials, these formal algebraic operations are in fact a short-hand for deeper arguments justified by the implicit and/or inverse function theorems. But, again, that's advanced calculus.

$$\frac{\partial F}{\partial y} = 2xy \quad \Rightarrow \quad F(x,y) = \int 2xy\,dy = xy^2 + C_2(x)$$

*It follows that $F(x,y) = xy^2$ should suffice. Indeed a short calculation shows that the given differential equation in just $dF = 0$ hence the solutions have the form $xy^2 = k$. One special solution is $x = 0$ and $y$ free, this is allowed by the given differential equation, but sometimes you might not count this a solution. You can also find the explicit solutions here without too much trouble: $y^2 = k/x$ hence $y = \pm\sqrt{k/x}$ . These solutions foliate the plane into disjoint families in the four quadrants:*

$$k > 0 \text{ and } + \text{ in I}, \qquad k < 0 \text{ and } + \text{ in II}, \qquad k < 0 \text{ and } - \text{ in III}, \qquad k > 0 \text{ and } - \text{ in IV}$$

*The coordinate axes separate these cases and are themselves rather special solutions for the given DEqn.*

The explicit integration to find $F$ is not really necessary if you can make an educated guess. That is the approach I adopt for most problems.

**Example 2.3.5. Problem:** *find the solutions of* $2xy^2dx + (2x^2y - \sin(y))dy = 0$

**Solution:** *observe that the function* $F(x,y) = x^2y^2 + \cos(y)$ *has*

$$\frac{\partial F}{\partial x} = 2xy^2 \qquad \& \qquad \frac{\partial F}{\partial y} = 2x^2y - \sin(y)$$

*Consequently, the given differential equation is nothing more than $dF = 0$ which has obvious solutions of the form $x^2y^2 + \cos(y) = k$.*

I invite the reader to find explicit local solutions for this problem. I think I'll stick with the level curve view-point for examples like this one.

**Example 2.3.6. Problem:** *find the solutions of* $\frac{xdx+ydy}{x^2+y^2} = 0$

**Solution:** *observe that the function* $F(x,y) = \frac{1}{2}\ln(x^2+y^2)$ *has*

$$\frac{\partial F}{\partial x} = \frac{x}{x^2+y^2} \qquad \& \qquad \frac{\partial F}{\partial y} = \frac{y}{x^2+y^2}$$

*Consequently, the given differential equation is nothing more than $dF = 0$ which has curious solutions of the form $\frac{1}{2}\ln(x^2+y^2) = k$. If you expoentiate this equation it yields $\sqrt{x^2+y^2} = e^k$. We can see that the unit-circle corresponds to $k = 0$ whereas generally the $k$-level curve has radius $e^k$.*

Notice that $2x + 2y = 0$ and $\frac{xdx+ydy}{x^2+y^2} = 0$ share nearly the same set of solutions. The origin is the only thing which distinguishes these examples. This raises a question we should think about. When are two differential equations equivalent? I would offer this definition: two differential equations are equivalent if they share the same solution set. This is the natural extension of the concept we already know from algebra. Naturally the next question to ask is: how can we modify a given differential equation to obtain an equivalent differential equation? This is something we have to think about as the course progresses. Whenever we perform some operation to a differential equation we ought to ask, *did I just change the solution set?*. For example, multiplying $2x+2y=0$ by $\frac{1}{2(x^2+y^2)}$ removed the origin from the solution set of $\frac{xdx+ydy}{x^2+y^2}=0$.

### 2.3.1 conservative vector fields and exact equations

You should recognize the search for $F$ in the examples above from an analgous problem in multivariable calculus[4] Suppose $\vec{G}=\langle M,N\rangle$ is conservative on $U$ with potential function $F$ such that $\vec{G}=\nabla F$. Pick a point $(x_o,y_o)$ and let $C$ be the level curve of $F$ which starts at $(x_o,y_o)$[6]. Recall that the tangent vector field of the level curve $F(x,y)=k$ is perpendicular to the gradient vector field $\nabla F$ along $C$. It follows that $\int_C \nabla F \bullet d\vec{r}=0$. Or, in the differential notation for line-integrals, $\int_C Mdx+Ndy=0$.

Continuing our discussion, suppose $(x_1,y_1)$ is the endpoint of $C$. Let us define the line-segment $L_1$ from $(x_o,y_o)$ to $(x_o,y_1)$ and the line-segment $L_2$ from $(x_o,y_1)$ to $(x_1,y_1)$. The curve $L_1\cup L_2$ connects $(x_o,y_o)$ to $(x_1,y_1)$. By path-independence of conservative vector fields we know that $\int_{L_1\cup L_2}\vec{G}\bullet d\vec{r}=\int_C\vec{G}\bullet d\vec{r}$. It follows that[7]:

$$\begin{aligned} 0=\int_{L_1\cup L_2}\vec{G}\bullet d\vec{r} &= \int_{L_1} N\,dy+\int_{L_2} M\,dx \\ &= \int_{y_o}^{y_1} N(x_o,t)\,dt+\int_{x_o}^{x_1} M(t,y_1)\,dt \end{aligned}$$

Let $x_1=x$ and $y_1=y$ and observe that the equation

$$0=\int_{y_o}^{y} N(x_o,t)\,dt+\int_{x_o}^{x} M(t,y)\,dt$$

ought to provide the level-curve solution of the exact equation $Mdx+Ndy=0$ which passes through the point $(x_o,y_o)$. For future reference let me summarize our discussion here:

[4] Let us briefly review the results we derived for conservative vector fields in multivariable calculus. Recall that $\vec{G}=\langle M,N\rangle$ is conservative iff there exists a potential function[5] $F$ such that $\vec{G}=\nabla F=\langle\partial_x F,\partial_y F\rangle$ on $dom(\vec{G})$. Furthermore, it is known that $\vec{G}=\langle M,N\rangle$ is conservative on a simply connected domain iff $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ for all points in the domain. The Fundamental Theorem of Calculus for line-integrals states if $C$ is a curve from $P$ to $Q$ then $\int_C\nabla F\bullet d\vec{r}=F(Q)-F(P)$. It follows that conservative vector fields have the property of path-independence. In particular, if $\vec{G}$ is conservative on $U$ and $C_1,C_2$ are paths beginning and ending at the same points then $\int_{C_1}\vec{G}\bullet d\vec{r}=\int_{C_1}\vec{G}\bullet d\vec{r}$.

[6] the level curve extends past $C$, we just want to make $(x_o,y_o)$ the starting point

[7] Recall that if $C$ is parametrized by $\vec{r}(t)=\langle x(t),y(t)\rangle$ for $t_1\le t\le t_2$ then the line-integral of $\vec{G}=\langle M,N\rangle$ is by definition:

$$\int_C Mdx+Ndy=\int_{t_1}^{t_2}\left[M(x(t),y(t))\frac{dx}{dt}+N(x(t),y(t))\frac{dy}{dt}\right]dt$$

I implicitly make use of this definition in the derivation that follows.

**Proposition 2.3.7.** *solution by line-integral for exact equations:*

Suppose the differential equation $Mdx + Ndy = 0$ is exact on a simply connected region $U$ then the solution through $(x_o, y_o) \in U$ is given implicitly by

$$\int_{x_o}^{x} M(t,y)\,dt + \int_{y_o}^{y} N(x_o,t)\,dt = 0.$$

Perhaps you doubt this result. We can check it by taking the total differential of the proposed solution:

$$\begin{aligned} d\left[\int_{x_o}^{x} M(t,y)\,dt\right] &= \frac{\partial}{\partial x}\left[\int_{x_o}^{x} M(t,y)\,dt\right]dx + \frac{\partial}{\partial y}\left[\int_{x_o}^{x} M(t,y)\,dt\right]dy \\ &= M(x,y)dx + \left[\int_{x_o}^{x} \frac{\partial M}{\partial y}(t,y)\,dt\right]dy \\ &= M(x,y)dx + \left[\int_{x_o}^{x} \frac{\partial N}{\partial x}(t,y)\,dt\right]dy \qquad \text{since } \partial_x N = \partial_y M \\ &= M(x,y)dx + \big[N(x,y) - N(x_o,y)\big]dy \end{aligned}$$

On the other hand,

$$d\left[\int_{y_o}^{y} N(x_o,t)\,dt\right] = \frac{\partial}{\partial x}\left[\int_{y_o}^{y} N(x_o,t)\,dt\right]dx + \frac{\partial}{\partial y}\left[\int_{y_o}^{y} N(x_o,t)\,dt\right]dy = N(x_o,y)dy$$

Add the above results together to see that $M(x,y)dx + N(x,y)dy = 0$ is a differential consequence of the proposed solution. In other words, it works.

**Example 2.3.8. Problem:** *find the solutions of* $(2xy+e^y)dx+(2y+x^2+e^y)dy = 0$ *through* $(0,0)$*.*

**Solution:** *note* $M(x,y) = 2xy+e^y$ *and* $N(x,y) = 2y+x^2+e^y$ *has* $\partial_y M = \partial_x N$*. Apply Proposition 2.3.7*

$$\begin{aligned} \int_0^x M(t,y)\,dt + \int_0^y N(0,t)\,dt = 0 \quad &\Rightarrow \quad \int_0^x (2ty+e^y)\,dt + \int_0^y (2t+e^t)\,dt = 0 \\ &\Rightarrow \quad \left(t^2y+te^y\right)\Big|_0^x + \left(t^2+e^t\right)\Big|_0^y = 0 \\ &\Rightarrow \quad \boxed{x^2y + xe^y + y^2 + e^y - 1 = 0.} \end{aligned}$$

*You can easily verify that* $(0,0)$ *is a point on the curve boxed above.*

The technique illustrated in the example above is missing from many differential equations texts, I happened to discover it in the excellent text by Ritger and Rose *Differential Equations with Applications*. I suppose the real power of Proposition 2.3.7 is to capture formulas for an arbitrary point with a minimum of calculation:

**Example 2.3.9.** **Problem:** *find the solutions of $2x\,dx + 2y\,dy = 0$ through $(x_o, y_o)$.*

**Solution:** *note $M(x,y) = 2x$ and $N(x,y) = 2y$ has $\partial_y M = \partial_x N$. Apply Proposition 2.3.7*

$$\begin{aligned} \int_0^x M(t,y)\,dt + \int_0^y N(0,t)\,dt = 0 \quad &\Rightarrow \quad \int_{x_o}^x 2t\,dt + \int_{y_o}^y 2t\,dt = 0 \\ &\Rightarrow \quad t^2\Big|_{x_o}^x + t^2\Big|_{y_o}^y = 0 \\ &\Rightarrow \quad x^2 - x_o^2 + y^2 - y_o^2 = 0. \\ &\Rightarrow \quad \boxed{x^2 + y^2 = x_o^2 + y_o^2.} \end{aligned}$$

*The solutions are circles with radius $\sqrt{x_o^2 + y_o^2}$.*

You can solve exact equations without Proposition 2.3.7, but I like how this result ties the math back to multivariable calculus.

**Example 2.3.10.** **Problem:** *find the solutions of $E_1\,dx + E_2\,dy = 0$ through $(x_o, y_o)$. Assume $\partial_x E_2 = \partial_y E_1$.*

**Solution:** *the derivatiion of Proposition 2.3.7 showed that the solution of $E_1\,dx + E_2\,dy = 0$ is given by level curves of the potential function for $\vec{E} = \langle E_1, E_2\rangle$. In particular, if $\vec{E} = -\nabla V$, where the minus is customary in physics, then the solution is simply given by the equipotential curve $V(x,y) = V(x_o, y_o)$. In other words, we could interpret the examples in terms of voltage and electric fields. That is an important, real-world, application of this mathematics.*

### 2.3.2 inexact equations and integrating factors

Consider once more the Pfaffian form $Mdx + Ndy = 0$. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ at some point $P$ then we cannot find a potential function for a set which contains $P$. It follows that we can state the following no-go proposition for the problem of exact equations.

**Proposition 2.3.11.** *inexact equations:*

If differential equation $Mdx + Ndy = 0$ has $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $Mdx + Ndy = 0$ is **inexact**. In other words, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then there does not exist $F$ such that $dF = Mdx + Ndy$.

Pfaff was one of Gauss' teachers at the beginning of the nineteenth century. He was one of the first mathematicians to pursue solutions to exact equations. One of the theorems he discovered is that almost any first order differential equation $Mdx + Ndy = 0$ can be multiplied by an integrating factor $I$ to make the equation $IMdx + INdy = 0$ an exact equation. In other words, we can find $I$ such that there exists $F$ with $dF = IMdx + INdy$. I have not found a simple proof of this claim[8]

[8] this may be a pretty deep result, I would like to better understand the geometry of Pfaff's Theorem.

Given the proposition above, it is clear we must seek an integrating factor $I$ such that

$$\frac{\partial}{\partial y}\Big[\ IM\ \Big] = \frac{\partial}{\partial x}\Big[\ IN\ \Big].$$

Often for a particular problem we add some restriction to make the search for $I$ less daunting. In each of the examples below I add a restriction on the search for $I$ which helps us narrow the search[9]

**Example 2.3.12. Problem:** *find the solutions of*

$$\left(\ 3x + \frac{2}{y}\ \right)dx + \left(\ \frac{x^2}{y}\ \right)dy = 0 \qquad (\star)$$

*by finding an integrating factor to make the equation exact.*

**Solution:** *Since the problem only involves simple polynomials and rational functions the factor* $I = x^A y^B$ *may suffice. Let us give it a try and see if we can choose a particular value for* $A, B$ *to make* $I$ *a proper integrating factor for the given problem. Multiply* $\star$ *by* $I = x^A y^B$*,*

$$\left(\ 3x^{A+1}y^B + 2x^A y^{B-1}\ \right)dx + \left(\ x^{A+2}y^{B-1}\ \right)dy = 0 \qquad (I\star)$$

*Let* $M = 3x^{A+1}y^B + 2x^A y^{B-1}$ *and* $N = x^{A+2}y^{B-1}$*. We need* $\partial_y M = \partial_x N$*, this yields:*

$$3Bx^{A+1}y^{B-1} + 2(B-1)x^A y^{B-2} = (A+2)x^{A+1}y^{B-1}$$

*It follows that* $3B = A + 2$ *and* $2(B-1) = 0$*. Thus* $B = 1$ *and* $A = 1$*. We propose* $I = xy$ *serves as an integrating factor for* $\star$*. Multiply by* $\star$ *by* $xy$ *to obtain*

$$\left(\ 3x^2y + 2x\ \right)dx + \left(\ x^3\ \right)dy = 0 \qquad (xy\star)$$

*note that* $F(x,y) = x^3y + x^2 = k$ *has* $\partial_x F = 3x^2y + 2x$ *and* $\partial_y F = x^3$ *therefore* $F(x,y) = x^3y + x^2 = k$ *yield solutions to* $xy\star$*. These are also solutions for* $\star$*. However, we may have removed several solutions from the solution set when we multiplied by* $I$*. If* $I = 0$ *or if* $I$ *is undefined for some points in the plane then we must consider those points separately and directly with* $\star$*. Note that* $I = xy$ *is zero for* $x = 0$ *or* $y = 0$*. Clearly* $y = 0$ *is not a solution for* $\star$ *since it is outside the domain of definition for* $\star$*. On the other hand,* $x = 0$ *does solve* $\star$ *and is an* **extraneous solution**. *Let us summarize:* $\star$ *has solutions of the form* $x^3y + x^2 = k$ *or* $x \equiv 0$.

[9] It turns out that there are infinitely many integrating factors for a given inexact equation and we just need to find one that works.

**Example 2.3.13.** **Problem:** *find the solutions of* $\frac{dy}{dx} + Py = Q$ *by the method of exact equations. Assume that* $P, Q$ *are differentiable functions of* $x$.

**Solution:** *in Pfaffian form this DEqn takes the form* $dy + Pydx = Qdx$ *or* $(Py - Q)dx + dy = 0$. *Generally,* $P, Q$ *are not given such that this equation is exact. We seek an integrating factor* $I$ *such that* $I(Py - Q)dx + Idy = 0$ *is exact. We need:*

$$\frac{\partial}{\partial y}\big[I(Py - Q)\big] = \frac{\partial}{\partial x}\big[I\big]$$

*Assume that* $I$ *is not a function of* $y$ *for the sake of discovery, and it follows that* $IP = \frac{dI}{dx}$. *This is solved by separation of variables:* $\frac{dI}{I} = Pdx$ *implies* $\ln|I| = \int P\,dx$ *yielding* $I = exp(\int P\,dx)$. *This means the integrating factor is an integrating factor. We gave several examples in the previous section.*

The nice feature of the integrating factor $I = exp(\int P\,dx)$ is that when we multiply the linear differential equation $\frac{dy}{dx} + Py = Q$ we lose no solutions since $I \neq 0$. There are no extraneous solutions in this linear case.

You can read Nagel Saff and Snider pages 70-71 for further analysis of special integrating factors. This is a fascinating topic that we could easily spend a semester developing better tools to solve such problems. In particular, if you wish to do further reading I recommend the text by Peter Hydon on symmetries and differential equations. Or, if you want a deeper discussion which is still primarily computational you might look at the text by Brian Cantwell. The basic idea is that if you know a *symmetry* of the differential equation it allows you to find special coordinates where the equation is easy to solve. Ignoring the symmetry part, this is what we did in this section, we found an integrating factor which transforms the given inexact equation to the simple exact equation $dF = 0$. I'll conclude this section with a theorem borrowed from Ritger and Rose page 53 of §2.5:

**Theorem 2.3.14.** *integrating factors are not unique*

If $u(x, y)$ is an integrating factor of $M\,dx + N\,dy = 0$ and if $dv = uM\,dx + uN\,dy$ then $u(x, y)F(v(x, y))$ is also an integrating factor for any continuous function $F$

To see how this is true, integrate $F$ to obtain $G$ such that $G'(v) = F(v)$. Observe $dG = G'(v)dv = F(v)dv$. However, we know $dv = uM\,dx + uN\,dy$ hence $dG = F(v)[uM\,dx + uN\,dy] = uFM\,dx + uFN\,dy$ which shows the DEqn $M\,dx + N\,dy = 0$ is made exact upon multiplication by $uF$. This makes $uF$ and integrating factor as the theorem claims.

## 2.4 substitutions

In this section we discuss a few common substitutions. The idea of substitution is simply to transform a given problem to one we already know how to solve. Let me sketch the general idea before we get into examples: we are given

$$\frac{dy}{dx} = f(x,y)$$

We propose a new dependent variable $v$ which is defined by $y = h(x,v)$ for some function $h$. Observe, by the multivariate chain-rule,

$$\frac{dy}{dx} = \frac{d}{dx}h(x,v) = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial v}\frac{dv}{dx}$$

Hence, the substitution yields:

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial v}\frac{dv}{dx} = f(x, h(x,v))$$

which, if we choose wisely, is simpler to solve.

**Example 2.4.1. Problem:** *solve* $\frac{dy}{dx} = (x+y-6)^2$. *(call this* $\star$*)*

**Solution:** *the substitution* $v = x+y-6$ *looks promising. We obtain* $y = v-x+6$ *hence* $\frac{dy}{dx} = \frac{dv}{dx} - 1$ *thus the DEQn* $\star$ *transforms to*

$$\frac{dv}{dx} - 1 = v^2 \quad \Rightarrow \quad \frac{dv}{dx} = v^2+1 \quad \Rightarrow \quad \frac{dv}{1+v^2} = dx \quad \Rightarrow \quad \tan^{-1}(v) = x + C$$

*Hence,* $\tan^{-1}(x+y-6) = x+C$ *is the general, implicit, solution to* $\star$. *In this case we can solve for* $y$ *to find the explicit solution* $y = 6 + \tan(x+C) - x$.

**Remark 2.4.2.**

Generally the example above gives us hope that a DEqn of the form $\frac{dy}{dx} = F(ax+by+c)$ is solved through the substitution $v = ax+by+c$.

**Example 2.4.3. Problem:** *solve* $\frac{dy}{dx} = \frac{y/x+1}{y/x-1}$. *(call this* $\star$*)*

**Solution:** *the substitution* $v = y/x$ *looks promising. Note that* $y = xv$ *hence* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *by the product rule. We find* $\star$ *transforms to:*

$$v + x\frac{dv}{dx} = \frac{v+1}{v-1} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{v+1-v(v-1)}{v-1} = \frac{-v^2+2v+1}{v-1}$$

*Hence, separating variables,*

$$\frac{(v-1)\,dv}{-v^2+2v+1} = \frac{dx}{x} \quad \Rightarrow \quad -\frac{1}{2}\ln|v^2-2v-1| = \ln|x| + \tilde{C}$$

*Thus,* $\ln|v^2-2v-1| = \ln(1/x^2) + C$ *and after exponentiation and multiplication by* $x^2$ *we find the implicit solution* $\boxed{y^2 - 2xy - x^2 = K.}$

A differential equation of the form $\frac{dy}{dx} = F(y/x)$ is called **homogeneous**[10] . If we change coordinates by rescaling both $x$ and $y$ by the same scale then the ratio $y/x$ remains invariant; $\bar{x} = \lambda x$ and $\bar{y} = \lambda y$ gives $\frac{\bar{y}}{\bar{x}} = \frac{\lambda y}{\lambda x} = \frac{y}{x}$. It turns out this is the reason the example above worked out so nicely, the coordinate $v = y/x$ is invariant under the rescaling symmetry.

**Remark 2.4.4.**

Generally the example above gives us hope that a DEqn of the form $\frac{dy}{dx} = F(y/x)$ is solved through the substitution $v = y/x$.

**Example 2.4.5. Problem:** *Solve $y' + xy = xy^3$. (call this $\star$)*

**Solution:** *multiply by $y^{-3}$ to obtain $y^{-3}y' + xy^{-2} = x$. Let $z = y^{-2}$ and observe $z' = -2y^{-3}y'$ thus $y^{-3}y' = \frac{-1}{2}z'$. It follows that:*

$$\frac{-1}{2}\frac{dz}{dx} + xz = x \quad \Rightarrow \quad \frac{dz}{dx} - 2xz = -2x$$

*Identify this is a linear ODE and calculate the integrating factor is $e^{-x^2}$ hence*

$$e^{-x^2}\frac{dz}{dx} - 2xe^{-x^2}z = -2xe^{-x^2} \quad \Rightarrow \quad d(e^{-x^2}z) = -2xe^{-x^2}dx$$

*Conquently, $e^{-x^2}z = e^{-x^2} + C$ which gives $z = y^{-2} = 1 + Ce^{x^2}$. Finally, solve for $y$*

$$\boxed{y = \frac{\pm 1}{\sqrt{1 + Ce^{x^2}}}.}$$

*Given an initial condition we would need to select either $+$ or $-$ as appropriate.*

**Remark 2.4.6.**

This type of differential equation actually has a name; a differential equation of the type $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called a **Bernoulli DEqn**. The procedure to solve such problems is as follows:

1. multiply $\frac{dy}{dx} + P(x)y = Q(x)y^n$ by $y^{-n}$ to obtain $y^{-n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$,
2. make the substitution $z = y^{-n+1}$ and observe $z' = (1-n)y^{-n}y'$ hence $y^{-n}y' = \frac{1}{1-n}z'$,
3. solve the linear ODE in $z$; $\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$,
4. replace $z$ with $y^{-n+1}$ and solve if worthile for $y$.

[10] this term is used several times in this course with differing meanings. The more common use arises in the discussion of linear differential equations.

Substututions which change both the dependent and independent variable are naturally handled in the differential notation. If we replace $x = f(s,t)$ and $y = g(s,t)$ then $dx = f_s ds + f_t dt$ and $dy = g_s ds + g_t dt$. If we wish to transform $M(x,y)dx + N(x,y)dy$ into $s,t$ coordinates we simply substitute the natural expressions:

$$M(x,y)dx + N(x,y)dy = M(f(s,t),g(s,t))\left[\frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt\right] + N(f(s,t),g(s,t))\left[\frac{\partial g}{\partial s}ds + \frac{\partial g}{\partial t}dt\right].$$

Let us see how this works in a particular example:

**Example 2.4.7. Problem:** *solve* $(x+y+2)dx + (x-y)dy = 0$. *(call this* $\star$*)*

**Solution:** *the substitution* $s = x+y+2$ *and* $t = x-y$ *looks promising. Algebra yields* $x = \frac{1}{2}(s+t-2)$ *and* $y = \frac{1}{2}(s-t-2)$ *hence* $dx = \frac{1}{2}(ds+dt)$ *and* $dy = \frac{1}{2}(ds-dt)$ *thus* $\star$ *transforms to:*

$$s\frac{1}{2}(ds+dt) + t\frac{1}{2}(ds-dt) = 0 \quad \Rightarrow \quad (t+s)ds + (s-t)dt = 0 \quad \Rightarrow \quad \frac{dt}{ds} = \frac{t+s}{t-s}.$$

*It follows, for* $s \neq 0$,

$$\frac{dt}{ds} = \frac{t/s+1}{t/s-1}$$

*Recall we solved this in Example 2.4.3 hence:*

$$t^2 - 2st - s^2 = K \quad \Rightarrow \quad \boxed{(x-y)^2 - 2(x+y+2)(x-y) - (x+y+2)^2 = K.}$$

*You can simplify that to* $-2x^2 - 4xy - 8x + 2y^2 - 4 = K$. *On the other hand, this DEqn is exact so it is considerably easier to see that* $2x + \frac{x^2-y^2}{2} + xy = C$ *is the solution. Multiply by* $-4$ *to obtain* $-8x - 2x^2 + 2y^2 - 4xy = -4C$. *It is the same solution as we just found through a much more laborious method. I include this example here to illustrate the method, naturally the exact equation approach is the better solution. Most of these problems do not admit the exact equation short-cut.*

In retrospect, we were fortunate the transformed $\star$ was homogeneous. In Nagel Saff and Snider on pages 77-78 of the 5-th ed. a method for choosing $s$ and $t$ to insure homogeneity of the transformed DEqn is given.

**Example 2.4.8. Problem:** *solve* $\left[\frac{x}{\sqrt{x^2+y^2}} + y^2\right]dx + \left[\frac{y}{\sqrt{x^2+y^2}} - xy\right]dy = 0$. *(call this* $\star$*)*

**Solution:** *polar coordinates look promising here. Let* $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$,

$$dx = \cos(\theta)dr - r\sin(\theta)d\theta, \qquad dy = \sin(\theta)dr + r\cos(\theta)d\theta$$

*Furthermore,* $r = \sqrt{x^2+y^2}$. *We find* $\star$ *in polar coordinates,*

$$\left[\cos(\theta) + r^2\sin^2(\theta)\right]\left[\cos(\theta)dr - r\sin(\theta)d\theta\right] + \left[\sin(\theta) - r^2\cos(\theta)\sin(\theta)\right]\left[\sin(\theta)dr + r\cos(\theta)d\theta\right] = 0$$

*Multiply, collect terms, a few things cancel and we obtain:*

$$dr + \left[-r^3\sin^3(\theta) - r^3\sin(\theta)\cos^2(\theta)\right]d\theta = 0$$

*Hence,*

$$dr - r^3\sin(\theta)d\theta = 0 \quad \Rightarrow \quad \frac{dr}{r^3} = \sin(\theta)d\theta \quad \Rightarrow \quad \frac{-1}{2r^2} = -\cos(\theta) + C.$$

*Returning to Cartesian coordinates we find the implicit solution:*

$$\boxed{\frac{1}{2(x^2+y^2)} = \frac{x}{\sqrt{x^2+y^2}} - C.}$$

Sometimes a second-order differential equation is easily reduced to a first-order problem. The examples below illustrate a technique called **reduction of order**.

**Example 2.4.9. Problem:** *solve $y'' + y' = x^2$. (call this $\star$)*

**Solution:** *Let $y' = v$ and observe $y'' = v'$ hence $\star$ transforms to*

$$\frac{dv}{dx} - v = e^{-x}$$

*multiply the DEqn above by the integrating factor $e^x$:*

$$e^x\frac{dv}{dx} - ve^x = 1 \quad \Rightarrow \quad \frac{d}{dx}\Big[\, e^x v \,\Big] = 1$$

*thus $e^x v = x + c_1$ and we find $v = xe^{-x} + c_1e^{-x}$. Then as $v = \frac{dy}{dx}$ we can integrate once more to find the solution:*

$$y = \int \left[xe^{-x} + c_1e^{-x}\right]dx = -xe^{-x} - e^{-x} - c_1e^{-x} + c_2$$

*cleaning it up a bit,*

$$\boxed{y = -e^{-x}(x - 1 + c_1) + c_2.}$$

**Remark 2.4.10.**

Generally, given a differential equation of the form $y'' = F(y', x)$ we can solve it by a two-step process:

1. substitute $v = y'$ to obtain the first-order problem $v' = F(v, x)$. Solve for $v$.
2. recall $v = y'$, integrate to find $y$.

There will be two constants of integration. This is a typical feature of second-order ODE.

**Example 2.4.11.** **Problem:** *solve* $\frac{d^2y}{dt^2} + y = 0$. *(call this* $\star$*)*

**Solution:** *once more let* $v = \frac{dy}{dt}$. *Notice that*

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}$$

*thus* $\star$ *transforms to the first-order problem:*

$$v\frac{dv}{dy} + y = 0 \quad \Rightarrow \quad vdv + ydy = 0 \quad \Rightarrow \quad \frac{1}{2}v^2 + \frac{1}{2}y^2 = \frac{1}{2}C^2.$$

*assume the constant* $C > 0$, *note nothing is lost in doing this except the point solution* $y = 0, v = 0$. *Solving for* $v$ *we obtain* $v = \pm\sqrt{C - y^2}$. *However,* $v = \frac{dy}{dt}$ *so we find:*

$$\frac{dy}{\sqrt{C^2 - y^2}} = \pm dt \quad \Rightarrow \quad \sin^{-1}(y/C) = \pm t + \phi$$

*Thus,* $y = C\sin(\pm t + \phi)$. *We can just as well write* $y = A\sin(t + \phi)$. *Moreover, by trigonometry, this is the same as* $y = B\cos(t + \gamma)$, *it's just a matter of relabeling the constants in the general solution.*

**Remark 2.4.12.**

Generally, given a differential equation of the form $y'' = F(y)$ we can solve it by a two-step process:

1. substitute $v = y'$ and use the identity $\frac{dv}{dt} = v\frac{dv}{dy}$ to obtain the first-order problem $v\frac{dv}{dy} = F(y)$. Solve for $v$.
2. recall $v = y'$, integrate to find $y$.

There may be several cases possible as we solve for $v$, but in the end there will be two constants of integration.

## 2.5 physics and applications

I've broken this section into two parts. The initial subsection examines how we can use differential-equations techniques to better understand Newton's Laws and energy in classical mechanics. This sort of discussion is found in many of the older classic texts on differential equations. The second portion of this section is a collection of isolated application examples which are focused on a particular problems from a variety of fields.

### 2.5.1 physics

In physics we learn that $\vec{F}_{net} = m\vec{a}$ or, in terms of momentum $\vec{F}_{net} = \frac{d\vec{p}}{dt}$. We consider the one-dimensional problem hence we have no need of the vector notation and we generally are faced with the problem:

$$F_{net} = m\frac{dv}{dt} \qquad \text{or} \qquad F_{net} = \frac{dp}{dt}$$

where the momentum $p$ for a body with mass $m$ is given by $p = mv$ where $v$ is the velocity as defined by $v = \frac{dx}{dt}$. The acceleration $a$ is defined by $a = \frac{dv}{dt}$. It is also customary to use the dot and double dot notation for problems of classical mechanics. In particular: $v = \dot{x}$, $a = \dot{v} = \ddot{x}$. Generally the net-force can be a function of position, velocity and time; $F_{net} = F(x, v, t)$. For example,

1. the spring force is given by $F = -kx$
2. the force of gravity near the surface of the earth is given by $F = \pm mg$ ($\pm$ depends on interpretation of $x$)
3. force of gravity distance $x$ from center of mass $M$ given by $F = -\frac{GmM}{x^2}$
4. thrust force on a rocket depends on speed and rate at which mass is ejected
5. friction forces which depend on velocity $F = \pm bv^n$ ($\pm$ needed to insure friction force is opposite the direction of motion)
6. an external force, could be sinusoidal $F = A\cos(\omega t)$, ...

Suppose that the force only depends on $x$; $F = F(x)$ consider Newton's Second Law:

$$m\frac{dv}{dt} = F(x)$$

Notice that we can use the identity $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$ hence

$$mv\frac{dv}{dx} = F(x) \quad \Rightarrow \quad \int_{v_o}^{v_f} mv\,dv = \int_{x_o}^{x_f} F(x)\,dx \quad \Rightarrow \quad \boxed{\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_o^2 = \int_{x_o}^{x_f} F(x)\,dx.}$$

The equation boxed above is the **work-energy theorem**, it says the change in the kinetic energy $K = \frac{1}{2}mv^2$ is given by $\int_{x_o}^{x_f} F(x)\,dx$. which is the **work** done by the force $F$. This result holds for any net-force, however, in the case of a conservative force we have $F = -\frac{dU}{dx}$ for the **potential energy** function $U$ hence the work done by $F$ simplifies nicely

$$\int_{x_o}^{x_f} F(x)\,dx = -\int_{x_o}^{x_f} \frac{dU}{dx}\,dx = -U(x_f) + U(x_o)$$

and we obtain the **conservation of total mechanical energy** $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 = -U(x_f) + U(x_o)$ which is better written in terms of energy $E(x,v) = \frac{1}{2}mv^2 + U(x)$ as $E(x_o, v_o) = E(x_f, v_f)$. The total energy of a conservative system is constant. We can also see this by a direct-argument on the differential equation below:

$$m\frac{dv}{dt} = -\frac{dU}{dx} \quad \Rightarrow \quad m\frac{dv}{dt} + \frac{dU}{dx} = 0$$

multiply by $\frac{dx}{dt}$ and use the identity $\frac{d}{dt}\left[\frac{1}{2}v^2\right] = v\frac{dv}{dt}$:

$$m\frac{dx}{dt}\frac{dv}{dt} + \frac{dx}{dt}\frac{dU}{dx} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2\right] + \frac{dU}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = 0 \quad \Rightarrow \quad \boxed{\frac{dE}{dt} = 0.}$$

Once more we have derived that the energy is constant for a system with a net-force which is conservative. Note that as time evolves the expression $E(x,v) = \frac{1}{2}mv^2 + U(x)$ is invariant. It follows that the motion of the system is in described by an **energy-level** curve in the $xv$-plane. This plane is commonly called the **phase plane** in physics literature. Much information can be gleaned about the possible motions of a system by studying the energy level curves in the phase plane. I'll return to that topic later in the course.

We now turn to a mass $m$ for which the net-force is of the form $F(x,v) = -\frac{dU}{dx} \mp b|v|^n$. Here we insist that $-$ is given for $v > 0$ whereas the $+$ is given for the case $v < 0$ since we assume $b > 0$ and this friction force ought to point opposite the direction of motion. Once more consider Newton's Second Law:

$$m\frac{dv}{dt} = -\frac{dU}{dx} \mp bv^n \quad \Rightarrow \quad m\frac{dv}{dt} - \frac{dU}{dx} = \mp b|v|^n$$

multiply by the velocity and use the identity as we did in the conservative case:

$$m\frac{dx}{dt}\frac{dv}{dt} - \frac{dx}{dt}\frac{dU}{dx} = \mp bv|v|^n \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = \mp bv|v|^n \quad \Rightarrow \quad \boxed{\frac{dE}{dt} = \mp bv|v|^n.}$$

The friction force reduces the energy. For example, if $n = 1$ then we have $\frac{dE}{dt} = -bv^2$.

**Remark 2.5.1.**

The concept of energy is implicit within Example 2.4.11. I should also mention that the trick of multiplying by the velocity to reveal a conservation law is used again and again in the junior-level classical mechanics course.

### 2.5.2 applications

**Example 2.5.2. Problem:** *Suppose $x$ is the position of a mass undergoing one-dimensional, constant acceleration motion. You are given that initially we have velocity $v_o$ at position $x_o$ and later we have velocity $v_f$ at position $x_f$. Find how the initial and final velocities and positions are related.*

**Solution:** *recall that $a = \frac{dv}{dt}$ but, by the chain-rule we can write $a = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$. We are given that $a$ is a constant. Separate variables, and integrate with respect to the given data*

$$a = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \quad \Rightarrow \quad a\,dx = v\,dv \quad \Rightarrow \quad \int_{x_o}^{x_f} a\,dx = \int_{v_o}^{v_f} v\,dv \quad \Rightarrow \quad a(x_f - x_o) = \frac{1}{2}\big(v_f^2 - v_o^2\big).$$

*Therefore,* $\boxed{v_f^2 = v_o^2 + 2a(x_f - x_o)}$. *I hope you recognize this equation from physics.*

**Example 2.5.3. Problem:** *suppose the population $P$ grows at a rate which is directly proportional to the population. Let $k$ be the proportionality constant. Find the population at time $t$ in terms of the initial population $P_o$.*

**Solution:** *the given problem translates into the differential equation $\frac{dP}{dt} = kP$ with $P(0) = P_o$. Seperate variables and integrate, note $P > 0$ so I drop the absolute value bars in the integral,*

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad \int \frac{dP}{P} = \int k\,dt \quad \Rightarrow \quad \ln(P(t)) = kt + C$$

*Apply the initial condition;* $\ln(P(0)) = k(0) + C$ *hence* $C = \ln(P_o)$. *Consequently* $\ln(P(t)) = \ln(P_o) + kt$. *Exponentiate to derive* $\boxed{P(t) = P_o e^{kt}}$.

In the example above I have in mind $k > 0$, but if we allow $k < 0$ that models exponential population decline. Or, if we think of $P$ as the number of radioactive particles then the same mathematics for $k < 0$ models radioactive decay.

**Example 2.5.4. Problem:** *the voltage dropped across a resistor $R$ is given by the product of $R$ and the current $I$ through $R$. The voltage dropped across a capacitor $C$ depends on the charge $Q$ according to $C = Q/V$ (this is actually the definition of capacitance). If we connect $R$ and $C$ end-to-end making a loop then they are in parallel hence share the same voltage: $IR = \frac{Q}{C}$. As time goes on the charge on $C$ flows off the capacitor and through the resistor. It follows that $I = -\frac{dQ}{dt}$. If the capacitor initially has charge $Q_o$ then find $Q(t)$ and $I(t)$ for the* **discharging capacitor**

**Solution:** *We must solve*

$$-R\frac{dQ}{dt} = \frac{Q}{C}$$

*Separate variables, integrate, apply $Q(0) = Q_o$:*

$$\frac{dQ}{Q} = -\frac{dt}{RC} \quad \Rightarrow \quad \ln|Q| = -\frac{t}{RC} + c_1 \quad \Rightarrow \quad Q(t) = \pm e_1^c e^{-t/RC} \quad \Rightarrow \quad \boxed{Q(t) = Q_o e^{-t/RC}}$$

Another application of first order differential equations is simply to search for curves with particular properties. The next example illustrates that concept.

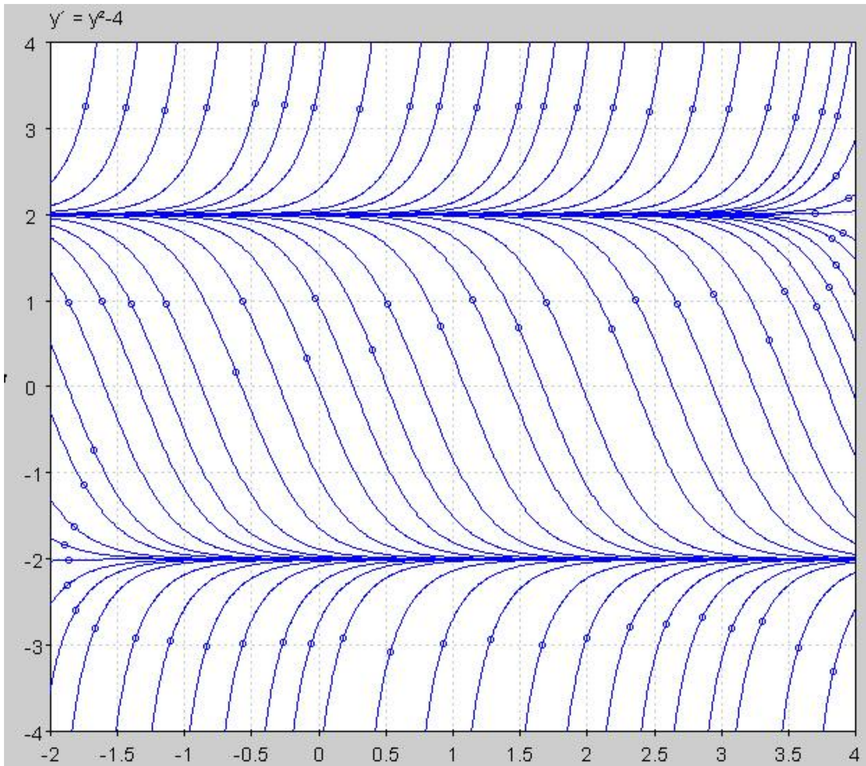
**Example 2.5.5. Problem:** *find a family of curves which are increasing whenever $y < -2$ or $y > 2$ and are decreasing whenever $-2 < y < 2$.*

**Solution:** *while many examples exist, the simplest example is one for which the derivative is quadratic in $y$. Think about the quadratic $(y+2)(y-2)$. This expression is positive for $|y| > 2$ and negative for $|y| < 2$. It follows that solutions to the differential equation $\frac{dy}{dx} = (y+2)(y-2)$ will have the desired properties. Note that $y = \pm 2$ are exceptional solutions for the give DEqn. Proceed by separation of variables, recall the technique of partial fractions,*

$$
\begin{aligned}
\frac{dy}{(y+2)(y-2)} = dx \quad &\Rightarrow \quad \int \left[ \frac{1}{4(y-2)} - \frac{1}{4(y+2)} \right] dy = \int dx \quad \star \\
&\Rightarrow \quad \ln|y-2| - \ln|y+2| = 4x + C \\
&\Rightarrow \quad \ln\left| \frac{y-2}{y+2} \right| = 4x + C \\
&\Rightarrow \quad \ln\left| \frac{y+2}{y+2} - \frac{4}{y+2} \right| = 4x + C \\
&\Rightarrow \quad \ln\left| 1 - \frac{4}{y+2} \right| = 4x + C \\
&\Rightarrow \quad \left| 1 - \frac{4}{y+2} \right| = e^{4x+C} = e^C e^{4x} \\
&\Rightarrow \quad 1 - \frac{4}{y+2} = \pm e^C e^{4x} = Ke^{4x} \\
&\Rightarrow \quad \frac{1}{y+2} = \frac{1 - Ke^{4x}}{4} \\
&\Rightarrow \quad \boxed{y = -2 + \frac{4}{1 - Ke^{4x}}, \quad \textit{for } K \neq 0.}
\end{aligned}
$$

*It is neat that $K = 0$ returns the exceptional solution $y = 2$ whereas the other exceptional solution is lost since we have division by $y + 2$ in the calculation above. If we had multiplied $\star$ by $-1$ then the tables would turn and we would recover $y = -2$ in the general formula.*

The plot of the solutions below was prepared with pplane which is a feature of Matlab. To plot solutions to $\frac{dy}{dx} = f(x, y)$ you can put $x' = 1$ and $y' = f(x, y)$. This is an under-use of pplane. We discuss some of the deeper features towards the end of this chapter. Doubtless Mathematica will do these things, however, I don't have 10 hours to code it so, here it is:




If you study the solutions in the previous example you'll find that all solutions tend to either $y = 2$ or $y = -2$ in some limit. You can also show that all the solutions which cross the x-axis have inflection points at their $x$-intercept. We can derive that from the differential equation directly:

$$\frac{dy}{dx} = (y+2)(y-2) = y^2 - 4 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y(y+2)(y-2).$$

We can easily reason when solutions have $y > 2$ or $-2 < y < 0$ they are concave up whereas solutions with $0 < y < 2$ or $y < -2$ are concave down. It follows that a solution crossing $y = 0, -2$ or 2 is at a point of inflection. Careful study of the solutions show that solutions do not cross $y = -2$ or $y = 2$ thus only $y = 0$ has solutions with genuine points of inflection.

**Example 2.5.6. Problem:** *suppose you are given a family $S$ of curves which satisfy $\frac{dy}{dx} = f(x,y)$. Find a differential equation for a family of curves which are orthogonal to the given set of curves. In other words, find a differential equation whose solution consists of curves $S^{\perp}$ whose tangent vectors are perpendicular to the tangent vectors of curves in $S$ at points of intersection.*

**Solution:** *Consider a point $(x_o, y_o)$, note that the solution to $\frac{dy}{dx} = f(x,y)$ has slope $f(x_o, y_o)$ at that point. The perpendicular to the tangent has slope $-1/f(x_o, y_o)$. Thus, we should use the differential equation $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ to obtain orthogonal trajectories.*

Let me give a concrete example of orthogonal trajectories:

**Example 2.5.7. Problem:** *find orthogonal trajectories of $xdx + ydy = 0$.*

**Solution:** *we find $\frac{dy}{dx} = \frac{-x}{y}$ hence the orthogonal trajectories are found in the solution set of $\frac{dy}{dx} = \frac{-y}{x}$. Separate variables to obtain:*

$$\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln|y| = \ln|x| + C \quad \Rightarrow \quad y = \pm e^C x.$$

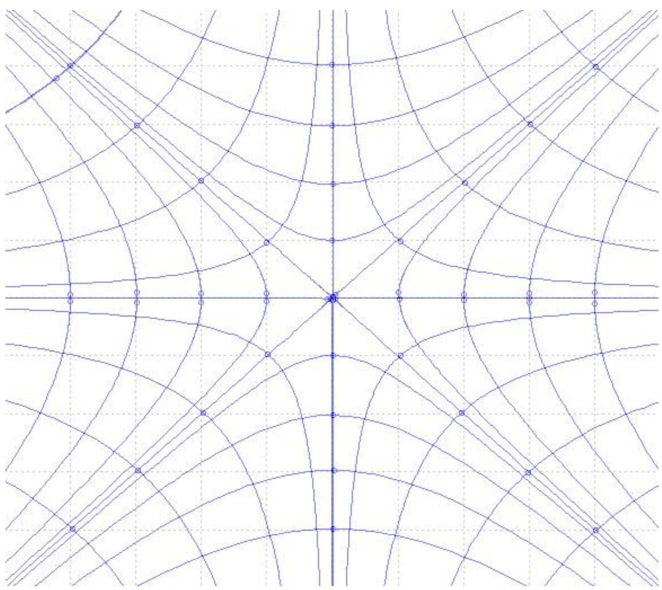
*In other words, the orthogonal trajectories are lines through the origin $y = kx$. Technically, by our derivation, we ought not allow $k = 0$ but when you understand the solutions of $xdx + ydy = 0$ are simply circles $x^2 + y^2 = R^2$ it is clear that $y = 0$ is indeed an orthogonal trajectory.*

**Example 2.5.8. Problem:** *find orthogonal trajectories of $x^2 - y^2 = 1$.*

**Solution:** *observe that the hyperbola above is a solution of the differential equation $2x - 2y\frac{dy}{dx} = 0$ hence $\frac{dy}{dx} = \frac{x}{y}$. Orthogonal trajectories are found from $\frac{dy}{dx} = \frac{-y}{x}$. Separate variables,*

$$\frac{dy}{y} = \frac{-dx}{x} \quad \Rightarrow \quad \ln|y| = -\ln|x| + C \quad \Rightarrow \quad y = k/x.$$

*Once more, the case $k = 0$ is exceptional, but it is clear that $y = 0$ is an orthogonal trajectory of the given hyperbola.*



Orthogonal trajectories are important to the theory of electrostatics. The field lines which are *integral curves* of the electric field form orthogonal trajectories to the *equpotential* curves. Or, in the study of heatflow, the isothermal curves are orthgonal to the curves which line-up with the flow of heat.

**Example 2.5.9. Problem:** *Suppose the force of friction on a speeding car is given by $F_f = -bv^2$. If the car has mass $m$ and initial speed $v_o$ and position $x_o$ then find the velocity and position as a function of $t$ as the car glides to a stop. Assume that the net-force is the friction force since the normal force and gravity cancel.*

**Solution:** *by Newton's second law we have $m\frac{dv}{dt} = -bv^2$. Separate variables, integrate. apply initital condition,*

$$\frac{dv}{v^2} = -\frac{bdt}{m} \quad \Rightarrow \quad \frac{-1}{v} = \frac{-bt}{m} + c_1 \quad \Rightarrow \quad \frac{-1}{v_o} = \frac{-b(0)}{m} + c_1 \quad \Rightarrow \quad c_1 = \frac{-1}{v_o}$$

*Thus,*

$$\frac{1}{v(t)} = \frac{bt}{m} + \frac{1}{v_o} \quad \Rightarrow \quad v(t) = \frac{1}{\frac{bt}{m} + \frac{1}{v_o}} \quad \Rightarrow \quad \boxed{v(t) = \frac{v_o}{\frac{btv_o}{m} + 1}}.$$

*Since $v = \frac{dx}{dt}$ we can integrate the velocity to find the position*

$$x(t) = c_1 + \frac{m}{b}\ln\left|1 + \frac{bv_o t}{m}\right| \quad \Rightarrow \quad x(0) = c_1 + \ln(1) = x_o \quad \Rightarrow \quad \boxed{x(t) = x_o + \frac{m}{b}\ln\left|1 + \frac{bv_o t}{m}\right|}.$$

*Notice the slightly counter-intuitive nature of this solution, the position is unbounded even though the velocity tends to zero. Common sense might tell you that if the car slows to zero for large time then the total distance covered must be finite. Well, common sense fails, math wins. The point is that the velocity actually goes too zero too slowly to give bounded motion.*

**Example 2.5.10. Problem:** *Newton's Law of Cooling states that the change in temperature $T$ for an object is proportional to the difference between the ambient temperature $R$ and $T$; in particular: $\frac{dT}{dt} = -k(T - R)$ for some constant $k$ and $R$ is the room-temperature. Suppose that $T(0) = 150$ and $T(1) = 120$ if $R = 70$, find $T(t)$*

**Solution:** *To begin let us examine the differential equation for arbitrary $k$ and $R$,*

$$\frac{dT}{dt} = -k(T - R) \quad \Rightarrow \quad \frac{dT}{dt} + kT = kR$$

*Identify that $p = k$ hence $I = e^{kt}$ and we find*

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = ke^{kt}R \quad \Rightarrow \quad \frac{d}{dt}\big[\, e^{kt}T \,\big] = ke^{kt}R \quad \Rightarrow \quad e^{kt}T = Re^{kt} + C \quad \Rightarrow \quad \boxed{T(t) = R + Ce^{-kt}.}$$

*Now we may apply the given data to find both $C$ and $k$, we already know $R = 70$ from the problem statement;*

$$T(0) = 70 + C = 150 \qquad \& \qquad T(1) = 70 + Ce^{-k} = 120$$

*Hence $C = 80$ which implies $e^{-k} = 5/8$ thus $e^k = 8/5$ and $k = \ln(8/5)$. Therefore, $\boxed{T(t) = 70 + 80e^{t\ln(5/8)}}$. To understand this solution note that $\ln(5/8) < 0$ hence the term $80e^{t\ln(5/8)} \to 0$ as $t \to \infty$ hence $T(t) \to 70$ as $t \to \infty$. After a long time, Newton's Law of Cooling predicts objects will assume room temperature.*

**Example 2.5.11.** *Suppose you decide to have coffee with a friend and you both get your coffee ten minutes before the end of a serious presentation by your petty boss who will be offended if you start drinking during his fascinating talk on maximal efficiencies for production of widgets. You both desire to drink your coffee with the same amount of cream and you both like the coffee as hot as possible. Your friend puts the creamer in immediately and waits quitely for the talk to end. You on the other hand think you wait to put the cream in at the end of talk. Who has hotter coffee and why?* **Discuss.**

**Example 2.5.12. Problem:** *the voltage dropped across a resistor $R$ is given by the product of $R$ and the current $I$ through $R$. The voltage dropped across an inductor $L$ depends on the change in the current according to $L\frac{dI}{dt}$. An inductor resists a change in current whereas a resistor just resists current. If we connect $R$ and $L$ in series with a voltage source $\mathcal{E}$ then the Kirchoff's voltage law yields the differential equation*

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

*Given that $I(0) = I_o$ find $I(t)$ for the circuit.*

**Solution:** *Identify that this is a linear DE with independent variable $t$,*

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}}{L}$$

*The integrating factor is simply $\mu = e^{\frac{Rt}{L}}$ (using $I$ here would be a poor notation). Multiplying the DEqn above by $\mu$ to obtain,*

$$e^{\frac{Rt}{L}}\frac{dI}{dt} + \frac{R}{L}e^{\frac{Rt}{L}}I = \frac{\mathcal{E}}{L}e^{\frac{Rt}{L}} \quad \Rightarrow \quad \frac{d}{dt}\big[\,e^{\frac{Rt}{L}}I\,\big] = \frac{\mathcal{E}}{L}e^{\frac{Rt}{L}}$$

*Introduce a dummy variable of integration $\tau$ and integrate from $\tau = 0$ to $\tau = t$,*

$$\int_0^t \frac{d}{d\tau}\big[\,e^{\frac{R\tau}{L}}I\,\big]d\tau = \int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}d\tau \quad \Rightarrow \quad e^{\frac{Rt}{L}}I(t) - I_o = \int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}d\tau.$$

*Therefore, $I(t) = I_oe^{\frac{-Rt}{L}} + e^{\frac{-Rt}{L}}\int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}d\tau$. If the voltage source is constant then $\mathcal{E}(t) = \mathcal{E}_o$ for all $t$ and the solution yields to $I(t) = I_oe^{\frac{-Rt}{L}} + e^{\frac{-Rt}{L}}\frac{\mathcal{E}_o}{L}\frac{L}{R}\big(e^{\frac{Rt}{L}} - 1\big)$ which simplifies to*

$$\boxed{I(t) = \left[I_o - \frac{\mathcal{E}_o}{R}\right]e^{\frac{-Rt}{L}} + \frac{\mathcal{E}_o}{R}}.$$

*The* **steady-state** *current found from letting $t \to \infty$ where we find $I(t) \to \frac{\mathcal{E}_o}{R}$. After a long time it is approximately correct to say the inductor is just a short-circuit. What happens is that as the current changes in the inductor a magnetic field is built up. The magnetic field contains energy and the maximum energy that can be stored in the field is governed by the voltage source. So, after a while, the field is approximately maximal and all the voltage is dropped across the resistor. You could think of it like saving money in a piggy-bank which cannot fit more than $\mathcal{E}_o$ dollars. If every week you get an allowance then eventually you have no choice but to spend the money if the piggy-bank is full and there is no other way to save.*

**Example 2.5.13. Problem:** *Suppose a tank of salty water has $15kg$ of salt disolved in $1000L$ of water at time $t = 0$. Furthermore, assume pure water enters the tank at a rate of $10L/min$ and salty water drains out at a rate of $10L/min$. If $y(t)$ is the number of $kg$ of salt at time $t$ then find $y(t)$ for $t > 0$. Also, how much salt is left in the tank when $t = 20$ (minutes). We suppose that this tank is arranged such that the concentration of salt is constant throughout the liquid in this* **mixing tank.**

**Solution:** *Generally when we work such a problem we are interested in the rate of salt enetering the tank and the rate of salt exiting the tank ; $\frac{dy}{dt} = R_{in} - R_{out}$. However, this problem only has a nonzero out-rate: $R_{out} = \frac{10\,L}{min}\frac{y}{1000L} = \frac{y}{100min}$. We omit the "min" in the math below as we assume $t$ is in minutes,*

$$\frac{dy}{dt} = -\frac{y}{100} \quad \Rightarrow \quad \frac{dy}{y} = -\frac{dt}{100} \quad \Rightarrow \quad \ln|y| = -\frac{t}{100} + C \quad \Rightarrow \quad y(t) = ke^{-\frac{t}{100}}.$$

*However, we are given that $y(0) = 15$ hence $k = 15$ and we find*[11]*:*

$$\boxed{y(t) = 15e^{-0.01t}.}$$

*Evaluating at $t = 20min$ yields $y(20) = 12.28\,kg$.*

**Example 2.5.14. Problem:** *Suppose a water tank has $100L$ of pure water at time $t = 0$. Suppose salty water with a concentration of $1.5kg$ of salt per $L$ enters the tank at a rate of $8L/min$ and gets quickly mixed with the initially pure water. There is a drain in the tank where water drains out at a rate of $6L/min$. If $y(t)$ is the number of $kg$ of salt at time $t$ then find $y(t)$ for $t > 0$. If the water tank has a maximum capacity of $1000L$ then what are the physically reasonable values for the solution? For what $t$ does your solution cease to be reasonable?*

**Solution:** *Generally when we work such a problem we are interested in the rate of salt enetering the tank and the rate of salt exiting the tank ; $\frac{dy}{dt} = R_{in} - R_{out}$. The input-rate is constant and is easily found from multiplying the given concentration by the flow-rate:*

$$R_{in} = \frac{1.5\,kg}{L}\frac{8\,L}{min} = \frac{12\,kg}{min}$$

*notice how the units help us verify we are setting-up the model wisely. That said, I omit them in what follows to reduce clutter for the math. The output-rate is given by the product of the flow-rate $6L/min$ and the salt-concentration $y(t)/V(t)$ where $V(t)$ is the volume of water in $L$ at time $t$. Notice that the $V(t)$ is given by $V(t) = 100 + 2t$ for the given flow-rates, each minute the volume increases by $2L$. We find (in units of $kg$ and $min$):*

$$R_{out} = \frac{6y}{100 + 2t}$$

*Therefore, we must solve:*

$$\frac{dy}{dt} = 12 - \frac{6y}{100 + 2t} \quad \Rightarrow \quad \frac{dy}{dt} + \frac{3dt}{50 + t}y = 12.$$

[11] to be physically explicit, $y(t) = (15kg)exp(\frac{-0.01t}{min})$, but the units clutter the math here so we omit them

*This is a linear ODE, we can solve it by the integrating factor method.*

$$I(t) = exp\left(\int \frac{3dt}{50+t}\right) = exp\left(3\ln(50+t)\right) = (50+t)^3.$$

*Multiplying by $I$ yields:*

$$(50+t)^3\frac{dy}{dt} + 3(50+t)^2 y = 12(50+t)^3 \quad \Rightarrow \quad \frac{d}{dt}\Big[(50+t)^3 y\Big] = 12(50+t)^3$$

*Integrating yields $(50+t)^3 y(t) = 3(50+t)^4 + C$ hence $y(t) = 3(50+t) + \frac{C}{(50+t)^3}$. The water is initially pure thus $y(0) = 0$ thus $0 = 150 + C/50^3$ which gives $C = -150(50)^3$. The solution is*[12]

$$y(t) = 3(50+t) - 150\left(\frac{50}{50+t}\right)^3$$

*Observe that $V(t) \leq 1000\,L$ thus we need $100 + 2t \leq 1000$ which gives $t \leq 450$. The solution is only appropriate physically for $0 \leq t \leq 450$.*

**Example 2.5.15. Problem:** *suppose the population $P$ grows at a rate which is directly proportional to the population. Let $k_1$ be the proportionality constant for the growth rate. Suppose further that as the population grows the death-rate is proportional to the square of the population. Suppose $k_2$ is the proportionality constant for the death-rate. Find the population at time $t$ in terms of the initial population $P_o$.*

**Solution:** *the given problem translates into the IVP of $\frac{dP}{dt} = k_1 P - k_2 P^2$ with $P(0) = P_o$. Observe that $k_1 P - k_2 P^2 = k_1 P(1 - k_2 P/k_1)$. Introduce $C = k_1/k_2)$. Separate variables:*

$$\frac{dP}{P(1-P/C)} = k_1 dt$$

*Recall the technique of partial fractions:*

$$\frac{1}{P(1-P/C)} = \frac{-C}{P(P-C)} = \frac{A}{P} + \frac{B}{P-C} \quad \Rightarrow \quad -C = A(P-C) + BP$$

*Set $P = 0$ to obtain $-C = -AC$ hence $A = 1$ and set $P = C$ to obtain $-C = BC$ hence $B = -1$ and we find:*

$$\int \left[\frac{1}{P} - \frac{1}{P-C}\right] dP = k_1 dt \quad \Rightarrow \quad \ln|P| - \ln|P-C| = k_1 t + c_1$$

*It follows that letting $c_2 = e^{c_1}$ and $c_3 = \pm c_2$*

$$\left|\frac{P}{P-C}\right| = c_2 e^{k_1 t} \quad \Rightarrow \quad P = (P-C)c_3 e^{k_1 t}$$

[12]following the formatting of Example 7 of § 2.7 of Rice & Strange's Ordinary Differential Equations with Applications

*hence,* $P[1 - c_3 e^{k_1 t}] = -c_3 C e^{k_1 t}$

$$P(t) = \frac{c_3 C e^{k_1 t}}{c_3 e^{k_1 t} - 1} \quad \Rightarrow \quad P(t) = \frac{C}{1 - c_4 e^{-k_1 t}}$$

*where I let* $c_4 = 1/c_3$ *for convenience. Let us work on writing this general solution in-terms of the initial population* $P(0) = P_o$*:*

$$P_o = \frac{C}{1 - c_4} \quad \Rightarrow \quad P_o(1 - c_4) = C \quad \Rightarrow \quad P_o - C = P_o c_4 \quad \Rightarrow \quad c_4 = \frac{P_o - C}{P_o}.$$

*This yields,*

$$P(t) = \frac{C}{1 - \frac{P_o - C}{P_o} e^{-k_1 t}} \quad \Rightarrow \quad \boxed{P(t) = C\left[\frac{P_o}{P_o - [P_o - C]e^{-k_1 t}}\right]}$$

*The quantity* $C$ *is called the* **carrying capacity** *for the system. As we defined it here it is given by the quotient of the birth-rate and death-rate constants* $C = k_1/k_2$*. Notice that as* $t \to \infty$ *we find* $P(t) \to C$*. If* $P_o > C$ *then the population decreases towards* $C$ *whereas if* $P_o < C$ *then the population increases towards* $C$*. If* $P_o = C$ *then we have a special solution where* $\frac{dP}{dt} = 0$ *for all* $t$*, the* **equilbrium solution***. A a bit of fun trivia, these models are notoriously incorrect for human populations. For example, in 1920 a paper by R. Pearl and L. J. Reed found* $P(t) = \frac{210}{1+51.5e^{-0.03t}}$*. The time* $t$ *is the number of years past 1790 (*$t = 60$ *for 1850 for example). As discussed in Ritger and Rose page 85 this formula does quite well for 1950 where is well-approximates the population as 151 million. However, the carrying capacity of 210 million people is not even close to correct. Why? Because there are many factors which influence population which are simply not known. The same problem exists for economic models. You can't model game-changing events such as an interfering government. It doesn't flow from logic or optimal principles, political convenience whether it benefits or hurts a given market cannot be factored in over a long-term. Natural disasters also spoil our efforts to model populations and markets. That said, the exponential and logarthmic population models are important to a wide-swath of reasonably isolated populations which are free of chaotic events.*

**Example 2.5.16. Problem:** *Suppose a raindrop falls through a cloud and gathers water from the cloud as it drops towards the ground. Suppose the mass of the raindrop is* $m$ *and the suppose the rate at which the mass increases is proportional to the mass;* $\frac{dm}{dt} = km$ *for some constant* $k > 0$*. Find the equation of the velocity for the drop.*

**Solution:** *Newton's equation is* $-mg = \frac{dp}{dt}$*. This follows from the assumption that, on average, there is no net-momentum of the water vapor which adheres to the raindrop thus the momentum change is all from the gravitational force. Since* $p = mv$ *the product rule gives:*

$$-mg = \frac{dm}{dt}v + m\frac{dv}{dt} \quad \Rightarrow \quad -mg = kmv + m\frac{dv}{dt}$$

*Consequently, dividing by* $m$ *and applying the integrating factor method gives:*

$$\frac{dv}{dt} + kv = -g \quad \Rightarrow \quad e^{kt}\frac{dv}{dt} + ke^{kt}v = -ge^{kt} \quad \Rightarrow \quad \frac{d}{dt}\left[e^{kt}v\right] = -ge^{kt}$$

*Integrate to obtain* $e^{kt}v = \frac{-g}{k}e^{kt} + C$ *from which it follows* $\boxed{v(t) = -\frac{g}{k} + Ce^{-kt}.}$ *Consider the limit* $t \to \infty$, *we find* $v_\infty(t) = -\frac{g}{k}$. *This is called the* **terminal velocity**. *Physically this is a very natural result; the velocity is constant when the forces balance. There are two forces at work here (1.) gravity* $-mg$ *and (2.) water friction* $-kmv$ *and we look at*

$$m\frac{dv}{dt} = -mg - kmv$$

*If* $v = -\frac{g}{k}$ *then you obtain* $ma = 0$. *You might question if we should call the term* $-kmv$ *a "force". Is it really a force? In any event, you might note we can find the terminal velocity without solving the DEqn, we just have to look for an equilbrium of the forces.*

Not all falling objects have a terminal velocity... well, at least if you believe the following example. To be honest, I'm not so sure it is very physical. I would be interested in your thoughts on the analysis if your thoughts happen to differ from my own.

**Example 2.5.17. Problem:** *Suppose a raindrop falls through a cloud and gathers water from the cloud as it drops towards the ground. Suppose the mass of the raindrop is* $m$ *and the suppose the drop is spherical and the rate at which the mass adheres to the drop is proportional to the cross-sectional area relative the vertical drop (*$\frac{dm}{dt} = k\pi R^2$*). Find the equation of the velocity for the drop.*

**Solution:** *we should assume the water in the could is motionless hence the water collected from cloud does not impart momentum directly to the raindrop. It follows that Newton's Law is* $-mg = \frac{dp}{dt}$ *where the momentum is given by* $p = mv$ *and* $v = \dot{y}$ *and* $y$ *is the distance from the ground. The mass* $m$ *is a function of time. However, the density of water is constant at* $\rho = 1000kg/m^3$ *hence we can relate the mass* $m$ *to the volume* $V = \frac{4}{3}\pi R^3$ *we have*

$$\rho = \frac{4\pi R^3}{3m}$$

*Solve for* $R^2$,

$$R^2 = \left[\frac{3\rho m}{4\pi}\right]^{2/3}$$

*As the drop falls the rate of water collected should be proportional to the cross-sectional area* $\pi R^2$ *the drop presents to cloud. It follows that:*

$$\frac{dm}{dt} = km^{2/3}$$

*Newton's Second Law for varying mass,*

$$-mg = \frac{d}{dt}\big[mv\big] = \frac{dm}{dt}v + m\frac{dv}{dt} = km^{2/3}v + m\frac{dv}{dt}$$

*This is a linear ODE in velocity,*

$$\frac{dv}{dt} + \left(\frac{k}{m^{1/3}}\right)v = -g$$

*We should find the mass as a function of time,*

$$\frac{dm}{dt} = km^{2/3} \quad \Rightarrow \quad \frac{dm}{m^{2/3}} = kdt \quad \Rightarrow \quad 3m^{1/3} = kt + C_1 \quad \Rightarrow \quad m = \frac{1}{27}\big[kt + C_1\big]^3$$

*where* $m_o$ *is the initial mass of the droplet.*

$$\frac{dv}{dt} + \frac{3kv}{kt + C_1} = -g$$

*The integrating factor is found from integrating the coefficient of* $v$*,*

$$I = exp\bigg[\int \frac{3kdt}{kt + C_1}\bigg] = exp\bigg[3\ln(kt + C_1)\bigg] = (kt + C_1)^3$$

*Hence,*

$$(kt + C_1)^3\frac{dv}{dt} + 3(kt + C_1)^2 v = -g(kt + C_1)^3 \quad \Rightarrow \quad \frac{d}{dt}\bigg[(kt + C_1)^3 v\bigg] = -g(kt + C_1)^3$$

*Hence* $\boxed{v(t) = -\frac{gt}{4} - C_3 + C_2/(kt + C_1)^3.}$ *The constants* $C_1, C_2, C_3$ *have to do with the geometry of the drop, its initial mass and its initial velocity. Suppose* $t = 0$ *marks the intiial formation of the raindrop, it is interesting to consider the case* $t \to \infty$*, we find*

$$v_\infty(t) = -\frac{gt}{4} - C_3$$

*which says that the drop accelerates at approximately constant acceleration* $-g/4$ *as it falls through the cloud. There is no terminal velocity in contrast to the previous example. You can integrate* $v(t) = \frac{dy}{dt}$ *to find the equation of motion for* $y$*.*

**Example 2.5.18. Problem:** *Rocket flight. Rockets fly by ejecting mass with momentum to form thrust. We analyze the upward motion of a vertically launched rocket in this example. In this case Netwon's Second Law takes the form:*

$$\frac{d}{dt}\bigg[\ mv\ \bigg] = F_{external} + F_{thrust}$$

*the external force could include gravity as well as friction and the thrust arises from conservation of momentum. Suppose the rocket expells gas downward at speed* $u$ *relative the rocket. Suppose that the rocket burns mass at a uniform rate* $m(t) = m_o - \alpha t$ *and find the resulting equation of motion. Assume air friction is neglible.*

**Solution:** *If the rocket has velocity* $v$ *then the expelled gas has velocity* $v - u$ *relative the ground's frame of reference. It follows that:*

$$F_{thrust} = (v - u)\frac{dm}{dt}$$

*Since* $F_{external} = -mg$ *and* $\frac{dm}{dt} = -\alpha$ *we must solve*

$$\frac{d}{dt}\left[ mv \right] = -mg + (v-u)\frac{dm}{dt} \quad \Rightarrow \quad \frac{dm}{dt}v + m\frac{dv}{dt} = -mg + v\frac{dm}{dt} - u\frac{dm}{dt}$$

*Thus,*

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg$$

*Suppose, as was given, that* $m(t) = m_o - \alpha t$ *hence* $\frac{dm}{dt} = -\alpha$

$$(m_o - \alpha t)\frac{dv}{dt} = \alpha u - (m_o - \alpha t)g \quad \Rightarrow \quad \frac{dv}{dt} = \frac{\alpha u}{m_o - \alpha t} - g$$

*We can solve by integration: assume* $v(0) = 0$ *as is physically reasonable,*

$$v(t) = -u\ln(m_o - \alpha t) + u\ln(m_o) - gt = -u\ln\left(1 - \frac{\alpha t}{m_o}\right) - gt.$$

*The initial mass* $m_o$ *consists of fuel and the rocket itself:* $m_o = m_f + m_r$*. This model is only physical for time* $t$ *such that* $m_r \leq m_f + m_r - \alpha t$ *hence* $0 \leq t \leq m_f/\alpha$*. Once the fuel is finished the empty rocket completes the flight by projectile motion. You can integrate* $v = dy/dt$ *to find the equation of motion. In particular:*

$$\begin{aligned} y(t) &= \int_0^t \big[-u\ln(m_o - \alpha\tau) + u\ln(m_o) - g\tau\big]d\tau \qquad (2.1) \\ &= \left(-\frac{u}{\alpha}\left[(\alpha\tau - m_o)\ln(m_o - \alpha\tau) - \alpha\tau\right] + u\tau\ln(m_o) - \frac{1}{2}g\tau^2\right)\Big|_0^t \\ &= -\frac{u}{\alpha}\left[(\alpha t - m_o)\ln(m_o - \alpha t) - \alpha t\right] + ut\ln(m_o) - \frac{1}{2}gt^2 - \frac{m_o u}{\alpha}\ln(m_o) \\ &= ut - \frac{1}{2}gt^2 - u\frac{m_o}{\alpha}\left(1 - \frac{\alpha t}{m_o}\right)\ln\left(1 - \frac{\alpha t}{m_o}\right) \end{aligned}$$

*Suppose* $-\int_0^{\frac{m_f}{\alpha}} u\ln\left(1 - \frac{\alpha t}{m_o}\right)dt = A$ *then* $y(t) = A - \frac{1}{2}g\left(t - \frac{m_f}{\alpha}\right)^2$ *for* $t > \frac{m_f}{\alpha}$ *as the rocket freefalls back to earth having exhausted its fuel.*

Technically, if the rocket flies more than a few miles vertically then we ought to use the variable force of gravity which correctly accounts for the weaking of the gravitational force with increasing altitude. Mostly this example is included to show how variable mass with momentum transfer is handled.

Other interesting applications include chemical reactions, radioactive decay, blood-flow, other population models, dozens if not hundreds of modifications of the physics examples we've considered, rumor propogation, etc... the math here is likely found in any discipline which uses math to quantiatively describe variables. I'll conclude this section with an interesting example I found in Edwards and Penny's *Elementary Differential Equations with Boundary Value Problems*, the 3rd Ed.

**Example 2.5.19. Problem:** *Suppose a flexiible rope of length 4ft has 3ft coiled on the edge of a balcony and 1ft hangs over the edge. If at $t = 0$ the rope begins to uncoil further then find the velocity of the rope as it falls. Also, how long does it take for the rope to fall completely off the balcony. Suppose that the force of friction is neglible.*

**Solution:** *let $x$ be the length of rope hanging off and suppose $v = dx/dt$. It follows that $x(0) = 1$ and $v(0) = 0$. The force of gravity is $mg$, note that if $\lambda$ is the mass per unit length of the rope then $m = \lambda x$, thus:*

$$\frac{d}{dt}[\ mv\ ] = mg \quad \Rightarrow \quad \frac{d}{dt}[\ \lambda x v\ ] = \lambda x g \quad \Rightarrow \quad \lambda\frac{dx}{dt}v + \lambda x\frac{dv}{dt} = \lambda x g$$

*The mass-density $\lambda$ cancels and since $v = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$ we find:*

$$v^2 + xv\frac{dv}{dx} = xg \quad \Rightarrow \quad \left(\frac{v^2}{x} - g\right)dx + v\,dv = 0$$

*In your text, in the discussion of special integrating factors, it is indicated that when $\frac{\partial_y M - \partial_x N}{N} = A(x)$ then $I = \int A\,dx$ is an integrating factor. Observe $M(x,v) = \frac{v^2}{x} - g$ and $N(x,v) = v$ hence $\partial_v M - \partial_x N = 2/x$ hence we calculate $I = exp(\int \frac{2dx}{x}) = exp(2\ln|x|) = x^2$. Don't believe it? Well, believe this:*

$$\left(xv^2 - gx^2\right)dx + x^2 v\,dv = 0 \quad \Rightarrow \quad \frac{1}{2}x^2v^2 - \frac{1}{3}gx^3 = C.$$

*Apply the intiial condtions $x(0) = 1$ and $v(0) = 0$ gives $-\frac{1}{3}g = C$ thus $\frac{1}{2}x^2v^2 - \frac{1}{3}gx^3 = -\frac{1}{3}g$. and we solve for $v > 0$,*

$$v = \sqrt{\frac{2g}{3}\left(\frac{x^3-1}{x^2}\right)}$$

*However, $v = \frac{dx}{dt}$ consequently, separating and integrating:*

$$T = \sqrt{\frac{3}{2g}}\int_1^4 \frac{x\,dx}{\sqrt{x^3-1}} \approxeq 2.5\sqrt{\frac{3}{2g}} = 0.541\ s$$

*by Wolfram Alpha. See Section 1.7 Example 6 of Edwards and Penny's Elementary Differential Equations with Boundary Value Problems, the 3rd Ed. for another approach involving Simpson's rule with 100 steps.*

## 2.6 visualizations, existence and uniqueness

Given a curve in the $\mathbb{R}^2$ we have two general methods to describe the curve:

(1.) $F(x, y) = k$ as a **level curve** (2.) $\vec{r}(t) = \langle x(t), y(t)\rangle$ as a **parametrized curve**

As an example, we can either write $x^2 + y^2 = 1$ or $x = \cos(t), y = \sin(t)$. The parametric view has the advantage of capturing the direction or orientation of the curve. We have studied solutions of $Mdx + Ndy = 0$ in terms of cartesian coordinates and naturally our solutions were level curves. We now turn to ask what conditions ought to hold for the parametrization of the solution to $Mdx + Ndy = 0$.

Given a differentiable function of two variables $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ we may subsitute a differentiable path $\vec{r} : I \subseteq \mathbb{R} \to D$ to form the composite function $F \circ \vec{r} : I \subseteq \mathbb{R} \to \mathbb{R}$. If we denote $\vec{r}(t) = \langle x(t), y(t)\rangle$ then the multivariate chain-rule says:

$$\frac{d}{dt}F(x(t), y(t)) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}.$$

Suppose we have a level curve $C$ which is the solution set of $F(x, y) = k$ and suppose $C$ is the solution of $Mdx + Ndy = 0$ (call this $(\star xy)$. It follows that the level-function $F$ must have $\partial_x F = M$ and $\partial_y F = N$. Continuing, suppose a parametrization of $C$ is given by the set of functions $x, y : I \subseteq \mathbb{R} \to \mathbb{R}$ where $F(x(t), y(t)) = k$ for all $t \in I$. Notice that when we differentiate $k$ with respect to $t$ we obtain zero hence, applying the general chain rule to our context,

$$\frac{\partial F}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial F}{\partial y}(x(t), y(t))\frac{dy}{dt} = 0$$

for any parametrization of $C$. But, $\partial_x F = M$ and $\partial_y F = N$ hence

$$M(x(t), y(t))\frac{dx}{dt} + N(x(t), y(t))\frac{dy}{dt} = 0 \qquad (\star t)$$

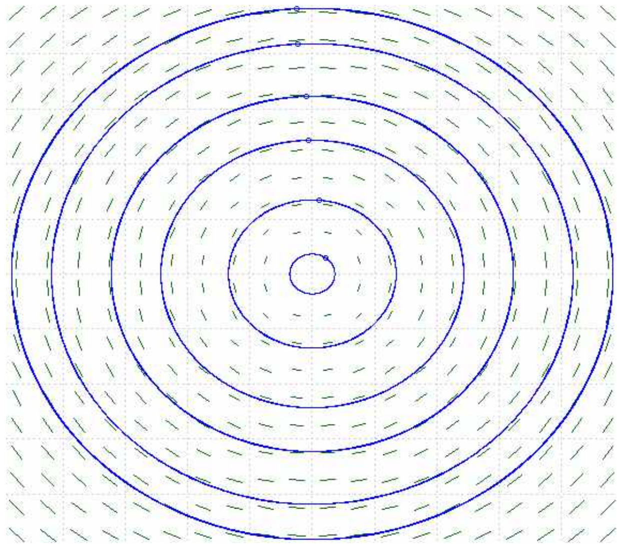
I probably cheated in class and just "divided by $dt$" to derive this from $Mdx + Ndy = 0$. However, that is just an abbreviation of the argument I present here. How should we solve $(\star t)$? Observe that the conditions

$$\frac{dx}{dt} = -N(x(t), y(t)) \qquad \& \qquad \frac{dy}{dt} = M(x(t), y(t))$$

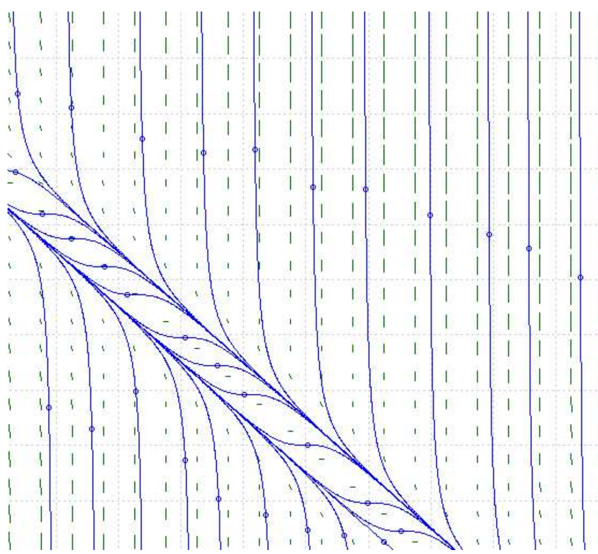
will suffice. Moreover, these conditions show that the solution of $Mdx + Ndy = 0$ is an streamline (or integral curve) of the vector field $\vec{G} = \langle -N, M\rangle$. Naturally, we see that $\vec{F} = \langle M, N\rangle$ is orthogonal to $\vec{G}$ as $\vec{F} \bullet \vec{G} = 0$. The solutions of $Mdx+Ndy = 0$ are perpendicular to the vector field $\langle M, N\rangle$.

There is an ambiguity we should face. Given $Mdx + Ndy = 0$ we can either view solutions as streamlines to the vector field $\langle -N, M\rangle$ or we could use $\langle N, -M\rangle$. The solutions of $Mdx+Ndy = 0$ do not have a natural direction unless we make some other convention or have some larger context. Therefore, as we seek to visualize the solutions of $Mdx + Ndy = 0$ we should either ignore the direction of the vector field $\langle -N, M\rangle$ or simply not plot the arrowheads. A plot of $\langle -N, M\rangle$ with directionless vectors is called an **isocline** plot for $Mdx + Ndy = 0$. Perhaps you looked at some isoclines in your second semester calculus course.

**Example 2.6.1. Problem:** *plot the isocline field for $xdx + ydy = 0$ and a few solutions.*
**Solution:** *use pplane with $x' = -y$ and $y' = x$ for the reasons we just derived in general.*

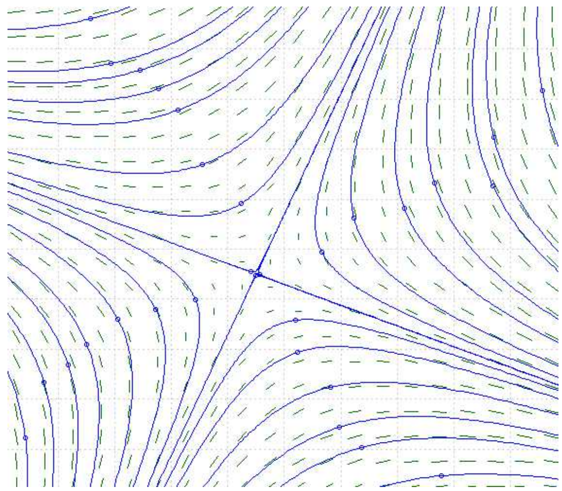


**Example 2.6.2. Problem:** *plot the isocline field for $\frac{dy}{dx} = (x + y - 6)^2$ and a few solutions.*
**Solution:** *in Pfaffian form we face $(x + y - 6)^2 dx - dy$ hence we use pplane with $x' = -1$ and $y' = (x + y - 6)^2$.*



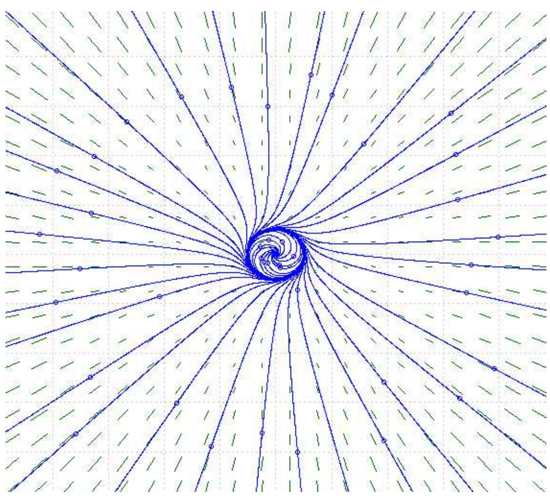
*Recall that we found solutions $y = 6 + \tan(x + C) - x$ in Example 2.4.1. This is the plot of that.*

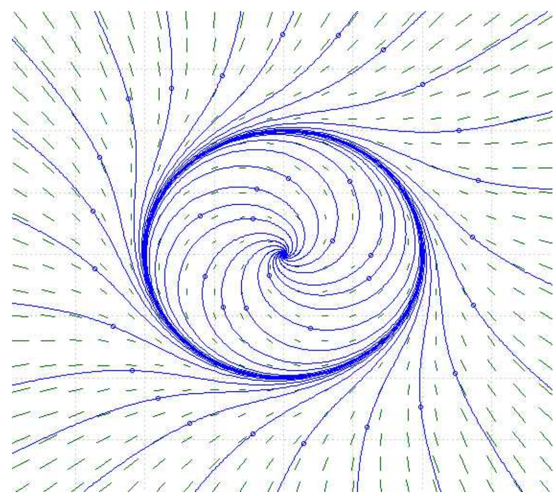
**Example 2.6.3.** **Problem:** *plot the isocline field for* $(x + y + 2)dx + (x - y)dy = 0$ *and a few solutions.*
**Solution:** *in Pfaffian form we face* $(x + y - 6)^2 dx - dy$ *hence we use pplane with* $x' = -1$ *and* $y' = (x + y - 6)^2$.



*Recall that we found solutions* $2x + \frac{x^2-y^2}{2} + xy = C$ *in Example 2.4.7. This is the plot of that.*

**Example 2.6.4.** **Problem:** *plot the isocline field for* $\frac{dy}{dx} = \frac{y^3+x^2y-y-x}{xy^2+x^3+y-x}$ *and a few solutions.*
**Solution:** *in Pfaffian form we face* $(y^3 + x^2y - y - x)dx - (xy^2 + x^3 + y - x)dy$ *hence we use pplane with* $x' = xy^2 + x^3 + y - x$ *and* $y' = y^3 + x^2y - y - x$.





*See my handwritten notes (38-40) for the solution of this by algebraic methods. It is a beautiful example of how polar coordinate change naturally solves a first order ODE with a rotational symmetry. Also, notice that all solutions asymptotically are drawn to the unit circle. If the solution begins inside the circle it is drawn outwards to the circle whereas all solutions outside the circle spiral inward.*

If you study the plots I just gave you will notice that at most points there is just one solution the flows through. However, at certain points there are multiple solutions that intersect. When there is just one solution at a given point $(x_o, y_o)$ then we say that the solution is **unique**. It turns out there are simple theorems that capture when the solution is unique for a general first order ODE of the form $\frac{dy}{dx} = f(x, y)$. I will not prove these here[13]

**Theorem 2.6.5.** *existence of solution(s)*

> Suppose $f$ is continuous on a rectangle $R \subset \mathbb{R}^2$ then **at least** one solution exists for $\frac{dy}{dx} = f(x, y)$ at each point in $R$. Moreover, these solutions exist on all of $R$ in the sense that they reach the edge of $R$.

This is a dumbed-down version of the theorem given in the older texts like Rabenstein or Ritger & Rose. See pages 374-378 of Rabenstein or Chapter 4 of Ritger & Rose. You can read those if you wish to see the man behind the curtain here.

**Theorem 2.6.6.** *uniqueness of solution*

> Suppose $f$ is continuous on a rectangle $R \subset \mathbb{R}^2$ and $\frac{\partial f}{\partial y}(x_o, y_o) \neq 0$ then **there exists a unique** solution near $(x_o, y_o)$.

Uniqueness can be lost as we get too far away from the point where $\frac{\partial f}{\partial y}(x_o, y_o) \neq 0$. The solution is separated from other solutions near $(x_o, y_o)$, but it may intersect other solutions as we travel away from the given point.

**Example 2.6.7.** *Solve $\frac{dy}{dx} = y^2$ and analyze how the uniqueness and existence theorems are validated. This nonlinear DEqn is easily solved by separation of variables: $dy/y^2 = dx$ hence $-1/y = x + C$ or $y = \frac{-1}{x+C}$. We also have the solution $y = 0$. Consider,*

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y^2] = 2y$$

*Thus all points off the $y = 0$ (aka $x$-axis) should have locally unique solutions. In fact, it turns out that the solution $y = 0$ is also unique in this case. Notice that the theorem does not forbid this. The theorem on uniqueness only indicates that it is* **possible** *for multiple solutions to exist at a point where $\frac{\partial f}{\partial y}(x_o, y_o) = 0$. It is important to not over-extend the theorem. On the other hand, the existence theorem says that solutions should extend to the edge of $\mathbb{R}^2$ and that is clearly accomplished by the solutions we found. You can think of $y = 0$ as reaching the horizontal infinities of the plane whereas the curves $y = \frac{-1}{x+C}$ have vertical asymptotes which naturally extend to the vertical infinities of the plane. (these comments are heuristic !)*

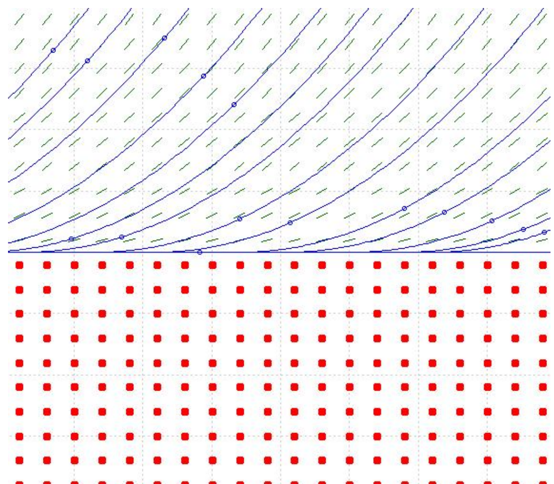
**Example 2.6.8.** *Solve $\frac{dy}{dx} = 2\sqrt{y}$ and analyze how the uniqueness and existence theorems are validated. Observe that $\frac{dy}{2\sqrt{y}} = dx$ hence $\sqrt{y} = x + C$ and we find $y = (x + C)^2$. Note that the*

[13] see Rosenlicht's *Introduction to Analysis for a proof of this theorem, you need ideas from advanced calculus and real analysis to properly understand the proof.*

*solutions reach points on $\mathbb{R}\times[0,\infty)$ however the solutions do not have $y<0$. The existence theorem suggests solutions should exist on $\mathbb{R}\times[0,\infty)$ and this is precisely what we found. On the other hand, for uniqueness, consider: $f(x,y)=2\sqrt{y}$*

$$\frac{\partial f}{\partial y}=\frac{1}{\sqrt{y}}$$

*We can expect unique solutions at points with $y\neq 0$, however, we* **may** *find mutiple solutions at points with $y=0$. Indeed, note that $y=0$ is a solution and at any point $(a,0)$ we also have the solution $y=(x-a)^2$. At each point along the $x$-axis we find two solutions intersect the point. Moreover, if you look at an open interval centered at $(a,0)$ you'll find infinitely many solutions which flow off the special solution $y=0$. Note, in the plot below, the pplane tool illustrates the points outside the domain of defintion by the red dots:*



**Example 2.6.9.** *Consider the first order linear ODE $\frac{dy}{dx}+P(x)y=Q(x)$. Identify that $\frac{dy}{dx}=Q(x)-P(x)y=f(x,y)$. Therefore, $\frac{\partial f}{\partial y}=P(x)$. We might find there are multiple solutions for points with $P(x)=0$. However, will we? Discuss.*

If a first order ODE does not have the form[14] $\frac{dy}{dx}+P(x)y=Q(x)$ then it is said to be **nonlinear**. Often the nonlinear ODEs we have studied have possessed unique solutions at most points. However, the unique solutions flow into some exceptional solution like $y=0$ in Example 2.6.8 or the unit circle $x^2+y^2=1$ or origin $(0,0)$ in Example 2.6.4. These exceptional solutions for nonlinear problems are called **singular solutions** and a point like the origin in Example 2.6.4 is naturally called a **singular point**. That said, we will discuss a more precise idea of singular point for systems of ODEs later in this course. The study of nonlinear problems is a deep an interesting subject which we have only scratched the surface of here. I hope you see by now that the resource of pplane allows you to see things that would be very hard to see with more naive tools like a TI-83 or uncoded Mathematica.

[14] could also write $\frac{dy}{dx}=Q(x)-P(x)y$ or $dy=(Q(x)-P(x)y)dx$ etc... the key is that the expression has $y$ and $y'$ appearing linearly when the DEqn is written in $\frac{dy}{dx}$ notation

# Chapter 3

# ordinary $n$-th order problem

To be honest, we would more properly title this chapter "the ordinary **linear** $n$-th order problem". Our focus is entirely in that direction here. We already saw that some things are known about solutions to nonlinear problems. For example: the Bernoulli, Ricatti, or Clairaut equations in the $n = 1$ case are all nonlinear ODEs. Generally the nonlinear problem is more challenging and we do not seek to describe what is known in these notes. This introductory chapter instead focuses on the linear case. Linearity is a very strong condition which at once yields very strong theorems about the existence and format of the general solution. In particular, if we are given a linear $n$-th order ODE as well as $n$-initial conditions then a unique solution usually exists. I'll try to be a bit more precise on what I mean by "usually" and beyond that we'll discuss techniques to derive the solution.

The approach here is less direct than the $n = 1$ case. We use operators and a series of propositions and theorems to think through these problems. Don't ignore the theory here, it is your guide to success. Once the theory is completed we turn out attention to methods for deriving solutions. Section 2 gives the rules for operator calculus as well as a little complex notation we will find useful in this course. In Section 3 we see how any constant coefficient ODE can be disassembled into a polynomial $P$ of the derivative operator $D$. Moreover, this polynomial possesses all the usual algebraic properties and once we factor a given $P(D)$ then the solution of $P(D)[y] = 0$ is simple to write down. Then in Section 4 we turn to nonhomogeneous problems of the simple type. We'll see how the annihilator method allows us to propose the form of the particular solution and a little algebra finishes the job. The technique of Section 4, while beatiful, is inadequate to face general forcing functions. In Section 5 we find a method known as variation of parameters. The generality of variation of parameters ultimately rests on the existence of the Wronskian and a little linear algebra we introduce along the way. Section 6 discusses the general concept of reduction of order. A particular formula is derived for the $n = 2$ case which allows us to find a second solution once we've found a first solution. In Section 7 we see how a factored operator allows for a nested integral solution of the given DEqn. However, when the operators are not commuting some interesting features occur. In section 8 the cauchy euler problem is studied and we use it as a laboratory to explore the factorization of operators concept further. It turns out the cauchy euler problem can be cast as $P(xD)[y] = 0$. Finally, we study springs and RLC circuits.

I am indebted to Nagel Saff and Snider and Rabenstein's texts primarily for this chapter. However, I wouldn't blaim them for any of the mistakes. Those are probably mine.[1]

[1] Ritger and Rose has added another tool to my toolbox for constant coefficient ODEs, I will show it to you before much of the material in this chapter, sadly I do not have time to add it to these notes. It is crucial you take notes in lecture.

## 3.1 linear differential equations

We say $I$ is an **interval** iff $I = (a,b), [a,b), [a,b), [a,b], [a,\infty), (a,\infty), (-\infty,a], (-\infty,a), (-\infty,\infty)$.

**Definition 3.1.1.** *n-th order linear differential equation*

Let $I$ be an interval of real numbers. Let $a_o, a_1, \dots, a_n, f$ be real-valued functions on $I$ such that $a_o(x) \neq 0$ for all $x \in I$. We say

$$a_o \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f$$

is an **$n$-th order linear differential equation** on $I$ with **coefficient functions** $a_o, a_1, \dots, a_n$ and **forcing** function $f$. If $f(x) = 0$ for all $x \in I$ then we say the differential equation is **homogeneous**. However, if $f(x) \neq 0$ for at least one $x \in I$ then the differential equation is said to be **nonhomogeneous**.

In the prime notation our generic linear ODE looks like:

$$a_o y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f$$

If the independent variable was denoted by $t$ then we could emphasize that by writing

$$a_o y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}y'(t) + a_n y(t) = f(t).$$

Typically we either use $x$ or $t$ as the independent variable in this course. We can denote differentiation as an **operator** $D$ and, depending on the context, either $D = d/dx$ or $D = d/dt$. In this operator notation we can write the generic differential equation as

$$a_o D^n[y] + a_1 D^{n-1}[y] + \cdots + a_{n-1}D[y] + a_n y = f$$

or, introducing $L = a_o D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$ we find our generic linear ODE is simply $L[y] = f$. We'll calculate in a future section that this operator $L$ is a **linear transformation** on function space. You input a function $y$ into $L$ and you get out a new function $L[y]$. A function-valued function of functions is called an operator, it follows that $L$ is a **linear operator**. If all the coefficient functions were smooth then we could say that $L$ is a **smooth operator**. Enough math-jargon. The bottom line is that $L$ enjoys the following beautiful properties:

$$L[y_1 + y_2] = L[y_1] + L[y_2] \qquad \& \qquad L[cy_1] = cL[y_1]$$

for any $n$-fold differentiable functions[2] $y_1, y_2$ on $I$ and constant $c$.

[2] in fact these could be complex-valued functions and $c$ could be complex, but I don't want to discuss the details of that just yet

**Theorem 3.1.2.** *unique solutions to the initial value problem for $L[y] = f$.*

Suppose $a_o, a_1, \dots, a_n, f$ are continuous on an interval $I$ with $a_o(x) \neq 0$ for all $x \in I$. Suppose $x_o \in I$ and $y_o, y_1, \dots, y_{n-1} \in \mathbb{R}$ then there exists a **unique** function $\phi$ such that:

$$\text{(1.)} \quad a_o(x)\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \cdots + a_{n-1}(x)\phi'(x) + a_n(x)\phi(x) = f(x)$$

for all $x \in I$ and

$$\text{(2.)} \quad \phi(x_o) = y_o, \ \phi'(x_o) = y_1, \ \phi''(x_o) = y_2, \ \dots, \ \phi^{(n-1)}(x_o) = y_{n-1}.$$

The linear differential equation $L[y] = f$ on an interval $I$ paired with the $n$-conditions $y(x_o) = y_o, \ y'(x_o) = y_1, \ y''(x_o) = y_2, \ \dots, \ y^{(n-1)}(x_o) = y_{n-1}$ is called an **initial value problem** (IVP). The theorem above simply says that there is a unique solution to the initial value problem for any linear $n$-th order ODE with continuous coefficients. The proof of this theorem can be found in many advanced calculus or differential equations texts. See Chapter 13 of Nagle Saff and Snider for some discussion. We can't cover it here because we need ideas about convergence of sequences of functions. If you are interested you should return to this theorem after you have completed the real analysis course. Proof aside, we will see how this theorem works dozens if not hundreds of times as the course continues.

I'll illustrate the theorem with some examples.

**Example 3.1.3.** *The solution of of $y' = y$ with $y(0) = 1$ is given by $y = e^x$. Here $L[y] = y' - y$ and the coefficient functions are $a_0 = 1$ and $a_1 = 1$. These constant coefficients are continuous on $\mathbb{R}$ and $a_o = 1 \neq 0$ on $\mathbb{R}$ as well. It follows from Theorem 3.1.2 that the unique solution with $y(0) = 1$ should exist on $\mathbb{R}$.*

**Example 3.1.4.** *The general solution of $y'' + y$ is given by*

$$y = c_1 \cos(x) + c_2 \sin(x)$$

*by the method of reduction of order shown in Example 2.4.11. Theorem 3.1.2 indicates that there is a unique choice of $c_1, c_2$ to produce a particular set of initial conditions. For example: the solution of of $y'' + y = 0$ with $y(0) = 1, y'(0) = 1$ is given by $y = \cos(x) + \sin(x)$. Here $L[y] = y'' + y$ and the coefficient functions are $a_0 = 1, a_1 = 0$ and $a_1 = 1$. These constant coefficients are continuous on $\mathbb{R}$ and $a_o = 1 \neq 0$ on $\mathbb{R}$ as well. Once more we see from Theorem 3.1.2 that the unique solution with $y(0) = 1, y'(0) = 1$ should exist on $\mathbb{R}$, and it does!*

**Example 3.1.5.** *The general solution of $x^3y'' + xy' - y = 0$ is given by*

$$y = c_1 x + c_2 x e^{1/x}$$

*Observe that $a_o(x) = x^3$ and $a_1(x) = x$ and $a_o(x) = -1$. These coefficient functions are continuous on $\mathbb{R}$, however, $a_o(0) = 0$. We can only expect, from Theorem 3.1.2, that solutions to exist on $(0, \infty)$ or $(-\infty, 0)$. This is precisely the structure of the general solution. I leave it to the reader to verify that the initial value problem has a unique solution on either $(0, \infty)$ or $(-\infty, 0)$.*

**Example 3.1.6.** *Consider $y^{(n)}(t) = 0$. If we integrate $n$-times then we find (absorbing any fractions of integration into the constants for convenience)*

$$y(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

*is the general solution. Is there a unique solution to the initial value problem here? Theorem 3.1.2 indicates yes since* $a_o = 1$ *is nonzero on* $\mathbb{R}$ *and all the other coefficient functions are clearly continuous. Once more I leave the proof to the reader*[3]*, but as an example* $y''' = 0$ *with* $y(0) = 1, y'(0) = 1$ *and* $y''(0) = 2$ *is solved uniquely by* $y(t) = t^2 + t + 1$.

We see that there seem to be $n$-distinct functions forming the solution to an $n$-th order linear ODE. We need to develop some additional theory to make this idea of *distinct* a bit more precise. For example, we would like to count $e^x$ and $2e^x$ as the same function since multiplication by 2 in our context could easily be absorbed into the constant. On the other hand, $e^{2x}$ and $e^{3x}$ are distinct functions.

**Definition 3.1.7.** *linear independence of functions on an interval* $I$*.*

Let $I$ be an interval of real numbers. We say the set of functions $\{f_1, f_2, f_3, \dots, f_m\}$ are **linearly independent (LI)** on $I$ iff

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all $x \in I$ implies $c_1 = c_2 = \cdots = c_m = 0$. Conversely, if $\{f_1, f_2, f_3, \dots, f_m\}$ are not linearly independent on $I$ then they are said to be **linearly dependent** on $I$.

It is not hard to show that if $\{f_1, f_2, f_3, \dots, f_m\}$ is linearly dependent set of functions on $I$ then there is at least one function, say $f_j$ such that

$$f_j = c_1 f_1 + c_2 f_2 + \cdots c_{j-1} f_{j-1} + c_{j+1} f_{j+1} + \cdots + c_n f_n.$$

This means that the function $f_j$ is redundant. If these functions are solutions to $L[y] = 0$ then we don't really need $f_j$ since the other $n-1$ functions can produce the same solutions under linear combinations. On the other hand, if the set of solutions is linearly independent then every function in the set is needed to produce the general solution. As a point of conversational convenience let us adopt the following convention: $f_1$ **and** $f_2$ **are independent on** $I$ **iff** $\{f_1, f_2\}$ **are linearly independent on** $I$.

I may discuss direct application of the defintion above in lecture, however it is better to think about the construction to follow here. We seek a convenient computational characterization of linear independence of functions. Suppose that $\{y_1, y_2, y_3, \dots, y_m\}$ is linearly **independent** set of functions on $I$ which are at least $(n-1)$-times differentiable. Furthermore, suppose for all $x \in I$

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_m y_m(x) = 0.$$

Differentiate to obtain for all $x \in I$:

$$c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_m y_m'(x) = 0.$$

Differentiate again to obtain for all $x \in I$:

$$c_1 y_1''(x) + c_2 y_2''(x) + \cdots + c_m y_m''(x) = 0.$$

Continue differentiating until we obtain for all $x \in I$:

$$c_1 y_1^{(m-1)}(x) + c_2 y_2^{(m-1)}(x) + \cdots + c_m y_m^{(m-1)}(x) = 0.$$

[3] this makes a nice linear algebra problem

Let us write these $m$-equations in matrix notation[4]

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_m(x) \\ y_1'(x) & y_2'(x) & \cdots & y_m'(x) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(m-1)}(x) & y_2^{(m-1)}(x) & \cdots & y_m^{(m-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In linear algebra we show that the linear equation $A\vec{x} = \vec{b}$ has a unique solution iff $det(A) \neq 0$. Since we have assumed linear independence of $\{y_1, y_2, y_3, \dots, y_m\}$ on $I$ we know $c_1 = c_2 = \cdots = c_m = 0$ is the only solution of the system above for each $x \in I$. Therefore, the coefficient matrix must have nonzero determinant[5] on all of $I$. This determinant is called the **Wronskian**.

**Definition 3.1.8.** *Wronskian of functions $y_1, y_2, \dots, y_m$ at $x$.*

$$W(y_1, y_2, \dots, y_m; x) = det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_m(x) \\ y_1'(x) & y_2'(x) & \cdots & y_m'(x) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(m-1)}(x) & y_2^{(m-1)}(x) & \cdots & y_m^{(m-1)}(x) \end{bmatrix}.$$

It is clear from the discussion preceding this definition that we have the following proposition:

**Theorem 3.1.9.** *nonzero Wronskian on $I$ implies linear independence on $I$.*

If $W(y_1, y_2, \dots, y_m; x)$ for each $x \in I$ then $\{y_1, y_2, y_3, \dots, y_m\}$ is linearly independent on $I$.

Let us pause to introduce the formulas for the determinant of a square matrix. We define,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Then the $3 \times 3$ case is defined in terms of the $2 \times 2$ formula as follows:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

and finally the $4 \times 4$ determinant is given by

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \cdot \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \cdot \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix}$$
$$+ c \cdot \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \cdot \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

[4] don't worry too much if you don't know matrix math just yet, we will cover some of the most important matrix computations a little later in this course, for now just think of it as a convenient notation

[5] have no fear, I will soon remind you how we calculate determinants, you saw the pattern before with cross products in calculus III

Expanding the formula for the determinant in terms of lower order determinants is known as *Laplace's expansion by minors*. It can be shown, after considerable effort, this is the same as defining the deteminant as the completely antisymmetric multilinear combination of the rows of $A$:

$$det(A) = \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}.$$

See my linear algebra notes if you want to learn more. For the most part we just need the $2 \times 2$ or $3 \times 3$ for examples.

**Example 3.1.10.** *Consider $y_1 = e^{ax}$ and $y_2 = e^{bx}$ for $a, b \in \mathbb{R}$ with $a \neq b$. The Wronskian is*

$$W(e^{ax}, e^{bx}, x) = det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \begin{bmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{bmatrix} = e^{ax}be^{bx} - e^{bx}ae^{ax} = (a-b)e^{(a+b)x}.$$

*Since $a - b \neq 0$ it follows $W(e^{ax}, e^{bx}, x) \neq 0$ on $\mathbb{R}$ and we find $\{e^{ax}, e^{bx}\}$ is LI on $\mathbb{R}$.*

**Example 3.1.11.** *Consider $y_1(t) = 1$ and $y_2(t) = t$ and $y_3(t) = t^2$. The Wronskian is*

$$W(1, t, t^2, t) = det \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{bmatrix} = (1)(1)(2) = 2.$$

*Clearly $W(1, t, t^2; t) \neq 0$ for all $t \in \mathbb{R}$ and we find $\{1, t, t^2\}$ is LI on $\mathbb{R}$.*

**Example 3.1.12.** *Consider $y_1(x) = x$ and $y_2(x) = \sinh(x)$ and $y_3(x) = \cosh(x)$. Calculate $W(x, \cosh(x), \sinh(x); x) =$*

$$\begin{aligned}
&= det \begin{bmatrix} x & \cosh(x) & \sinh(x) \\ 1 & \sinh(x) & \cosh(x) \\ 0 & \cosh(x) & \sinh(x) \end{bmatrix} \\
&= x det \begin{bmatrix} \sinh(x) & \cosh(x) \\ \cosh(x) & \sinh(x) \end{bmatrix} - \cosh(x) det \begin{bmatrix} 1 & \cosh(x) \\ 0 & \sinh(x) \end{bmatrix} + \sinh(x) det \begin{bmatrix} 1 & \sinh(x) \\ 0 & \cosh(x) \end{bmatrix} \\
&= x\big[\sinh^2(x) - \cosh^2(x)\big] - \cosh(x)\big[1\sinh(x) - 0\cosh(x)\big] + \sinh(x)\big[1\cosh(x) - 0\sinh(x)\big] \\
&= -x.
\end{aligned}$$

*Clearly $W(x, \cosh(x), \sinh(x); x) \neq 0$ for all $x \neq 0$. It follows $\{x, \cosh(x), \sinh(x)\}$ is LI on any interval which does not contain zero.*

The interested reader is apt to ask: is $\{x, \cosh(x), \sinh(x)\}$ linearly dependent on an interval which does contain zero? The answer is no. In fact:

**Theorem 3.1.13.** *Wronskian trivia.*

Suppose $\{y_1, y_2, \dots, y_m\}$ are $(n-1)$-times differentiable on an interval $I$. If $\{y_1, y_2, \dots, y_m\}$ is linearly dependent on $I$ then $W(y_1, y_2, \dots, y_m; x) = 0$ for all $x \in I$. Conversely, if there exists $x_o \in I$ such that $W(y_1, y_2, \dots, y_m; x) \neq 0$ then $\{y_1, y_2, \dots, y_m\}$ is LI on $I$.

The still interested reader might ask: "what if the Wronskian is zero at all points of some interval? Does that force linear dependence?". Again, no. Here's a standard example that probably dates back to a discussion by Peano and others in the late 19-th century:

**Example 3.1.14.** *The functions $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent on $\mathbb{R}$. You can see this from supposing $c_1x^2 + c_2x|x| = 0$ for all $x \in \mathbb{R}$. Take $x = 1$ to obtain $c_1 + c_2 = 0$ and take $x = -1$ to obtain $c_1 - c_2 = 0$ which solved simultaneously yield $c_1 = c_2 = 0$. However,*

$$W(x, |x|; x) = det \begin{bmatrix} x^2 & x|x| \\ 2x & 2|x| \end{bmatrix} = 0.$$

The Wronskian is useful for testing linear-dependence of complete solution sets of a linear ODE.

**Theorem 3.1.15.** *Wronskian on a solution set of a linear ODE.*

Suppose $L[y] = 0$ is an $n$-th order linear ODE on an interval $I$ and $y_1, y_2, \dots, y_n$ are solutions on $I$. If there exists $x_o \in I$ such that $W(y_1, y_2, \dots, y_n; x_o) \neq 0$ then $\{y_1, y_2, \dots, y_n\}$ is LI on $I$. On the other hand, if there exists $x_o \in I$ such that $W(y_1, y_2, \dots, y_n; x_o) = 0$ then $\{y_1, y_2, \dots, y_n\}$ is linearly dependent on $I$

Notice that the number of solutions considered must match the order of the equation. It turns out the theorem does not apply to smaller sets of functions. It is possible for the Wronskian of two solutions to a third order ODE to vanish even though the functions are linearly independent. The most interesting proof of the theorem above is given by Abel's formula. I'll show how to derive it in the $n = 2$ case to begin:

Let $a_o, a_1, a_2$ be continuous functions on an interval $I$ with $a_o(x) \neq 0$ for each $x \in I$. Suppose $a_oy'' + a_1y' + a_2y = 0$ has solutions $y_1, y_2$ on $I$. Consider the Wronskian $W(x) = y_1y_2' - y_2y_1'$. Something a bit interesting happens as we calculate the derivative of $W$,

$$W' = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''.$$

However, $y_1$ and $y_2$ are solutions of $a_oy'' + a_1y' + a_2y = 0$ hence

$$y_1'' = -\frac{a_1}{a_o}y_1' - \frac{a_2}{a_o}y_1 \qquad \& \qquad y_2'' = -\frac{a_1}{a_o}y_2' - \frac{a_2}{a_o}y_2$$

Therefore,

$$W' = y_1\left(-\frac{a_1}{a_o}y_2' - \frac{a_2}{a_o}y_2\right) - y_2\left(-\frac{a_1}{a_o}y_1' - \frac{a_2}{a_o}y_1\right) = \frac{a_1}{a_o}\left(y_1y_2' - y_2y_1'\right) = \frac{a_1}{a_o}W$$

We can solve $\frac{dW}{dx} = \frac{a_1}{a_o}W$ by separation of variables:

$$\boxed{W(x) = C\, exp\left[\int \frac{a_1}{a_o}\, dx\right]} \qquad \Leftarrow \text{ Abel's Formula.}$$

It follows that either $C = 0$ and $W(x) = 0$ for all $x \in I$ or $C \neq 0$ and $W(x) \neq 0$ for all $x \in I$.

It is a bit surprising that Abel's formula does not involve $a_2$ directly. It is fascinating that this continues to be true for the $n$-th order problem: if $y_1, y_2, \dots, y_n$ are solutions of $a_oy^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$ and $W$ is the Wronskian of the given $n$-functions then $W$ is gvien by Abel's formula $W(x) = C\, exp\left[\int \frac{a_1}{a_o}\, dx\right]$. You can skip the derivation that follows if you wish. What follows is an example of tensor calculus: let $Y = [y_1, y_2, \dots, y_n]$ thus $Y' = [y_1', y_2', \dots, y_n']$ and $Y^{(n-1)} = [y_1^{(n-1)}, y_2^{(n-1)}, \dots, y_n^{(n-1)}]$. The Wronskian is given by

$$W = \sum_{i_1, i_2, \dots, i_n = 1}^{n} \epsilon_{i_1 i_2 \dots i_n} Y_{i_1} Y_{i_2}' \cdots Y_{i_n}^{(n-1)}$$

Apply the product rule for $n$-fold products on each summand in the above sum,

$$W' = \sum_{i_1,i_2,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n}\left( Y'_{i_1}Y'_{i_2}\cdots Y^{(n-1)}_{i_n} + Y_{i_1}Y''_{i_2}Y''_{i_3}\cdots Y^{(n-1)}_{i_n} + \cdots + Y_{i_1}Y'_{i_2}\cdots Y^{(n-2)}_{i_{n-1}}Y^{(n)}_{i_n}\right)$$

The term $Y'_{i_1}Y'_{i_2}\cdots Y^{(n-1)}_{i_n} = Y'_{i_2}Y'_{i_1}\cdots Y^{(n-1)}_{i_n}$ hence is symmetric in the pair of indices $i_1, i_2$. Next, the term $Y_{i_1}Y''_{i_2}Y''_{i_3}\cdots Y^{(n-1)}_{i_n}$ is symmetric in the pair of indices $i_2, i_3$. This patterns continues up to the term $Y_{i_1}Y'_{i_2}\cdots Y^{(n-1)}_{i_{n-2}}Y^{(n-2)}_{i_{n-1}}Y^{(n-1)}_{i_n}$ which is symmetric in the $i_{n-2}, i_{n-1}$ indices. In contrast, the completely antisymmetric symbol $\epsilon_{i_1 i_2 \dots i_n}$ is antisymmetric in each possible pair of indices. Note that if $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$ then

$$\sum_i \sum_j S_{ij}A_{ij} = \sum_i \sum_j -S_{ji}A_{ji} = -\sum_j \sum_i S_{ji}A_{ji} = -\sum_i \sum_j S_{ij}A_{ij} \quad \Rightarrow \quad \sum_i \sum_j S_{ij}A_{ij} = 0.$$

If we sum an antisymmetric object against a symetric object then the result is zero. It follows that only one term remains in calculation of $W'$:

$$W' = \sum_{i_1,i_2,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n} Y_{i_1}Y'_{i_2}\cdots Y^{(n-2)}_{i_{n-1}}Y^{(n)}_{i_n} \qquad (\star)$$

Recall that $y_1, y_2, \dots, y_n$ are solutions of $a_o y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$ hence

$$y_j^{(n)} = -\frac{a_1}{a_o}y_j^{(n-1)} - \cdots - \frac{a_{n-1}}{a_o}y'_j - \frac{a_n}{a_o}y_j = 0$$

for each $j = 1, 2, \dots, n$. But, this yields

$$Y^{(n)} = -\frac{a_1}{a_o}Y^{(n-1)} - \cdots - \frac{a_{n-1}}{a_o}Y' - \frac{a_n}{a_o}Y$$

Substitute this into $\star$,

$$\begin{aligned}
W' &= \sum_{i_1,i_2,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n} Y_{i_1}Y'_{i_2}\cdots Y^{(n-2)}_{i_{n-1}}\left[-\frac{a_1}{a_o}Y^{(n-1)} - \cdots - \frac{a_{n-1}}{a_o}Y' - \frac{a_n}{a_o}Y\right]_{i_n} \\
&= \sum_{i_1,i_2,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n}\left(-\frac{a_1}{a_o}Y_{i_1}Y'_{i_2}\cdots Y^{(n-1)}_{i_n} - \cdots - \frac{a_{n-1}}{a_o}Y_{i_1}Y'_{i_2}\cdots Y'_{i_n} - \frac{a_n}{a_o}Y_{i_1}Y'_{i_2}\cdots Y_{i_n}\right) \\
&= -\frac{a_1}{a_o}\sum_{i_1,i_2,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n} Y_{i_1}Y'_{i_2}\cdots Y^{(n-1)}_{i_n} \qquad \star\star \\
&= -\frac{a_1}{a_o}W.
\end{aligned}$$

The $\star\star$ step is based on the observation that the index pairs $i_1, i_n$ and $i_2, i_n$ etc... are symmetric in the line above it hence as they are summed against the completely antisymmetric symbol those terms vanish. Alternatively, and equivalently, you could apply the multilinearity of the determinant paired with the fact that a determinant with any two repeated rows vanishes. Linear algebra aside, we find $W' = -\frac{a_1}{a_o}W$ thus Abel's formula $W(x) = C\,exp\big[\int \frac{a_1}{a_o}\,dx\big]$ follows immediately.

Solution sets of functions reside in function space. As a vector space, function space is infinite dimensional. The matrix techniques you learn in the linear algebra course do not apply to the totallity of function space. Appreciate the Wronskian says what it says. In any event, we should continue our study of DEqns at this point since we have all the tools we need to understand LI in this course.

**Definition 3.1.16.** *fundamental solutions sets of linear ODEs.*

Suppose $L[y] = f$ is an $n$-th order linear differential equation on an interval $I$. We say $S = \{y_1, y_2, \dots, y_n\}$ is a **fundamental solution set** of $L[y] = f$ iff $S$ is linearly independent set of solutions to the homogeneous equation; $L[y_j] = 0$ for $j = 1, 2, \dots n$.

**Example 3.1.17.** *The differential equation $y'' + y = f$ has fundamental solution set $\{\cos(x), \sin(x)\}$. You can easily verify that $W(\cos(x), \sin(x); x) = 1$ hence linear independence is established the given functions. Moreover, it is simple to check $y'' + y = 0$ has sine and cosine as solutions. The formula for $f$ is irrelevant to the fundamental solution set. Generally, the fundamental solution set is determined by the structure of $L$ when we consider the general problem $L[y] = f$.*

**Theorem 3.1.18.** *existence of a fundmental solution set.*

If $L[y] = f$ is an $n$-th order linear differential equation with continuous coefficient functions on an interval $I$ then there exists a fundamental solution set $S = \{y_1, y_2, \dots, y_n\}$ on $I$.

**Proof:** Theorem 3.1.2 applies. Pick $x_o \in I$ and use the existence theorem to obtain the solution $y_1$ subject to

$$y_1(x_o) = 1, \quad y_1'(x_o) = 0, \quad y_1''(x_o) = 0, \; \dots \; , y_1^{(n-1)}(x_o) = 0.$$

Apply the theorem once more to select solution $y_2$ with:

$$y_2(x_o) = 0, \quad y_2'(x_o) = 1, \quad y_2''(x_o) = 0, \; \dots \; , y_2^{(n-1)}(x_o) = 0.$$

Then continue in this fashion selecting solutions $y_3, y_4, \dots, y_{n-1}$ and finally $y_n$ subject to

$$y_n(x_o) = 0, \quad y_n'(x_o) = 0, \quad y_n''(x_o) = 0, \; \dots \; , y_n^{(n-1)}(x_o) = 1.$$

It remains to show that the solution set $\{y_1, y_2, \dots, y_n\}$ is indeed linearly independent on $I$. Calculate the Wronskian at $x = x_o$ for the solution set $\{y_1, y_2, \dots, y_n\}$: abbreviate $W(y_1, y_2, \dots, y_n; x)$ by $W(x)$ for the remainder of this proof:

$$W(x_o) = det \begin{bmatrix} y_1(x_o) & y_2(x_o) & \cdots & y_m(x_o) \\ y_1'(x_o) & y_2'(x_o) & \cdots & y_m'(x_o) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_o) & y_2^{(n-1)}(x_o) & \cdots & y_n^{(n-1)}(x_o) \end{bmatrix} = det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = 1.$$

Therefore, by Abel's formula, the Wronskian is nonzero on the whole interval $I$ and it follows the solution set is LI. □

**Theorem 3.1.19.** *general solution of the homogeneous linear $n$-th order problem.*

If $L[y] = f$ is an $n$-th order linear differential equation with continuous coefficient functions on an interval $I$ with fundamental solution set $S = \{y_1, y_2, \dots, y_n\}$ on $I$. Then any solution of $L[y] = 0$ can be expressed as a linear combination of the fundamental solution set: that is, there exist constants $c_1, c_2, \dots, c_n$ such that:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

**Proof:** Suppose $S = \{y_1, y_2, \dots, y_n\}$ is a fundamental solution set of $L[y] = 0$ on $I$. Furthermore, suppose $y$ is a solution; $L[y] = 0$. We seek to find $c_1, c_2, \dots, c_n$ such that $y = c_1y_1 + c_2y_2 + \cdots c_ny_n$. Consider a particular point $x_o \in I$, we need that solution $y$ and its derivatives $(y', y'', ...)$ up to order $(n-1)$ match with the proposed linear combination of the solution set:

$$y(x_o) = c_1y_1(x_o) + c_2y_2(x_o) + \cdots + c_ny_n(x_o).$$

$$y'(x_o) = c_1y_1'(x_o) + c_2y_2'(x_o) + \cdots + c_ny_n'(x_o).$$

continuing, up to the $(n-1)$-th derivative

$$y^{(n-1)}(x_o) = c_1y_1^{(n-1)}(x_o) + c_2y_2^{(n-1)}(x_o) + \cdots + c_ny_n^{(n-1)}(x_o).$$

It is instructive to write this as a matrix problem:

$$\begin{bmatrix} y(x_o) \\ y'(x_o) \\ \vdots \\ y^{(n-1)}(x_o) \end{bmatrix} = \begin{bmatrix} y_1(x_o) & y_2(x_o) & \cdots & y_n(x_o) \\ y_1'(x_o) & y_2'(x_o) & \cdots & y_n'(x_o) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_o) & y_2^{(n-1)}(x_o) & \cdots & y_n^{(n-1)}(x_o) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The coefficient matrix has nonzero determinant (it is the Wronskian at $x = x_o$) hence this system of equations has a unique solution. Therefore, we can select constants $c_1, c_2, \dots, c_n$ such that the solution $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$. $\square$

In fact, the proof shows that these constants are unique for a given fundamental solution set. Each solution is uniquely specified by the constants $c_1, c_2, \dots, c_n$. When I think about the solution of a linear ODE, I always think of the constants in the general solution as the reflection of the reality that a given DEqn can be assigned many different initial conditions. However, once the initial condition is given the solution is specified uniquely.

Finally we turn to the nonhomogeneous problem. I present the theory here, however, the computational schemes are given much later in this chapter.

**Theorem 3.1.20.** *general solution of the nonhomogeneous linear $n$-th order problem.*

If $L[y] = f$ is an $n$-th order linear differential equation with continuous coefficient functions on an interval $I$ with fundamental solution set $S = \{y_1, y_2, \dots, y_n\}$ on $I$. Then any solution of $L[y] = f$ can be expressed as a linear combination of the fundamental solution set and a function $y_p$ with $L[y_p] = f$ known as the **particular solution** :

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p.$$

**Proof:** Theorem 3.1.2 applies, it follows there are many solutions to $L[y] = f$, one for each set of initial conditions. Suppose $y$ and $y_p$ are two solutions to $L[y] = f$. Observe that

$$L[y - y_p] = L[y] - L[y_p] = f - f = 0.$$

Therefore, $y_h = y - y_p$ is a solution of the homogeneous ODE $L[y] = 0$ thus Theorem 3.1.19 we can write $y_h$ as a linear comination of the fundamental solutions: $y_h = c_1y_1 + c_2y_2 + \cdots + c_ny_n$. But, $y = y_p + y_h$ and the theorem follows. $\square$

**Example 3.1.21.** *Suppose $L[y] = f$ is a second order linear ODE and $y = e^x + x^2$ and $z = \cos(x) + x^2$ are solutions. Then*

$$L[y - z] = L[y] - L[z] = f - f = 0$$

*hence $y - z = (e^x + x^2) - (\cos(x) - x^2) = e^x - \cos(x)$ gives a homogeneous solution $y_1(x) = e^x - \cos(x)$. Notice that $w = y + 2y_1 = e^x + x^2 + 2(e^x - \cos(x)) = 3e^x - \cos(x) + x^2$ is also a solution since $L[y + 2y_1] = L[y] + 2L[y_1] = f + 0 = f$. Consider that $w - z = (3e^x - \cos(x) + x^2) - (\cos(x) + x^2) = 3e^x$ is also a homogeneous solution. It follows that $\{3e^x, e^x - \cos(x)\}$ is a fundamental solution set of $L[y] = f$. In invite the reader to show $\{e^x, \cos(x)\}$ is also a fundamental solution set.*

The example above is important because it illustrates that we can extract homogeneous solutions from particular solutions. Physically speaking, perhaps you might be faced with the same system subject to several different forces. If solutions are observed for $L[y] = F_1$ and $L[y] = 2F_1$ then we can deduce the general solution set of $L[y] = 0$. In particular, this means you could deduce the mass and spring constant of a particular spring-mass system by observing how it responds to a pair of forces. More can be said here, we'll return to these thoughts as we later discuss the *principle of superposition.*

## 3.2 operators and calculus

In this section we seek to establish the calculus of operators. First we should settle a few rules:

**Definition 3.2.1.** *operators, operator equality, new operators from old:*

Suppose $\mathcal{F}$ is a set of functions then $T : \mathcal{F} \to \mathcal{F}$ is an operator on $\mathcal{F}$. If $T_1, T_2$ are operators on $\mathcal{F}$ then $T_1 = T_2$ iff $T_1[f] = T_2[f]$ for all $f \in \mathcal{F}$. In addition, $T_1 + T_2$, $T_1 - T_2$ and $T_1T_2$ are defined by

$$(T_1 + T_2)[f] = T_1[f] + T_2[f] \quad \& \quad (T_1 - T_2)[f] = T_1[f] - T_2[f] \quad \& \quad (T_1T_2)[f] = T_1[T_2[f]]$$

If $g \in \mathcal{F}$ and $T$ is an operator on $\mathcal{F}$ then $gT$ and $Tg$ are the operators defined by $(gT)[f] = gT[f]$ and $(Tg)[f] = T[f]g$ for all $f \in \mathcal{F}$. In addition, for $n \in \mathbb{N} \cup \{0\}$ we define $T^n$ by $T^{n-1}T$ where $T^0 = 1$; that is $T^n[f] = T[T[\cdots[T[f]]\cdots]]$ where that is an $n$-fold composition. We are often interested in differentiation thus it is convenient to denote differentiation by $D$; $D[f] = f'$ for all $f \in \mathcal{F}$. Finally, a polynomial of operators is naturally defined as follows: if $P(x) = a_ox^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $T$ is an operator then

$$P(T) = a_oT^n + a_1T^{n-1} + \cdots + a_{n-1}T + a_n.$$

We usually assume $\mathcal{F}$ is a set of smooth functions on an interval. In fact, sorry to be sloppy, but we will not worry too much about what $\mathcal{F}$ is for most examples. Moreover, the functions we consider can take their values in $\mathbb{R}$ or in $\mathbb{C}$ as we shall soon show.

**Proposition 3.2.2.**

Suppose $T$ is an operator and $g$ is a function then $gT = Tg$

**Proof:** let $f$ be a function. Observe $(gT)[f] = gT[f] = T[f]g = (Tg)[f]$ hence $gT = Tg$. $\square$

**Example 3.2.3.** *Suppose $T = xd/dx$ and $S = d/dx$. Let's see if $TS = ST$ in this case. Let $f$ be a function,*

$$(TS)[f] = T[S[f]] = x\frac{d}{dx}\left[\frac{df}{dx}\right] = x\frac{d^2f}{dx^2}$$

$$(ST)[f] = S[T[f]] = \frac{d}{dx}\left[x\frac{df}{dx}\right] = \frac{df}{dx} + x\frac{d^2f}{dx^2}$$

*Apparently $TS - ST = d/dx$. In any event, clearly not all operators* **commute***.*

On the other hand, we can show $(aD + b)(cD + d) = acD^2 + (ad + bc)D + bd$ for constants $a, b, c, d$:

$$[(aD + b)(cD + d)][f] = (aD + b)[cf' + df] = (aD + b)[cf'] + (aD + b)[df] = acf'' + bcf' + adf' + bdf.$$

Thus $(aD+b)(cD+d)][f] = (acD^2+(ad+bc)D+bd)[f]$. Clearly it follows that the operators $T = aD+b$ and $S = cD + d$ commute. In fact, this calculation can be extended to arbitrary polynomials of the derivative.

**Proposition 3.2.4.**

Suppose $P(D)$ and $Q(D)$ are polynomials of the differentiation operator $D = d/dx$ then $P(D)Q(D) = (PQ)(D)$ where $PQ$ denotes the usual multiplication of polynomials.

**Proof:** Suppose $P(D) = a_oD^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$ and the special case $Q(D) = D^k$. Multiplying $P$ and $Q$ as polynomials yields:

$$(PQ)(D) = a_oD^{n+k} + a_1D^{n+k-1} + \cdots + a_{n-1}D^{k+1} + a_nD^k$$

Thus, $(PQ)(D)[f] = a_oD^{n+k}[f] + a_1D^{n+k-1}[f] + \cdots + a_{n-1}D^{k+1}[f] + a_nD^k[f]$. On the other hand,

$$\begin{aligned} P(D)[D^k[f]] &= a_oD^n[D^k[f]] + a_1D^{n-1}[D^k[f]] + \cdots + a_{n-1}D[D^k[f]] + a_nD^k[f] \\ &= a_oD^{n+k}[f] + a_1D^{n+k-1}[f] + \cdots + a_{n-1}D^{k+1}[f] + a_nD^k[f] \\ &= (P(D)D^k)[f] \end{aligned}$$

The calculation above shows $(P(D)D^k)[f] = P(D)[D^k[f]]$. Also we can pull out constants,

$$\begin{aligned} P(D)[cf] &= a_oD^n[cf] + a_1D^{n-1}[cf] + \cdots + a_{n-1}D[cf] + a_n[cf] \\ &= c(a_oD^n[f] + a_1D^{n-1}[f] + \cdots + a_{n-1}D[f] + a_n[f]) \\ &= (cP(D))[f] = (P(D)c)[f] \qquad \text{using Proposition 3.2.2} \end{aligned}$$

Next, examine $P(D)[f+g]$,

$$\begin{aligned} P(D)[f+g] &= a_oD^n[f+g] + a_1D^{n-1}[f+g] + \cdots + a_{n-1}D[f+g] + a_o[f+g] \\ &= a_oD^n[f] + a_1D^{n-1}[f] + \cdots + a_{n-1}D[f] + a_n[f] \\ &\quad + a_oD^n[g] + a_1D^{n-1}[g] + \cdots + a_{n-1}D[g] + a_n[g] \\ &= P(D)[f] + P(D)[g]. \end{aligned}$$

We use the results above to treat the general case. Suppose $P(D) = a_oD^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$ and $Q(D) = b_oD^k + b_1D^{k-1} + \cdots + b_{k-1}D + b_k$. Calculate:

$$\begin{aligned} P(D)[Q(D)[f]] &= P(D)[b_oD^n[f] + b_1D^{n-1}[f] + \cdots + b_{n-1}D[f] + b_o[f]] \\ &= P(D)[b_oD^n[f]] + P(D)[b_1D^{n-1}[f]] + \cdots + P(D)[b_{n-1}D[f]] + P(D)[b_o[f]] \\ &= (P(D)b_oD^n)[f] + (P(D)b_1D^{n-1})[f] + \cdots + (P(D)b_{n-1}D)[f] + (P(D)b_o)[f] \\ &= (P(D)b_oD^n + P(D)b_1D^{n-1} + \cdots + P(D)b_{n-1}D + P(D)b_o)[f] \\ &= (P(D)(b_oD^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_o))[f] \\ &= (P(D)Q(D))[f] \\ &= (PQ(D))[f]. \end{aligned}$$

Therefore, the operator formed from polynomial multiplication of $P(D)$ and $Q(D)$ is the same as the operator formed from composing the operators. $\square$

The beauty of the proposition is the following observation:

$$\boxed{(x-\lambda) \text{ is a factor of } P(x) \quad \Rightarrow \quad (D-\lambda) \text{ is a factor of } P(D)}$$

For example, if $P(x) = x^2 + 3x + 2$ then we can write $P(x) = (x+1)(x+2)$. Hence it follows $P(D) = (D+1)(D+2)$. I'll show the power of this fact in the next section.

### 3.2.1 complex-valued functions of a real variable

**Definition 3.2.5.**

Suppose $f$ is a function from an interval $I \subseteq \mathbb{R}$ to the complex numbers $\mathbb{C}$. In particular, suppose $f(t) = u(t) + iv(t)$ where $i^2 = -1$ we say $Re(f) = u$ and $Im(f) = v$. Furthermore, define

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt} \qquad \& \qquad \int f(t)\,dt = \int u\,dt + i\int v\,dt.$$

Higher derivatives are similarly defined. I invite the reader to verify the following properties for complex-valued functions $f, g$:

$$\frac{d}{dt}(f+g) = \frac{df}{dt} + \frac{dg}{dt} \qquad \& \qquad \frac{d}{dt}(cf) = c\frac{df}{dt} \qquad \& \qquad \frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt}$$

These are all straightforward consequences of the corresponding properties for functions on $\mathbb{R}$. Note that the constant $c$ can be complex in the property above.

**Example 3.2.6.** *Let $f(t) = \cos(t) + ie^t$. In this case $Re(f) = \cos(t)$ and $Im(f) = e^t$. Note,*

$$\frac{df}{dt} = \frac{d}{dt}\Big(\cos(t) + ie^t\Big) = -\sin(t) + ie^t.$$

The proposition below is not the most general that can be offered in this direction, but it serves our purposes. The proof is left to the reader.

**Proposition 3.2.7.**

Suppose $L = a_o D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$ is an $n$-th order linear differerential operator with real-valued coefficient functions. If $y_1$ and $y_2$ are real-valued functions such that $L[y_1] = f_1$ and $L[y_2] = f_2$ then $L[y_1 + iy_2] = f_1 + if_2$. Conversely, if $z$ is a complex-valued solution with $L[z] = w$ then $L[Re(z)] = Re(w)$ and $L[Im(z)] = Im(w)$.

Of course the proposition above is also interesting in the homogeneous case as it says a nontrivial complex solution of $L[y] = 0$ will reveal a pair of real solutions. Naturally there is much more to discuss about complex numbers, but we really just need the following two functions for this course[6].

[6] if you look at my calculus 131 notes you'll find an extended discussion of the complex exponential, I make no attempt to motivate this formula here.

**Definition 3.2.8.** *complex exponential function.*

We define $exp : \mathbb{C} \to \mathbb{C}$ by the following formula:

$$exp(z) = exp(Re(z) + iIm(z)) = e^{Re(z)}\big[\cos(Im(z)) + i\sin(Im(z))\big].$$

We can show $exp(z+w) = exp(z)exp(w)$. Suppose that $z = x+iy$ and $w = a+ib$ where $x, y, a, b \in \mathbb{R}$. Observe:

$$\begin{aligned}
exp(z+w) &= exp(x+iy+a+ib) \\
&= exp(x+a+i(y+b)) \\
&= e^{x+a}\big(\cos(y+b) + i\sin(y+b)\big) && \text{defn. of complex exp.} \\
&= e^{x+a}\big(\cos y\cos b - \sin y \sin b + i[\sin y\cos b + \sin b\cos y]\big) && \text{adding angles formulas} \\
&= e^{x+a}\big(\cos y + i\sin y\big)\big(\cos b + i\sin b\big) && \text{algebra} \\
&= e^{x}e^{a}\big(\cos y + i\sin y\big)\big(\cos b + i\sin b\big) && \text{law of exponents} \\
&= e^{x+iy}e^{a+ib} && \text{defn. of complex exp.} \\
&= exp(z)exp(w).
\end{aligned}$$

You can also show that $e^{0+i(0)} = 1$ and $e^{-z} = \frac{1}{e^z}$ and $e^z \neq 0$ in the complex case. There are many similarities to the real exponential function. But be warned there is much more to say. For example, $exp(z+2n\pi i) = exp(z)$ because the sine and cosine functions are $2\pi$-periodic. But, this means that the exponential is not 1-1 and consequently one cannot solve the equation $e^z = e^w$ uniquely. This introduces all sorts of ambiguities into the study of complex equations. Given $e^z = e^w$, you cannot conclude that $z = w$, however you can conclude that there exists $n \in \mathbb{Z}$ and $z = w + 2n\pi i$. In the complex variables course you'll discuss local inverses of the complex exponential function, instead of just one natural logarithm there are infinitely many to use.

Often, though, things work as we wish they ought:

**Proposition 3.2.9.** *Let $\lambda = \alpha + i\beta$ for real constants $\alpha, \beta$. We have:*

$$\frac{d}{dt}\big[e^{\lambda t}\big] = \lambda e^{\lambda t}.$$

**Proof:** direct calculation.

$$\begin{aligned}
\frac{d}{dt}\big[e^{\alpha t + i\beta t}\big] &= \frac{d}{dt}\big[e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\big] \\
&= \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) + i\alpha e^{\alpha t}\sin(\beta t) + i\beta e^{\alpha t}\cos(\beta t) \\
&= \alpha e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + i\beta e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \\
&= (\alpha + i\beta)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \\
&= \lambda e^{\lambda t}. \qquad \square
\end{aligned}$$

This is a beautiful result. Let's examine how it works in an example.

**Example 3.2.10.** *Let $f(t) = e^{(2+i)t}$. In this case $f(t) = e^{2t}(\cos(t) + i\sin(t))$ thus $Re(f) = e^{2t}\cos(t)$ and $Im(f) = e^{2t}\sin(t)$. Note,*

$$\frac{df}{dt} = \frac{d}{dt}\left(e^{(2+i)t}\right) = (2+i)e^{(2+i)t}.$$

*Expanding $(2+i)e^{2t}(\cos(t)+i\sin(t)) = 2e^t\cos(t) - e^t\sin(t) + i(2e^{2t}\sin(t) + e^{2t}\cos(t))$. Which is what we would naturally obtain via direct differentiation of $f(t) = e^{2t}\cos(t) + ie^{2t}\sin(t)$. Obviously the complex notation hides many details.*

The example below is very important to the next section.

**Example 3.2.11.** *Let $\lambda \in \mathbb{C}$ and calculate,*

$$\frac{d}{dt}\left[te^{\lambda t}\right] = e^{\lambda t} + \lambda te^{\lambda t}.$$

*Thus $D[te^{\lambda t}] = e^{\lambda t} + \lambda te^{\lambda t}$ hence we could write $(D - \lambda)[te^{\lambda t}] = e^{\lambda t}$. Likewise, consider*

$$\frac{d}{dt}\left[t^n e^{\lambda t}\right] = nt^{n-1}e^{\lambda t} + \lambda t^n e^{\lambda t}.$$

*Therefore, $(D - \lambda)[t^n e^{\lambda t}] = nt^{n-1}e^{\lambda t}$.*

One other complex calculation is of considerable importance to a wide swath of examples; the complex power function (here we insist the base is real... complex bases are the domain of the complex variables course, we can't deal with those here!). Forgive me for shifting notation to $x$ as the independent variable at this juncture. I have no particular reason except that power functions seem more natural with $x^n$ in our memory.

**Definition 3.2.12.** *complex power function with real base.*

> Let $a, b \in \mathbb{R}$, define $x^{a+ib} = x^a(\cos(b\ln(x)) + i\sin(b\ln(x)))$.

**Motivation:** $x^c = e^{log(x^c)} = e^{clog(x)} = e^{a\ln(x)+ib\ln(x)} = e^{a\ln(x)}e^{ib\ln(x)} = x^a e^{ib\ln(x)}$.

I invite the reader to check that the power-rule holds for complex exponents:

**Proposition 3.2.13.** *let $c \in \mathbb{C}$ then for $x > 0$,*

$$\frac{d}{dx}\left[x^c\right] = cx^{c-1}.$$

It seems likely this is a homework problem. I worked the analogus problem for the complex exponential earlier this section. We will need this result when we consider the Cauchy-Euler problem later in this course.

## 3.3 constant coefficient homogeneous problem

Let $L = P(D)$ for some polynomial with real coefficients $P(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$. By the fundamental theorem of algebra we can factor $P$ into $n$-linear factors. In particular, if $P(x) = 0$ has solutions $r_1, r_2, \dots, r_k$ then the factor theorem implies that there are real constants $m_1, m_2, \dots, m_k$ with $m_1 + m_2 + \cdots + m_k = n$ and

$$P(x) = a_o(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k}$$

I include the possibility that $r_j$ could be complex. $P$ is a polynomial with real coefficients, it follows that if $r_j$ is a complex zero then the complex conjugate $r_j^*$ also has $P(r_j^*) = 0$. By Proposition 3.2.4 the polynomial of the differentiation operator $P(D)$ shares the same factorization:

$$L = P(D) = a_o(D - r_1)^{m_1}(D - r_2)^{m_2} \cdots (D - r_k)^{m_k}$$

We wish to solve the differential equation $P(D)[y] = 0$. The following facts hold for both real and complex zeros. However, understand that when $r_j$ is complex the corresponding solutions are likewise complex:

1. if $(D - r_j)^{m_j}[y] = 0$ then $P(D)[y] = 0$.
2. if $D = d/dt$ then the DEqn $(D - r)[y] = 0$ has solution $y = e^{rt}$.
3. if $D = d/dt$ then the DEqn $(D - r)^2[y] = 0$ has two solutions $y = e^{rt}, te^{rt}$.
4. if $D = d/dt$ then the DEqn $(D - r)^3[y] = 0$ has three solutions $y = e^{rt}, te^{rt}, t^2e^{rt}$.
5. if $D = d/dt$ then the DEqn $(D - r)^m[y] = 0$ has $m$-solutions $y = e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$
6. $\{e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}\}$ is a LI set of functions (on $\mathbb{R}$ or $\mathbb{C}$).

Let us unravel the complex case into real notation. Suppose $r = \alpha + i\beta$ then $r^* = \alpha - i\beta$. Note:

$$e^{rt} = e^{\alpha t + i\beta t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$$

$$e^{r^*t} = e^{\alpha t - i\beta t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t)$$

Observe that the both complex functions give the same real solution set:

$$Re(e^{\alpha t \pm i\beta t}) = e^{\alpha t}\cos(\beta t) \qquad \& \qquad Im(e^{\alpha t \pm i\beta t}) = \pm e^{\alpha t}\sin(\beta t)$$

If $(D - r)^m[y] = 0$ has $m$-**complex** solutions $y = e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$ then $(D - r)^m[y] = 0$ possesses the $2m$-**real solutions**

$$e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t),\ te^{\alpha t}\cos(\beta t),\ te^{\alpha t}\sin(\beta t),\ \dots,\ t^{m-1}e^{\alpha t}\cos(\beta t),\ t^{m-1}e^{\alpha t}\sin(\beta t).$$

It should be clear how to assemble the general solution to the general constant coefficient problem $P(D)[y] = 0$. I will abstain from that notational quagmire and instead illustrate with a series of examples.

**Example 3.3.1. Problem:** *Solve* $y'' + 3y' + 2y = 0$.
**Solution:** *Note the differential equation is* $(D^2 + 3D + 2)[y]$. *Hence* $(D+1)(D+2)[y] = 0$. *We find solutions* $y_1 = e^{-x}$ *and* $y_2 = e^{-2x}$ *therefore the general solution is* $\boxed{y = c_1e^{-x} + c_2e^{-2x}}$.

**Example 3.3.2. Problem:** *Solve* $y'' - 3y' + 2y = 0$.
**Solution:** *Note the differential equation is* $(D^2 - 3D + 2)[y]$. *Hence* $(D-1)(D-2)[y] = 0$. *We find solutions* $y_1 = e^{x}$ *and* $y_2 = e^{2x}$ *therefore the general solution is* $\boxed{y = c_1e^{x} + c_2e^{2x}.}$

**Example 3.3.3.** **Problem:** *Solve* $y^{(4)} - 5y'' + 4y = 0$.
**Solution:** *Note the differential equation is* $(D^4 - 5D + 4)[y]$. *Note that*

$$D^4 - 5D + 4 = (D^2 - 1)(D^2 - 4) = (D + 1)(D - 1)(D + 2)(D - 2)$$

*It follows that the differential equation factors to* $(D+1)(D+2)(D-1)(D-2)[y] = 0$ *and the general solution reads*

$$\boxed{y = c_1e^{-x} + c_2e^{-2x} + c_3e^{x} + c_4e^{2x}.}$$

You should notice that I do **not** state that $D = \pm 1$ or $D = \pm 2$ in the example above. Those equations are illogical nonsense. I am using the theory we've developed in this chapter to extract solutions from inspection of the factored form. If you really want to think in terms of roots instead of factors then I would advise that you use the following fact:

$$\boxed{P(D)[e^{\lambda t}] = P(\lambda)e^{\lambda t}.}$$

I exploited this identity to solve the second order problem in our first lecture on the $n$-th order problem. Solutions to $P(\lambda) = 0$ are called the **characteristic values** of the DEqn $P(D)[y] = 0$. The equation $P(\lambda) = 0$ is called the **characteristic equation**.

**Example 3.3.4.** **Problem:** *Solve* $y^{(4)} - 5y'' + 4y = 0$.
**Solution:** *Let* $P(D) = D^4 - 5D + 4$ *thus the DEqn is* $P(D)[y] = 0$. *Note that* $P(\lambda) = \lambda^4 - 5\lambda + 4$.

$$\lambda^4 - 5\lambda + 4 = (\lambda^2 - 1)(\lambda^2 - 4) = (\lambda + 1)(\lambda - 1)(\lambda + 2)(\lambda - 2)$$

*Hence, the solutions of* $P(\lambda) = 0$ *are* $\lambda_1 = -1, \lambda_1 = -2, \lambda_3 = 1$ *and* $\lambda_4 = 2$ *the* **characteristic values** *of* $P(D)[y] = 0$. *The general solution follows:*

$$\boxed{y = c_1e^{-x} + c_2e^{-2x} + c_3e^{x} + c_4e^{2x}.}$$

We can also group the exponential functions via the hyperbolic sine and cosine. Since

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \qquad \& \qquad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

we have $e^x = \cosh(x) + \sinh(x)$ and $e^{-x} = \cosh(x) - \sinh(x)$. Thus,

$$c_1e^{-x} + c_3e^{x} = c_1(\cosh(x) - \sinh(x)) + c_3(\cosh(x) + \sinh(x)) = (c_1 + c_3)\cosh(x) + (c_1 - c_3)\sinh(x).$$

For a given problem we can either use exponentials or hyperbolic sine and cosine.

**Example 3.3.5.** **Problem:** *Solve* $y'' - y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 2$.
**Solution:** *we find* $\lambda^2 - 1 = 0$. *Hence* $\lambda = \pm 1$. *We find general solution* $y = c_1\cosh(x) + c_2\sinh(x)$ *in view of the comments just above this example (worth remembering for later btw). Observe:*

$$y' = c_1\sinh(x) + c_2\cosh(x)$$

*Consequently,* $y(0) = c_1 = 1$ *and* $y'(0) = c_2 = 2$ *and we find* $\boxed{y = \cosh(x) + 2\sinh(x).}$

Believe it or not, the hyperbolic sine and cosine are easier to work with when we encounter this type of ODE in our study of boundary value problems in partial differential equations towards the conclusion of this course.

**Example 3.3.6. Problem:** *Solve* $y^{(4)} + 2y'' + y = 0$.
**Solution:** *the characteristic equation is* $\lambda^4 + 2\lambda^2 + 1 = 0$. *Hence* $(\lambda^2 + 1)^2 = 0$. *It follows that we have* $\lambda = \pm i$ *repeated. The general solution is found from the real and imaginary parts of* $e^{it}$ *and* $te^{it}$. *Since* $e^{it} = \cos(t) + i\sin(t)$ *we find:*

$$\boxed{y = c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t).}$$

Up to this point I have given examples where we had to factor the operator (or characteristic eqn.) to extract the solution. Sometimes we find problems where the operators are already factored. I consider a few such problems now.

**Example 3.3.7. Problem:** *Solve* $(D^2 + 9)(D - 2)^3[y] = 0$ *with* $D = d/dx$ *for a change.*
**Solution:** *I read from the expression above that we have* $\lambda = \pm 3i$ *and* $\lambda = 2$ *thrice. Hence,*

$$\boxed{y = c_1\cos(3x) + c_2\sin(3x) + c_3e^{2x} + c_4xe^{2x} + c_5x^2e^{2x}.}$$

**Example 3.3.8. Problem:** *Solve* $(D^2 + 4D + 5)[y] = 0$ *with* $D = d/dx$.
**Solution:** *Complete the square to see that* $P(D)$ *is not reducible;* $D^2 + 4D + 5 = (D + 2)^2 + 1$ *it follows that the characteristic values are* $\lambda = -2 \pm i$ *and the general solution is given from the real and imaginary parts of* $e^{-2x+ix} = e^{-2x}e^{ix}$

$$\boxed{y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x).}$$

**Example 3.3.9. Problem:** *Solve* $(D^2 + 4D - 5)[y] = 0$ *with* $D = d/dx$.
**Solution:** *Complete the square;* $D^2 + 4D - 5 = (D + 2)^2 - 9$ *it follows that the characteristic values are* $\lambda = -2 \pm 3$ *or* $\lambda_1 = 1$ *or* $\lambda_2 = -5$

$$\boxed{y = c_1e^{x} + c_2e^{-5x}.}$$

*Of course, if you love hyperbolic sine and cosine then perhaps you would prefer that we see from* $((D+2)^2 - 9)[y] = 0$ *the solutions*

$$y = b_1e^{-2x}\cosh(3x) + b_2e^{-2x}\sinh(3x)$$

*as the natural expression of the general solution. In invite the reader to verify the solution above is just another way to write the solution* $y = c_1e^{x} + c_2e^{-5x}$.

**Example 3.3.10. Problem:** *Solve* $(D^2 + 6D + 15)(D^2 + 1)(D^2 - 4)[y] = 0$ *with* $D = d/dx$.
**Solution:** *Completing the square gives* $((D + 3)^2 + 6)(D^2 + 1)(D^2 - 4)[y] = 0$ *hence we find characteristic values of* $\lambda = -3 \pm i\sqrt{6}, \pm i, \pm 2$. *The general solution follows:*

$$\boxed{y = c_1e^{-3x}\cos(\sqrt{6}x) + c_2e^{-3x}\sin(\sqrt{6}x) + c_3\cos(x) + c_4\sin(x) + c_5e^{2x} + c_6e^{-2x}.}$$

The example that follows is a bit more challenging since it involves both theory and a command of polynomial algebra.

**Example 3.3.11. Problem:** *Solve* $(D^5 - 8D^2 - 4D^3 + 32)[y] = 0$ *given that* $y = \cosh(2t)$ *is a solution.*
**Solution:** *Straightforward factoring of the polynomial is challenging here, but I gave an olive branch. Note that if* $y = \cosh(2t)$ *is a solution then* $y = \sinh(2t)$ *is also a solution. It follows that* $(D^2 - 4) = (D - 2)(D + 2)$

*is a factor of* $D^5 - 8D^2 - 4D^3 + 32$*. For clarity of thought lets work on* $x^5 - 8x^2 - 4x^3 + 32$ *and try to factor out* $x^2 - 4$*. Long division is a nice tool for this problem. Recall:*

$$\begin{array}{rl} & \quad\; x^3 \qquad\qquad - 8 \\ \cline{2-2} x^2-4 \big) & x^5 - 4x^3 - 8x^2 + 32 \\ & -x^5 + 4x^3 \\ \cline{2-2} & \qquad\qquad -8x^2 + 32 \\ & \qquad\qquad\;\; 8x^2 - 32 \\ \cline{2-2} & \qquad\qquad\qquad\quad 0 \end{array}$$

*Thus,*

$$x^5 - 8x^2 - 4x^3 + 32 = (x^2 - 4)(x^3 - 8)$$

*Clearly* $x^3 - 8 = 0$ *has solution* $x = 2$ *hence we can factor* $(x - 2)$*. I'll use long-division once more (of course, some of you might prefer synthetic division and/or have this memorized already... good)*

$$\begin{array}{rl} & \qquad\quad x^2 + 2x + 4 \\ \cline{2-2} x-2 \big) & x^3 \qquad\qquad\quad - 8 \\ & -x^3 + 2x^2 \\ \cline{2-2} & \qquad 2x^2 \\ & \quad -2x^2 + 4x \\ \cline{2-2} & \qquad\qquad 4x - 8 \\ & \qquad\quad -4x + 8 \\ \cline{2-2} & \qquad\qquad\qquad 0 \end{array}$$

*Consequently,* $x^5 - 8x^2 - 4x^3 + 32 = (x^2 - 4)(x - 2)(x^2 + 2x + 4)$*. It follows that*

$$(D^5 - 8D^2 - 4D^3 + 32)[y] = 0 \Rightarrow (D - 2)^2)(D + 2)((D + 1)^2 + 3)[y] = 0$$

*Which suggests the solution below:*

$$\boxed{y = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-2t} + c_4 e^{-t}\cos(\sqrt{3}t) + c_5 e^{-t}\sin(\sqrt{3}t).}$$

## 3.4 annihilator method for nonhomogeneous problems

In the previous section we learned how to solve **any** constant coefficient $n$-th order ODE. We now seek to extend the technique to the nonhomogeneous problem. Our goal is to solve:

$$L[y] = f$$

where $L = P(D)$ as in the previous section, it is a polynomial in the differentiation operator $D$. Suppose we find a differential operator $A$ such that $A[f] = 0$. This is called an **annihilator** for $f$. Operate on $L[y] = f$ to obtain $AL[y] = A[f] = 0$. Therefore, if we have an annihilator for the forcing function $f$ then the differential equation yields a corresponding homogeneous differential equation $AL[y] = 0$. Suppose $y = y_h + y_p$ is the general solution as discussed for Theorem 3.1.20 we have $L[y_h] = 0$ and $L[y_p] = f$. Observe:

$$AL[y_h + y_p] = A[L[y_h + y_p]] = A[f] = 0$$

It follows that the general solution to $AL[y] = 0$ will include the general solution of $L[y] = f$. The method we justify and implement in this section is commonly called the **method of undetermined coefficients**. The annihilator method shows us how to set-up the coefficients. To begin, we should work on finding annihilators to a few simple functions.

**Example 3.4.1.** **Problem:** *find an annihilator for $e^x$.*
**Solution:** *recall that $e^x$ arises as the solution of $(D-1)[y] = 0$ therefore a natural choice for the annihilator is $A = D - 1$. This choice is* **minimal**. *Observe that $A_2 = Q(D)(D-1)$ is also an annihilator of $e^x$ since $A_2[e^x] = Q(D)[(D-1)[e^x]] = Q(D)[0] = 0$. There are many choices, however, we prefer the minimal annihilator. It will go without saying that all the choices that follow from here on out are minimal.*

**Example 3.4.2.** **Problem:** *find an annihilator for $xe^{3x}$.*
**Solution:** *recall that $xe^{3x}$ arises as a solution of $(D-3)^2[y] = 0$ hence choose $A = (D-3)^2$.*

**Example 3.4.3.** **Problem:** *find an annihilator for $e^{3x}\cos(x)$.*
**Solution:** *recall that $e^{3x}\cos(x)$ arises as a solution of $((D-3)^2+1)[y] = 0$ hence choose $A = ((D-3)^2+1)$.*

**Example 3.4.4.** **Problem:** *find an annihilator for $x^2e^{3x}\cos(x)$.*
**Solution:** *recall that $x^2e^{3x}\cos(x)$ arises as a solution of $((D-3)^2+1)^3[y] = 0$ hence choose $A = ((D-3)^2+1)^3$.*

**Example 3.4.5.** **Problem:** *find an annihilator for $2e^x\cosh(2x)$.*
**Solution:** *observe that $2e^x\cosh(2x) = e^x(e^{2x} + e^{-2x}) = e^{3x} + e^{-x}$ and note that $(D-3)[e^{3x}] = 0$ and $(D+1)[e^{-x}] = 0$ thus $A = (D-3)(D+1)$ will do nicely.*

For those who love symmetric calculational schemes, you could also view $2e^x\cosh(2x)$ as the solution arising from $((D-1)^2-4)[y] - 0$. Naturally $(D-1)^2 - 4 = D^2 - 2D - 3 = (D-3)(D+1)$.

**Example 3.4.6.** **Problem:** *find an annihilator for* $x^2 + e^{3x}\cos(x)$.
**Solution:** *recall that* $e^{3x}\cos(x)$ *arises as a solution of* $((D-3)^2+1)[y] = 0$ *hence choose* $A_1 = ((D-3)^2+1)$. *Next notice that* $x^2$ *arises as a solution of* $D^3[y] = 0$ *hence we choose* $A_2 = D^3$. *Construct* $A = A_1A_2$ *and notice how this works: (use* $A_1A_2 = A_2A_1$ *which is true since these are constant coefficient operators)*

$$\begin{aligned} A_1[A_2[x^2 + e^{3x}\cos(x)]] &= A_1[A_2[x^2] + A_2[A_1[e^{3x}\cos(x)]] \\ &= A_1[0] + A_2[0] \\ &= 0 \end{aligned}$$

*because* $A_1A_2 = A_2A_1$ *for these constant coefficient operators. To summarize, we find* $A = D^3((D-3)^2+1)$ *is an annihilator for* $x^2 + e^{3x}\cos(x)$.

I hope you see the idea generally. If we are given a function which arises as the solution of a constant coefficient differential equation then we can use the equation to write the annihilator. You might wonder if there are other ways to find annihilators.... well, surely there are, but not usually for this course. I think the examples thus far give us a good grasp of how to kill the forcing function. Let's complete the method. We proceed by example.

**Example 3.4.7.** **Problem:** *find the general solution of* $y'' + y = 2e^x$
**Solution:** *observe* $L = D^2 + 1$ *and we face* $(D^2+1)[y] = e^x$. *Let* $A = D - 1$ *and operate on the given nonhomogeneous ODE,*

$$(D-1)(D^2+1)[y] = (D-1)[e^x] = 0$$

*We find general solution* $y = c_1e^x + c_2\cos(x) + c_3\sin(x)$. *Notice this is* **not** *the finished product. We should only have two constants in the general solution of this second order problem. But, remember, we insist that* $L[y] = f$ *in addition to the condition* $AL[y] = 0$ *hence:*

$$L[c_1e^x + c_2\cos(x) + c_3\sin(x)] = 2e^x$$

*which simplifes to* $L[c_1e^x] = 2e^x$ *since the functions* $\cos(x), \sin(x)$ *are solutions of* $L[y] = 0$. *Expanding* $L[c_1e^x] = 2e^x$ *in detail gives us:*

$$D^2[c_1e^x] + c_1e^x = 2e^x \quad \Rightarrow \quad 2c_1e^x = 2e^x \quad \Rightarrow \quad c_1 = 1.$$

*Therefore we find,* $\boxed{y = e^x + c_2\cos(x) + c_3\sin(x).}$

The notation used in the example above is not optimal for calculation. Usually I skip some of those steps because they're not needed once we understand the method. For example, once I write $y = c_1e^x + c_2\cos(x) + c_3\sin(x)$ then I usually look to see which functions are in the fundamental solution set. Since $\{\cos(x), \sin(x)\}$ is a natural fundamental solution set this tells me that only the remaining function $e^x$ is needed to construct the particular solution. Since $c_1$ is annoying to do algebra on, I instead use notation $y_p = Ae^x$. Next, calculate $y_p' = Ae^x$ and $y_p'' = Ae^x$ and plug these into the given ODE:

$$Ae^x + Ae^x = 2e^x \quad \Rightarrow \quad 2Ae^x = 2e^x \quad \Rightarrow \quad A = 1.$$

which brings us to the fact that $y_p = e^x$ and naturally $y_h = c_1\cos(x) + c_2\sin(x)$. The general solution is $y = y_h + y_p = c_1\cos(x) + c_2\sin(x) + e^x$.

**Example 3.4.8.** **Problem:** *find the general solution of* $y'' + 3y' + 2y = x^2 - 1$
**Solution:** *in operator notation the DEqn is* $(D^2 + 3D + 2)[y] = (D+1)(D+2)[y] = 0$. *Let* $A = D^3$ *and operate on the given nonhomogeneous ODE,*

$$D^3(D+1)(D+2)[y] = D^3[x^2 - 1] = 0$$

*The homogeneous ODE above has solutions* $1, x, x^2, e^{-x}, e^{-2x}$. *Clearly the last two of these form the homogeneous solution whereas the particular solution is of the form* $y_p = Ax^2 + Bx + C$. *Calculate:*

$$y_p' = 2Ax + B, \qquad y_p'' = 2A$$

*Plug this into the DEqn* $y_p'' + 3y_p' + 2y_p = x^2 - 1$,

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 - 1$$

*multiply it out and collect terms:*

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = x^2 - 1 \quad \Rightarrow \quad 2Ax^2 + (6A + 2B)x + 2A + 3B + 2C = x^2 - 1$$

*this sort of equation is actually really easy to solve. We have two polynomials. When are they equal? Simple. When the coefficients match, thus calculate:*

$$2A = 1, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = -1$$

*Clearly* $A = 1/2$ *hence* $B = -3A = -3/2$. *Solve for* $C = -1/2 - A - 3B/2 = -1/2 - 1/2 + 9/4 = 5/4$. *Therefore, the general solution is given by:*

$$\boxed{y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}.}$$

At this point you might start to get the wrong impression. It might appear to you that the form of $y_p$ has nothing to do with the form of $y_h$. That is a fortunate feature of the examples we have thus far considered. The next example features what I usually call **overlap**.

**Example 3.4.9. Problem:** *find the general solution of* $y' - y = e^x + x$
**Solution:** *Observe the annihilator is* $A = (D-1)D^2$ *and when we operate on* $(D-1)[y] = e^x + x$ *we obtain*

$$(D-1)D^2(D-1)[y] = (D-1)D^2[e^x + x] = 0 \quad \Rightarrow \quad (D-1)^2D^2[y] = 0$$

*Thus,* $e^x, xe^x, 1, x$ *are solutions. We find* $y_h = c_1e^x$ *whereas the particular solution is of the form* $y_p = Axe^x + Bx + C$. *Calculate* $y_p' = A(e^x + xe^x) + B$ *and substitute into the DEqn to obtain:*

$$A(e^x + xe^x) + B - (Axe^x + Bx + C) = e^x + x \quad \Rightarrow \quad (A - A)xe^x + Ae^x - Bx - C = e^x + x$$

*We find from* **equating coefficients** *of the linearly independent functions* $e^x, 1, x$ *that* $A = 1$ *and* $-B = 1$ *and* $-C = 0$. *Therefore,* $\boxed{y = c_1e^x + xe^x - x.}$

If you look in my linear algebra notes I give a proof which shows we can equate coefficients for linearly independent sets. Usually in the calculation of $y_p$ we find it useful to use the technique of equating coefficients to fix the **undetermined constants** $A, B, C$, etc...

**Remark 3.4.10.**

When this material is covered in a calculus II course the operator method is not typically discussed. Instead, when you try to solve $ay'' + by' + cy = f$ you simply look at $f$ and its derivatives $f, f', f'', \ldots$ then identify all the basic functions and form $y_p$ as a linear combination. For example, to solve $y'' + y' = cos(x)$ you would note $f = cos(x)$, $f' = -\sin(x)$ and $f'', f'''$ etc... yield the same functions once again, the naive guess for the particular solution is simply $y_p = A\cos(x) + B\sin(x)$. Consider $y'' + y' = x + e^{2x}$, this time $f = x + e^{2x}, f' = 1 + 2e^{2x}, f'' = 4e^{2x}$ so we would naively try $y_p = Ae^{2x} + Bx + C$. However, this construction of $y_p$ is only partially successful. If we use the operator method of this section we would instead be led to use $y_p = Ae^{2x} + Bx^2 + Cx$. The inclusion of these extra $x$ factors is sold to the calculus II students in terms of "overlap". This sort of idea is given in Chapter 5 of your text where you are given a general formula to anticipate the formula for $y_p$. In contrast, the annihilator method we use requires only that you know the homogeneous case in depth. The inclusion of the extra $x$ factors are derived not memorized.

**Example 3.4.11. Problem:** *find the general solution of* $y'' + y = 4\cos(t)$
**Solution:** *Observe the annihilator is* $A = D^2 + 1$ *and when we operate on* $(D^2 + 1)[y] = \cos(t)$ *we obtain*

$$(D^2 + 1)^2[y] = 0$$

*Thus,* $\cos(t), \sin(t), t\cos(t), t\sin(t)$ *are solutions. We find* $y_h = c_1\cos(t) + c_2\sin(t)$ *whereas the particular solution is of the form* $y_p = At\cos(t) + Bt\sin(t) = t(A\cos(t) + B\sin(t))$*. Calculate*

$$y_p' = A\cos(t) + B\sin(t) + t(-A\sin(t) + B\cos(t)) = (A + Bt)\cos(t) + (B - At)\sin(t)$$

$$y_p'' = B\cos(t) - A\sin(t) - (A + Bt)\sin(t) + (B - At)\cos(t) = (2B - At)\cos(t) - (2A + Bt)\sin(t)$$

*It is nice to notice that* $y_p'' = 2B\cos(t) - 2A\sin(t) - y_p$ *hence* $y_p'' + y_p = 4\cos(t)$ *yields:*

$$2B\cos(t) - 2A\sin(t) = 4\cos(t)$$

*thus* $2B = 4,\ -2A = 0$*. Consequently,* $A = 0, B = 2$ *and the general solution is found to be:*

$$\boxed{y = c_1\cos(t) + c_2\sin(t) + 2t\sin(t).}$$

From this point forward I omit the details of the annihilator method and simply propose the correct template for $y_p$.

**Example 3.4.12. Problem:** *find the general solution of* $y' + y = x$
**Solution:** *Observe* $y_h = c_1e^{-x}$ *for the given DEqn. Let* $y_p = Ax + B$ *then* $y_p' + y_p = A + Ax + B = x$ *implies* $A + B = 0$ *and* $A = 1$ *hence* $B = -1$ *and we find* $\boxed{y = c_1e^{-x} + x - 1}$.

**Example 3.4.13. Problem:** *find the general solution of* $y'' + 4y' = x$
**Solution:** *Observe* $\lambda^2 + 4\lambda = 0$ *gives solutions* $\lambda = 0, -4$ *hence* $y_h = c_1 + c_2e^{-4x}$ *for the given DEqn. Let*[7] $y_p = Ax^2 + Bx$ *then* $y_p' = 2Ax + B$ *and* $y_p'' = 2A$ *hence* $y_p'' + 4y_p' = x$ *yields* $2A + 4(2Ax + B) = x$ *hence* $8Ax + 2A + 4B = x$*. Equate coefficients of* $x$ *and* $1$ *to find* $8A = 1$ *and* $2A + 4B = 0$ *hence* $A = 1/8$ *and* $B = -1/16$*. We find*

$$\boxed{y = c_1 + c_2e^{-4x} + \frac{1}{8}x^2 - \frac{1}{16}x}.$$

[7] I know this by experience, but you can derive this by the annihilator method, of course the merit is made manifest in the successful selection of $A, B$ below to actually solve $y_p'' + 4y_p' = x$.

**Example 3.4.14.** **Problem:** *find the general solution of* $y'' + 4y' = \cos(x) + 3\sin(x) + 1$
**Solution:** *Observe* $\lambda^2 + 4\lambda = 0$ *gives solutions* $\lambda = 0, -4$ *hence* $y_h = c_1 + c_2 e^{-4x}$ *for the given DEqn. Let* $y_p = A\cos(x) + B\sin(x) + Cx$ *then* $y_p' = -A\sin(x) + B\cos(x) + C$ *and* $y_p'' == -A\cos(x) - B\sin(x)$ *hence* $y_p'' + 4y_p' = \cos(x) + 3\sin(x) + 1$ *yields*

$$-A\cos(x) - B\sin(x) + 4(-A\sin(x) + B\cos(x) + C) = \cos(x) + 3\sin(x) + 1$$

*Collecting like terms:*

$$\Rightarrow \quad (4B - A)\cos(x) + (-4A - B)\sin(x) + 4C = \cos(x) + 3\sin(x) + 1$$

*Equate coefficients of* $\cos(x), \sin(x), 1$ *to obtain:*

$$4B - A = 1, \quad -4A - B = 3, \quad 4C = 1$$

*Observe* $B = -4A - 3$ *hence* $4(-4A - 3) - A = 1$ *or* $-17A - 12 = 1$ *thus* $A = -13/17$. *Consequently,* $B = 52/17 - 3 = (52 - 51)/17 = 1/17$. *Obviously* $C = 1/4$ *thus we find*

$$\boxed{y = c_1 + c_2 e^{-4x} - \frac{13}{17}\cos(x) + \frac{1}{17}\sin(x) + \frac{1}{4}x}.$$

We have enough examples to appreciate the theorem given below:

**Theorem 3.4.15.** *superposition principle for linear differential equations.*

Suppose $L[y] = 0$ is an $n$-th order linear differential equation with continuous coefficient functions on an interval $I$ with fundamental solution set $S = \{y_1, y_2, \dots, y_n\}$ on $I$. Furthermore, suppose $L[y_{p_j}] = f_j$ for functions $f_j$ on $I$ then for any choice of constants $b_1, b_2, \dots, b_k$ the function $y = \sum_{j=1}^{k} b_j y_{p_j}$ forms the particular solution of $L[y] = \sum_{j=1}^{k} b_j f_j$ on the interval $I$.

**Proof:** we just use $k$-fold additivity and then homogeneity of $L$ to show:

$$L\left[\sum_{j=1}^{k} b_j y_{p_j}\right] = \sum_{j=1}^{k} L[b_j y_{p_j}] = \sum_{j=1}^{k} b_j L[y_{p_j}] = \sum_{j=1}^{k} b_j f_j. \qquad \square$$

The Superposition Theorem paired with Theorem 3.1.20 allow us to find general solutions for complicated problems by breaking down the problem into pieces. In the example that follows we already dealt with the pieces in previous examples.

**Example 3.4.16.** **Problem:** *find the general solution of* $y'' + 4y' = 17(\cos(x) + 3\sin(x) + 1) + 16x = f$ *(introduced* $f$ *for convenience here)*
**Solution:** *observe that* $L = D^2 + 4D$ *for both Example 3.4.13 and Example 3.4.14. We derived that the particular solutions* $y_{p_1} = \frac{1}{8}x^2 - \frac{1}{16}x$ *and* $y_{p_2} = -\frac{13}{17}\cos(x) + \frac{1}{17}\sin(x) + \frac{1}{4}x$ *satisfy*

$$L[y_{p_1}] = f_1 = x \qquad \& \qquad L[y_{p_2}] = f_2 = \cos(x) + 3\sin(x) + 1$$

*Note that* $f = 17f_2 + 16f_1$ *thus* $L[y] = f$ *has particular solution* $y = 17y_{p_2} + 16y_{p_1}$ *by the superposition principle. Therefore, the general solution is given by:*

$$y = c_1 + c_2 e^{-4x} - 13\cos(x) + \sin(x) + \frac{17}{4}x + 2x^2 - x.$$

*Or, collecting the x-terms together,*

$$y = c_1 + c_2 e^{-4x} - 13\cos(x) + \sin(x) + \frac{13}{4}x + 2x^2.$$

**Example 3.4.17. Problem:** *find the general solution of* $y'' + 5y' + 6y = 2\sinh(t)$
**Solution:** *It is easy to see that* $y'' + 5y' + 6y = e^t$ *has* $y_{p_1} = \frac{1}{12}e^t$. *On the other hand, it is easy to see that* $y'' + 5y' + 6y = e^{-t}$ *has solution* $y_{p_2} = \frac{1}{2}e^{-t}$. *The definition of hyperbolic sine gives* $2\sinh(t) = e^t - e^{-t}$ *hence, by the principle of superposition we find particular solution of* $y'' + 5y' + 6y = 2\sinh(t)$ *is simply* $y_p = 2y_{p_1} - 2y_{p_2}$. *Note* $\lambda^2 + 5\lambda + 6 = 0$ *factors to* $(\lambda + 2)(\lambda + 3) = 0$ *hence* $y_h = c_1 e^{-2t} + c_2 e^{-3t}$. *Therefore, the general solution of* $y'' + 5y' + 6y = 2\sinh(t)$ *is*

$$\boxed{y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{12}e^t - \frac{1}{2}e^{-t}}.$$

Naturally, you can solve the example above directly. I was merely illustrating the superposition principle.

**Example 3.4.18. Problem:** *find the general solution of* $y'' + 5y' + 6y = 2\sinh(t)$
**Solution:** *a natural choice for the particular solution is* $y_p = A\cosh(t) + B\sinh(t)$ *hence*

$$y_p' = A\sinh(t) + B\cosh(t), \quad , y_p'' = A\cosh(t) + B\sinh(t) = y_p$$

*Thus* $y_p'' + 5y_p' + 6y_p = 5y_p' + 7y_p = 2\sinh(t)$ *and we find*

$$(5A + 7B)\cosh(t) + (5B + 7A)\sinh(t) = 2\sinh(t)$$

*Thus* $5A + 7B = 0$ *and* $5B + 7A = 2$. *Algebra yields* $A = 7/12$ *and* $B = -5/12$. *Therefore, as the characteristic values are* $\lambda = -2, -3$ *the general solution is given as follows:*

$$\boxed{y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{7}{12}\cosh(t) - \frac{5}{12}\sinh(t)}.$$

I invite the reader to verify the answers in the previous pair of examples are in fact equivalent.

## 3.5 variation of parameters

The method of annihilators is deeply satisfying, but sadly most function escape its reach. For example, if the forcing function was $\sec(x)$ or $\tan(x)$ or $\ln(x)$ then we would be unable to annihilate these functions with some polynomial in $D$. Moreover, if the DEqn $L[y] = f$ has nonconstant coefficients then the problem of factoring $L$ into linear factors $L_1, L_2, \dots, L_n$ is notoriously difficult[8]. If we had a factorization and a way to annihilate the forcing function we might be able to extend the method of the last section, but, this is not a particularly easy path to implement in any generality. In contrast, the technique of variation of parameters is both general and amazingly simple.

We begin by assuming the existence of a fundamental solution set for $L[y] = f$; assume $\{y_1, y_2, \dots, y_n\}$ is a linearly independent set of solutions for $L[y] = 0$. We **propose** the particular solution $y_p$ can be written as a linear combination of the fundmental solutions with coefficients of functions $v_1, v_2, \dots, v_n$ (these are the "parameters")

$$y_p = v_1y_1 + v_2y_2 + \cdots + v_ny_n$$

Differentiate,

$$y_p' = v_1'y_1 + v_2'y_2 + \cdots + v_n'y_n + v_1y_1' + v_2y_2' + \cdots + v_ny_n'$$

**Let constraint 1** state that $v_1'y_1 + v_2'y_2 + \cdots + v_n'y_n = 0$ and differentiate $y_p'$ in view of this added constraint, once more we apply the product-rule $n$-fold times:

$$y_p'' = v_1'y_1' + v_2'y_2' + \cdots + v_n'y_n' + v_1y_1'' + v_2y_2'' + \cdots + v_ny_n''$$

**Let constraint 2** state that $v_1'y_1' + v_2'y_2' + \cdots + v_n'y_n' = 0$ and differentiate $y_p''$ in view of contraints 1 and 2,

$$y_p''' = v_1'y_1'' + v_2'y_2'' + \cdots + v_n'y_n'' + v_1y_1''' + v_2y_2''' + \cdots + v_ny_n'''$$

**Let constraint 3** state that $v_1'y_1'' + v_2'y_2'' + \cdots + v_n'y_n'' = 0$. We continue in this fashion adding constraints after each differentiation of the form $v_1'{y_1}^{(j)} + v_2'{y_2}^{(j)} + \cdots + v_n'{y_n}^{(j)} = 0$ for $j = 3, 4, \dots, n-2$. Note this brings us to

$$y_p^{(n-1)} = v_1y_1^{(n-1)} + v_2y_2^{(n-1)} + \cdots + v_ny_n^{(n-1)}.$$

Thus far we have given $(n-1)$-constraints on $[v_1', v_2', \dots, v_n']$. We need one more constraint to fix the solution. Remember we need $L[y_p] = f$; $a_oy_p^{(n)} + a_1y_p^{(n-1)} + \cdots + a_{n-1}y_p' + a_ny_p = f$ thus:

$$y_p^{(n)} = \frac{f}{a_o} - \frac{a_1}{a_o}y_p^{(n-1)} - \cdots - \frac{a_{n-1}}{a_o}y_p' - \frac{a_n}{a_o}y_p. \qquad (\star)$$

Differentiating $y_p = v_1y_1 + v_2y_2 + \cdots + v_ny_n$ and apply the previous contraints to obtain:

$$y_p^{(n)} = v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \cdots + v_n'y_n^{(n-1)} + v_1y_1^{(n)} + v_2y_2^{(n)} + \cdots + v_ny_n^{(n)}. \qquad (\star^2)$$

[8] we'll tackle the problem for the Cauchy Euler problem later this chapter, see Rabenstein for some more exotic examples of factorization of operators

Equate $\star$ and $\star^2$ to obtain:

$$\begin{aligned}
\frac{f}{a_o} &= \frac{a_1}{a_o}y_p^{(n-1)} + \cdots + \frac{a_{n-1}}{a_o}y_p' + \frac{a_n}{a_o}y_p + \\
&\quad v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \cdots + v_n'y_n^{(n-1)} + v_1y_1^{(n)} + v_2y_2^{(n)} + \cdots + v_ny_n^{(n)} \\
&= v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \cdots + v_n'y_n^{(n-1)} \\
&\quad + \frac{a_o}{a_o}\left(v_1y_1^{(n)} + v_2y_2^{(n)} + \cdots + v_ny_n^{(n)}\right) + \\
&\quad + \frac{a_1}{a_o}\left(v_1y_1^{(n-1)} + v_2y_2^{(n-1)} + \cdots + v_ny_n^{(n-1)}\right) + \\
&\quad + \cdots + \\
&\quad + \frac{a_{n-1}}{a_o}\left(v_1y_1' + v_2y_2' + \cdots + v_ny_n'\right) + \\
&\quad + \frac{a_n}{a_o}\left(v_1y_1 + v_2y_2 + \cdots + v_ny_n\right) \\
&= v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \cdots + v_n'y_n^{(n-1)} \\
&\quad + \frac{v_1}{a_o}\left(a_oy_1^{(n)} + a_1y_1^{(n-1)} + \cdots + a_{n-1}y_1' + a_ny_1\right) + \\
&\quad + \frac{v_2}{a_o}\left(a_oy_2^{(n)} + a_1y_2^{(n-1)} + \cdots + a_{n-1}y_2' + a_ny_2\right) + \\
&\quad + \cdots + \\
&\quad + \frac{v_n}{a_o}\left(a_oy_n^{(n)} + a_1y_n^{(n-1)} + \cdots + a_{n-1}y_n' + a_ny_n\right) \\
&= v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \cdots + v_n'y_n^{(n-1)}.
\end{aligned}$$

In the step before the last we used the fact that $L[y_j] = 0$ for each $y_j$ in the given fundamental solution set. With this calculation we obtain our $n$-th condition on the derivatives of the parameters. In total, we seek to impose

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f/a_o \end{bmatrix}. \qquad (\star^3)$$

Observe that the coefficient matrix of the system above is the **Wronskian Matrix**. Since we assumed $\{y_1, y_2, \dots, y_n\}$ is a fundamental solution set we know that the Wronskian is nonzero which means the equation above has a unique solution. Therefore, the constraints we proposed are consistent and attainable for **any** $n$-th order linear ODE.

Let us pause to learn a little matrix theory convenient to our current endeavors. Nonsingular system of linear equations by Cramer's rule. To solve $A\vec{v} = \vec{b}$ you can follow the procedure below: to solve for $v_k$ of $\vec{v} = (v_1, v_2, \dots, v_k, \dots, v_n)$ we

1. take the matrix $A$ and replace the $k$-th column with the vector $\vec{b}$ call this matrix $S_k$
2. calculate $det(S_k)$ and $det(A)$
3. the solution is simply $v_k = \frac{det(S_k)}{det(A)}$.

Cramer's rule is a horrible method for specific numerical systems of linear equations[9]. But, it has for us the advantage of giving a nice, neat formula for the matrices of functions we consider here.

**Example 3.5.1.** *Suppose you want to solve $x+y+z=6$, $x+z=4$ and $y-z=-1$ simultaneously. Note in matrix notation we have:*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}.$$

*We can swap out columns $1, 2$ and $3$ to obtain $S_1, S_2$ and $S_3$*

$$S_1 = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 4 & 1 \\ 0 & -1 & -1 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 1 & 1 & 6 \\ 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

*You can calculate $det(S_1) = 1$, $det(S_1) = 2$ and $det(S_3) = 3$. Likewise $det(A) = 1$. Cramer's Rule states the solution is $x = \frac{det(S_1)}{det(A)} = 1$, $y = \frac{det(S_2)}{det(A)} = 2$ and $z = \frac{det(S_3)}{det(A)} = 3$.*

In the notation introduced above we see $\star^3$ has

$$A = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \qquad \& \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f/a_o \end{bmatrix}.$$

Once more define $S_k$ as the matrix obtained by swapping the $k$-th column of $A$ for the column vector $\vec{b}$ and let $W$ be the Wronskian which is $det(A)$ in our current notation. We obtain the following solutions for $v_1', v_2', \dots, v_n'$ by Cramer's Rule:

$$v_1' = \frac{det(S_1)}{W}, \; v_2' = \frac{det(S_2)}{W}, \dots, \; v_n' = \frac{det(S_n)}{W}$$

Finally, we can integrate to find the formulas for the parameters. Taking $x$ as the independent parameter we note $v_k' = \frac{dv_k}{dx}$ hence:

$$v_1 = \int \frac{det(S_1)}{W}\,dx, \quad v_2 = \int \frac{det(S_2)}{W}\,dx, \; \dots, \; v_n = \int \frac{det(S_n)}{W}\,dx.$$

The matrix $S_k$ has a rather special form and we can simplify the determinants above in terms of the so-called **sub-Wronskian** determinants. Define $W_k = W(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n; x)$ then it follows by Laplace's Expansion by minors formula that $det(S_k) = (-1)^{n+k}\frac{f}{a_o}W_k$. Thus,

$$\boxed{v_1 = \int (-1)^{n+1}\frac{fW_1}{a_oW}\,dx, \quad v_2 = \int (-1)^{n+2}\frac{fW_2}{a_oW}\,dx, \; \dots, \; v_n = \int \frac{fW_n}{a_oW}\,dx}.$$

[9]Gaussian elimination is faster and more general, see my linear algebra notes or any text on the subject!

Of course, you don't have to think about subWronskians, we could just use the formula in terms of $det(S_k)$. Include the subWronskain comment in part to connect with formulas given in Nagel Saff and Snider (Ritger & Rose does not have detailed plug-and-chug formulas on this problem, see page 154). In any event, we should now enjoy the spoils of this conquest. Let us examine how to calculate $y_p = v_1 y_1 + \cdots + v_n y_n$ for particular $n$.

1. **(n=1)** $a_o \frac{dy}{dx} + a_1 y = f$ has $W(y_1; x) = y_1$ and $W_1 = 1$. It follows that the solution $y = y_1 v_1$ has $v_1 = \int \frac{f}{a_o y_1}\, dx$ where $y_1$ is the solution of $a_o \frac{dy}{dx} + a_1 y = 0$ which is given by $y_1 = exp(\int \frac{-a_1}{a_o}\, dx)$. In other words, variation of parameters reduces to the integrating factor method[10] for $n = 1$.

2. **(n=2)** Suppose $a_o y'' + a_1 y' + a_2 y = f$ has fundamental solution set $\{y_1, y_2\}$ then

$$W = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$$

furthermore, calculate:

$$det(S_1) = det \begin{bmatrix} 0 & y_2 \\ f/a_o & y_2' \end{bmatrix} = -\frac{f y_2}{a_o} \qquad \& \qquad det(S_2) = det \begin{bmatrix} y_1 & 0 \\ y_1' & f/a_o \end{bmatrix} = \frac{f y_1}{a_o}$$

Therefore,

$$\boxed{v_1 = \int \frac{-f y_2}{a_o(y_1 y_2' - y_2 y_1')}\, dx \qquad \& \qquad v_2 = \int \frac{f y_1}{a_o(y_1 y_2' - y_2 y_1')}\, dx}$$

give the particular solution $y_p = v_1 y_1 + v_2 y_2$. Note that if the integrals above are indefinite then the general solution is given by:

$$\boxed{y = y_1 \int \frac{-f y_2}{a_o}\, dx + y_2 \int \frac{f y_1}{a_o}\, dx}.$$

Formulas for $n = 3, 4$ are tedious to derive and I leave them to the reader in the general case. Most applications involve $n = 2$.

[10]note $\frac{dy}{dx} + \frac{a_1}{a_o} y = 0$ implies $I = exp(\int \frac{a_1}{a_o}\, dx)$ hence $\frac{d}{dx}(Iy) = 0$ and so $y = C/I$ and taking $C = 1$ derives $y_1$.

**Example 3.5.2.** *Solve $y'' + y = \sec(x)$. The characteristic equation $\lambda^2 + 1 = 0$ yields $\lambda = \pm i$ hence $y_1 = \cos(x), y_2 = \sin(x)$. Observe the Wronskian simplifies nicely in this case: $W = y_1y_2' - y_2y_1' = \cos^2(x) + \sin^2(x) = 1$. Hence,*

$$v_1 = \int \frac{-fy_2}{W}\,dx = \int -\sec(x)\sin(x)\,dx = -\int \frac{\sin(x)}{\cos(x)}\,dx = -\ln|\cos(x)| + c_1 = \ln|\sec(x)| + c_1.$$

*and,*

$$v_2 = \int \frac{fy_1}{W}\,dx = \int \sec(x)\cos(x)\,dx = \int dx = x + c_2.$$

*we find the general solution $y = y_1v_1 + y_2v_2$ is simply:*

$$\boxed{y = c_1\cos(x) + c_2\sin(x) + \cos(x)\ln|\sec(x)| + x\sin(x)}.$$

Sometimes variation of parameters does not include the $c_1, c_2$ in the formulas for $v_1$ and $v_2$. In that case the particular solution truly is $y_p = y_1v_1 + y_2v_2$ and the general solution is found by $y = y_h + y_p$ where $y_h = c_1y_1 + c_2y_2$. Whatever system of notation you choose, please understand that in the end there must be a term $c_1y_1 + c_2y_2$ in the general solution.

**Example 3.5.3.** *Solve $y'' - 2y' + y = f$. The characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$ yields $\lambda_1 = \lambda_2 = 1$ hence $y_1 = e^x, y_2 = xe^x$. Observe the Wronskian simplifies nicely in this case: $W = y_1y_2' - y_2y_1' = e^x(e^x + xe^x) - e^x xe^x = e^{2x}$. Hence,*

$$v_1 = \int \frac{-f(x)xe^x}{e^{2x}}\,dx \qquad \& \qquad v_2 = \int \frac{f(x)e^x}{e^{2x}}\,dx.$$

*we find the general solution $y = y_1v_1 + y_2v_2$ is simply:*

$$\boxed{y = c_1e^x + c_2xe^x - e^x\int \frac{f(x)xe^x}{e^{2x}}\,dx + xe^x\int \frac{f(x)e^x}{e^{2x}}\,dx}.$$

*In particular, if $f(x) = e^x\sin(x)$ then*

$$v_1 = \int -x\sin(x)\,dx = x\cos(x) - \sin(x) \qquad \& \qquad v_2 = \int \sin(x)\,dx = -\cos(x).$$

*Hence, $y_p = (x\cos(x) - \sin(x))e^x + xe^x(-\cos(x)) = -e^x\sin(x)$. The general solution is*

$$\boxed{y = c_1e^x + c_2xe^x - e^x\sin(x)}.$$

Notice that we could also solve $y'' - 2y' + y = e^x\sin(x)$ via the method of undetermined coefficients. In fact, any problem we can solve by undetermined coefficients we can also solve by variation of parameters. However, given the choice, it is usually easier to use undetermined coefficients.

**Example 3.5.4.** *Solve $y''' + y' = x\ln(x)$. The characteristic equation has $\lambda^3 + \lambda = \lambda(\lambda^2+1) = 0$ hence $\lambda = 0$ and $\lambda = \pm i$. The fundmental solutions are $y_1 = 1$, $y_2 = \cos(x)$, $y_3 = \sin(x)$. Calculate,*

$$W(1, \cos(x), \sin(x); x) = det \begin{bmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{bmatrix} = 1(\sin^2(x) + \cos^2(x)) = 1.$$

*Swapping the first column of the Wronskian matrix with $(0, 0, x\ln(x))$ gives us $S_1$ and we find*

$$det(S_1) = det \begin{bmatrix} 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ x\ln(x) & -\cos(x) & -\sin(x) \end{bmatrix} = x\ln(x).$$

*Swapping the second column of the Wronskian matrix with $(0, 0, x\ln(x))$ gives us $S_2$ and we find*

$$det(S_2) = det \begin{bmatrix} 1 & 0 & \sin(x) \\ 0 & 0 & \cos(x) \\ 0 & x\ln(x) & -\sin(x) \end{bmatrix} = -x\ln(x)\cos(x).$$

*Swapping the third column of the Wronskian matrix with $(0, 0, x\ln(x))$ gives us $S_3$ and we find*

$$det(S_3) = \det \begin{bmatrix} 1 & \cos(x) & 0 \\ 0 & -\sin(x) & 0 \\ 0 & -\cos(x) & x\ln(x) \end{bmatrix} = -x\ln(x)\sin(x).$$

*Note, integration by parts yields*[11] $v_1 = \int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$. *The integrals of $v_2 = \int -x\ln(x)\cos(x)\,dx$ and $v_3 = -x\ln(x)\sin(x)\,dx$ are not elementary. However, we can express the general solution as:*

$$\boxed{y = c_1 + c_2\cos(x) + c_3\sin(x) + \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 - \cos(x)\int x\ln(x)\cos(x)\,dx - \sin(x)\int x\ln(x)\sin(x)\,dx}.$$

If you use Mathematica directly, or Wolfram Alpha or other such software then some of the integrals will be given in terms of unusual functions such as *hypergeometric functions* or *polylogarithms* or *the cosine integral function* or the *exponential integral function* or the *sine integral function*, or *Bessel functions* and so forth... the list of nonstandard, but known, functions is very lenghthy at this point. What this means is that when you find an integral you cannot perform as part of an answer it may well be that the values of that integral are known, tabulated and often even automated as a built-in command. Moreover, if you randomly try other nonhomogeneous ODEs then you'll often find solutions appear in this larger class of named functions. More generally, the solutions appear as series of orthogonal functions. But, I suppose I'm getting a little ahead of the story here. In the next section we explore substitutions of a particular sort for the $n$-th order problem.

[11] I'm just calculating an antiderivative here since the homogeneous solution will account for the neccessary constants in the general solution

## 3.6 reduction of order

We return to the question of the homogeneous linear ODE $L[y] = 0$. Suppose we are **given** a solution $y_1$ with

$$a_o y_1^{(n)} + a_1 y_1^{(n-1)} + \cdots + a_{n-1}y_1' + a_n y_1 = 0$$

on an interval $I$. To find a second solution we **propose** there exists $v$ such that $y_2 = vy_1$ is a solution of $L[y] = 0$. I invite the reader to verify the following:

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

$$y_2''' = v'''y_1 + 3v''y_1' + 3v'y_1'' + vy_1'''$$

and, by an inductive argument, we arrive at

$$y_2^{(n)} = v^{(n)}y_1 + nv^{(n-1)}y_1' + \cdots + nv'y_1^{(n-1)} + vy_1^{(n)}$$

where the other coefficients are the binomial coefficients. I suppose it's worth mentioning the formula below is known as *Leibniz' product formula:*

$$\boxed{\frac{d^n}{dx^b}\Big[F(x)\,G(x)\Big] = \sum_{k=0}^{n}\binom{n}{k}F^{(n-k)}(x)G^{(k)}(x)}$$

Returning to the substitution $y_2 = vy_1$ we find that the condition $L[y_2] = 0$ gives

$$a_o(vy_1)^{(n)} + \cdots + a_{n-1}(vy_1)' + a_n vy_1 = 0$$

Thus,

$$a_o\big[v^{(n)}y_1 + nv^{(n-1)}y_1' + \cdots + nv'y_1^{(n-1)} + vy_1^{(n)}\big] + \cdots + +a_{n-1}\big[v'y_1 + vy_1'\big] + a_n vy_1 = 0$$

Notice how all the terms with $v$ collect together to give $v\big[y_1^{(n)} + \cdots + a_{n-1}y_1' + a_n y_1\big]$ which vanishes since $y_1$ is a solution. Therefore, the equation $L[y_2] = 0$ reduces to:

$$a_o\big[v^{(n)}y_1 + nv^{(n-1)}y_1' + \cdots + nv'y_1^{(n-1)}\big] + \cdots + +a_{n-1}v'y_1 = 0$$

If we substitute $z = v'$ then the equation is clearly an $(n-1)$-th order linear ODE for $z$;

$$a_o\big[z^{(n-1)}y_1 + nz^{(n-2)}y_1' + \cdots + nzy_1^{(n-1)}\big] + \cdots + +a_{n-1}zy_1 = 0.$$

I include this derivation to show you that the method extends to the $n$-th order problem. However, we are primarily interested in the $n = 2$ case. In that particular case we can derive a nice formula for $y_2$.

Let $a, b, c$ be functions and suppose $ay'' + by' + cy = 0$ has solution $y_1$. Let $y_2 = vy_1$ and seek a formula for $v$ for which $y_2$ is a solution of the $ay'' + by' + cy = 0$. Substitute $y_2 = vy_1$ and differentiate the product,

$$a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + cvy_1 = 0$$

Apply $ay_1'' + by_1' + cy_1 = 0$ to obtain:

$$a(v''y_1 + 2v'y_1') + bv'y_1 = 0$$

Now let $z = v'$ thus $z' = v''$

$$ay_1z' + 2ay_1'z + by_1z = 0 \quad \Rightarrow \quad \frac{dz}{dx} + \left[\frac{2ay_1' + by_1}{ay_1}\right]z = 0.$$

Apply the integrating factor method with $I = exp(\int \frac{2ay_1'+by_1}{ay_1}dx)$ we find

$$\frac{d}{dx}\left[Iz\right] = 0 \quad \Rightarrow \quad Iz = C \quad \Rightarrow \quad z = \frac{C}{I} = Cexp\Big(-\int \frac{2ay_1' + by_1}{ay_1}dx\Big)$$

Recall $z = \frac{dv}{dx}$ thus we integrate to find $v = \int Cexp\Big(-\int \frac{2ay_1'+by_1}{ay_1}dx\Big)dx$ thus

$$y_2 = y_1 \int Cexp\Big(-\int \frac{2ay_1' + by_1}{ay_1}dx\Big)dx$$

It is convenient to take $C = 1$ since we are just seeking a particular function to construct the solution set. Moreover, notice that the integral $\int \frac{-2}{y_1}\frac{dy_1}{dx}\,dx = -2\ln|y_1| = \ln(1/y_1^2)$ thus it follows

$$\boxed{y_2 = y_1 \int \frac{1}{y_1^2} exp\Big(-\int \frac{b}{a}\,dx\Big)\,dx}$$

**Example 3.6.1.** *Consider $y'' - 2y' + y = 0$. We found $y_1 = e^x$ by making a simple guess of $y = e^{\lambda x}$ and working out the algebra. Let us now find how to derive $y_2$ in view of the derivation preceding this example. Identify $a = 1, b = -2, c = 1$. Suppose $y_2 = vy_1$. We found that*

$$y_2 = y_1 \int \frac{1}{y_1^2} exp\Big(-\int \frac{b}{a}\,dx\Big)\,dx = e^x \int \frac{1}{e^{2x}} exp\Big(\int 2\,dx\Big)\,dx = e^x \int \frac{e^{2x}}{e^{2x}}\,dx = e^x \int dx$$

*Thus $y_2 = xe^x$.*

This example should suffice for the moment. We will use this formula in a couple other places. Notice if we have some method to find at least one solution for $ay'' + by' + cy = 0$ then this formula allows us to find a second, linearly independent[12] solution.

[12] no, I have not proved this, perhaps you could try

## 3.7 operator factorizations

In this section we consider a method to solve $L[y] = f$ given that $L = L_1 L_2 \cdots L_n$ and $L_j$ are all first order differential operators. Without loss of generality this means $L_j = a_j D + b_j$ for $j = 1, 2, \dots, n$. We do not suppose these operators commute. Let $z_1 = (L_2 L_3 \cdots L_n)[y]$ and note that in $z_1$ the $n$-th order ODE for $y$ simplifies to

$$L_1[z_1] = f \quad \Rightarrow \quad \frac{dz_1}{dx} + \frac{b_1}{a_1} z_1 = f \quad \Rightarrow \quad exp\left[\int \frac{b_1}{a_1} dx\right] \frac{dz_1}{dx} + exp\left[\int \frac{b_1}{a_1} dx\right] \frac{b_1}{a_1} z_1 = \frac{f}{a_1} exp\left[\int \frac{b_1}{a_1} dx\right]$$

Consequently,

$$\frac{d}{dx}\left[z_1 \, exp\left[\int \frac{b_1}{a_1} dx\right]\right] = \frac{f}{a_1} exp\left[\int \frac{b_1}{a_1} dx\right]$$

integrating and solving for $z_1$ yields:

$$z_1 = exp\left[-\int \frac{b_1}{a_1} dx\right]\left[c_1 + \int \frac{f}{a_1} exp\left[\int \frac{b_1}{a_1} dx\right] dx\right]$$

Next let $z_2 = (L_3 \cdots L_n)[y]$ observe that $L_1(L_2[z_2]) = f$ implies $z_1 = L_2[z_2]$ hence we should solve

$$a_2 \frac{dz}{dx} + b_2 z_2 = z_1$$

By the calculation for $z_1$ we find, letting $z_1$ play the role $f$ did in the previous calculation,

$$z_2 = exp\left[-\int \frac{b_2}{a_2} dx\right]\left[c_2 + \int \frac{z_1}{a_2} exp\left[\int \frac{b_2}{a_2} dx\right] dx\right]$$

Well, I guess you see where this is going, let $z_3 = (L_4 \cdots L_n)[y]$ and observe $(L_1 L_2)[L_3[z_3]] = f$ hence $L_3[z_3] = z_2$. We must solve $a_3 z_3' + b_3 z_3 = z_2$ hence

$$z_3 = exp\left[-\int \frac{b_3}{a_3} dx\right]\left[c_3 + \int \frac{z_2}{a_3} exp\left[\int \frac{b_3}{a_3} dx\right] dx\right].$$

Eventually we reach $y = z_n$ where $(L_1 L_2 \cdots L_n)[z_n] = f$ and $a_n z_n' + b_n z_n = z_{n-1}$ will yield

$$y = exp\left[-\int \frac{b_n}{a_n} dx\right]\left[c_n + \int \frac{z_{n-1}}{a_n} exp\left[\int \frac{b_n}{a_n} dx\right] dx\right].$$

If we expand $z_{n-1}, z_{n-2}, \dots, z_2, z_1$ we find the formula for the general solution of $L[y] = f$.

The trouble with this method is that its starting point is a factored differential operator. Many problems do not enjoy this structure from the outset. We have to do some nontrivial work to massage an arbitrary problem into this factored form. Rabenstein[13] claims that it is always possible to write $L$ in factored form, but even in the $n = 2$ case the problem of factoring $L$ is as difficult, if not more, then solving the differential equation!

[13] page 70, ok technically he only claims $n = 2$, I haven't found a general reference at this time

**Example 3.7.1.** *Let $L_1 = x\frac{d}{dx}$ and suppose $L_2 = 1 + \frac{d}{dx}$. Solve $(L_1L_2)[y] = 3$. We want to solve*

$$x\frac{d}{dx}\left[y + \frac{dy}{dx}\right] = 3$$

*Let $z = y + \frac{dy}{dx}$ and consider*

$$x\frac{dz}{dx} = 3 \quad \Rightarrow \quad \int dz = \int \frac{3dx}{x} \quad \Rightarrow \quad z = 3\ln|x| + c_1.$$

*Hence solve,*

$$y + \frac{dy}{dx} = 3\ln|x| + c_1$$

*Multiply by integrating factor $e^x$ and after a short calculation we find*

$$y = e^{-x}\int \left[3\ln|x|e^x + c_1e^x\right] dx$$

*Therefore,*

$$y = c_2e^{-x} + c_1 + e^{-x}\int \left[3\ln|x|e^x\right] dx$$

*Identify the fundamental solution set of $y_1 = e^{-x}$ and $y_2 = 1$. Note that $L_2[e^{-x}] = 0$ and $L_1[1] = 0$.*

Curious, we just saw a non-constant coefficient differential equation which has the same fundamental solution set as $y'' + y' = 0$. I'm curious how the solution will differ if we reverse the order of $L_1$ and $L_2$

**Example 3.7.2.** *Let $L_1, L_2$ be as before and solve $(L_2L_1)[y] = 3$. We want to solve*

$$\left[1 + \frac{d}{dx}\right]\left[x\frac{dy}{dx}\right] = 3$$

*Let $z = x\frac{dy}{dx}$ and seek to solve $z + \frac{dz}{dx} = 3$. This is a constant coefficient ODE with $\lambda = -1$ and it is easy to see that $z = 3 + c_1e^{-x}$. Thus consider, $x\frac{dy}{dx} = 3 + c_1e^{-x}$ yields $dy = \left(\frac{3}{x} + c_1\frac{e^{-x}}{x}\right) dx$ and integration yields:*

$$y = c_2 + c_1\int \frac{e^{-x}}{x}\, dx + 3\ln|x|.$$

*The fundamental solution set has $y_1 = 1$ and $y_2 = \int \frac{e^{-x}}{x}\, dx$.*

You can calculate that $L_1L_2 \neq L_2L_1$. This is part of what makes the last pair of examples interesting. On the other hand, perhaps you can start to appreciate the constant coefficient problem. In the next section we consider the next best thing; the *equidimensional* or *Cauchy Euler* problem. It turns out we can factor the differential operator for a Cauchy -Euler problem into commuting differential operators. This makes the structure of the solution set easy to catalogue.

## 3.8 cauchy euler problems

The general Cauchy-Euler problem is specified by $n$-constants $a_1, a_2, \dots, a_n$. If $L$ is given by

$$L = x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n$$

then $L[y] = 0$ is a Cauchy-Euler problem. Suppose the solution is of the form $y = x^R$ for some constant $R$. Note that

$$xD[x^R] = xRx^{R-1} = Rx^R$$

$$x^2 D^2[x^R] = x^2 R(R-1)x^{R-2} = R(R-1)x^R$$

$$x^3 D^3[x^R] = x^3 R(R-1)(R-2)x^{R-3} = R(R-1)(R-2)x^R$$

$$x^n D^n[x^R] = x^n R(R-1)(R-2)\cdots(R-n)x^{R-n} = R(R-1)(R-2)\cdots(R-n)x^R$$

Substitute into $L[y] = 0$ and obtain:

$$\Big(R(R-1)(R-2)\cdots(R-n) + a_1 R(R-1)(R-2)\cdots(R-n+1) + \cdots + a_{n-1}R + a_n\Big)x^R = 0$$

It follows that $R$ must satisfy the **characteristic equation**

$$\boxed{R(R-1)(R-2)\cdots(R-n) + a_1 R(R-1)(R-2)\cdots(R-n+1) + \cdots + a_{n-1}R + a_n = 0.}$$

Notice that it is not simply obtained by placing powers of $R$ next to the coefficients $a_1, a_2, \dots, a_n$. However, we do obtain an $n$-th order polynomial equation for $R$ and it follows that we generally have $n$-solutions, some repeated, some complex. Rather than attempting to say anything further on the general problem I now pause to consider three interesting second order problems.

**Example 3.8.1.** *Solve $x^2y'' + xy' + y = 0$. Let $y = x^R$ then we must have*

$$R(R-1) + R + 1 = 0 \quad \Rightarrow \quad R^2 + 1 = 0 \quad \Rightarrow \quad R = \pm i$$

*Hence $y = x^i$ is a complex solution. We defined, for $x > 0$, the complex power function $x^c = e^{c\ln(x)}$ hence*

$$x^i = e^{i\ln(x)} = \cos(\ln(x)) + i\sin(\ln(x))$$

*The real and imaginary parts of $x^i$ give real solutions for $x^2y'' + xy' + y = 0$. We find*

$$\boxed{y = c_1\cos(\ln(x)) + c_2\sin(\ln(x))}$$

**Example 3.8.2.** *Solve $x^2y'' + 4xy' + 2y = 0$. Let $y = x^R$ then we must have*

$$R(R-1) + 4R + 2 = 0 \quad \Rightarrow \quad R^2 + 3R + 2 = (R+1)(R+2) \quad \Rightarrow \quad R = -1, -2.$$

*We find fundamental solutions $y_1 = 1/x$ and $y_2 = 1/x^2$ hence the general solution is*

$$\boxed{y = c_1\frac{1}{x} + c_2\frac{1}{x^2}}$$

**Example 3.8.3.** *Solve $x^2y'' - 3xy' + 4y = 0$ for $x > 0$. Let $y = x^R$ then we must have*

$$R(R-1) - 3R + 4 = 0 \quad \Rightarrow \quad R^2 - 4R + 4 = (R-2)^2 \quad \Rightarrow \quad R = 2, 2.$$

*We find fundamental solution $y_1 = x^2$. To find $y_2$ we must use another method. We derived that the second solution of $ay'' + by' + cy = 0$ can be found from the first via:*

$$y_2 = y_1 \int \frac{1}{y_1^2} exp\biggl(\int \frac{-b}{a} dx\biggr) dx.$$

*In this problem identify that $a = x^2$ and $b = -3x$ whereas $y_1 = x^2$ and $y_1' = 2x$ thus:*

$$y_2 = x^2 \int \frac{1}{x^4} exp\biggl(\int \frac{3x}{x^2} dx\biggr) dx = x^2 \int \frac{1}{x^4} exp\Bigl(3\ln(x)\Bigr) dx = x^2 \int \frac{dx}{x} = x^2 \ln(x).$$

*The general solution is* $\boxed{y = c_1x^2 + c_2x^2\ln(x)}$.

The first order case is also interesting:

**Example 3.8.4.** *Solve $x\frac{dy}{dx} - ay = 0$. Let $y = x^R$ and find $R - a = 0$ hence* $\boxed{y = c_1x^a}$. *The operator $xD - a$ has characteristic equation $R - a = 0$ hence the characteristic value is $R = a$.*

Let us take two first order problems and construct a second order problem. Notice the operator in the last example is given by $xD - a$. We compose two such operators to construct,

$$(xD - a)(xD - b)[y] = 0$$

We can calculate,

$$(xD - a)[xy' - by] = xD[xy' - by] - axy' + aby = xy' + x^2y'' - bxy' - axy' + aby$$

In operator notation we find

$$(xD - a)(xD - b) = x^2D^2 + (1 - a - b)xD + ab$$

from which it is clear that $(xD - a)(xD - b) = (xD - b)(xD - a)$. Moreover,

$$(xD - a)(xD - b)[y] = 0 \qquad \Leftrightarrow \qquad (x^2D^2 + (1 - a - b)xD + ab)[y] = 0$$

**Example 3.8.5.** *To construct an cauchy-euler equation with characteristic values of $a = 2+3i$ and $b = 2-3i$ we simply note that $1 - a - b = -3$ and $ab = 4 + 9 = 13$. We can check that the cauchy-euler problem $x^2y'' - 3xy' + 13y = 0$ has complex solutions $y = x^{2\pm3i}$, suppose $y = x^R$ then it follows that $R$ must solve the characteristic equation:*

$$R(R-1) - 3R + 13 = R^2 - 4R + 13 = (R-2)^3 + 9 = 0 \quad \Rightarrow \quad R = 2 \pm 3i.$$

*Note $x^{2+3i} = e^{(2+3i)\ln(x)} = e^{ln(x^2)}e^{3i\ln(x)} = x^2(\cos(3\ln(x)) + i\sin(3\ln(x)))$ (you can just memorize it as I defined it, but these steps help me remember how this works) Thus, the DEqn $x^2y'' - 3xy' + 13y = 0$ has general solution*

$$\boxed{y = c_1x^2\cos(3\ln(x)) + c_2x^2\sin(3\ln(x))}$$

We saw that noncommuting operators are tricky to work with in a previous section. Define $[L_1, L_2] = L_1L_2 - L_2L_1$ and note that $L_2L_1 = L_1L_2 - [L_1, L_2]$. The $[L_1, L_2]$ is called the **commutator**, when it is zero then the inputs to $[,]$ are said to commute. If you think about the homogeneous problem $(L_1L_2)[y] = 0$ then contrast with $(L_2L_1)[y] = 0$ we can understand why these are not the same in terms of the commutator. For example, suppose $L_2[y] = 0$ then it is clearly a solution of $(L_1L_2)[y] = 0$ since $L_1[L_2[y]] = L_1[0] = 0$. On the other hand,

$$(L_2L_1)[y] = (L_1L_2 - [L_1, L_2])[y] = L_1[L_2[y]] - [L_1, L_2][y] = -[L_1, L_2][y]$$

and there is no reason in general for the solution to vanish on the commutator above. If we could factor a given differential operator into **commuting** operators $L_1, L_2, \dots, L_n$ then the problem $L[y] = 0$ nicely splits into $n$-separate problems $L_1[y] = 0, L_2[y] = 0, \dots, L_n[y] = 0$.

With these comments in mind return to the question of solving $L[y] = 0$ for

$$L = x^nD^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n$$

note in the case $n = 2$ we can solve $R(R-1) + a_R + a_o = 0$ for solutions $a, b$ and it follows that

$$x^2D^2 + a_1xD + a_2 = (xD - a)(xD - b)[y] = 0$$

The algebra to state $a, b$ as functions of $a_1, a_2$ is a quadratic equation. Notice that for the third order operator it starts to get ugly, the fourth unpleasant, and the fifth, impossible in closed form for an arbitrary equidimensional quintic operator.

All of this said, I think it is at least possible to *explicitly*[14] factor the operator whenever we can factor the characteristic equation. Suppose $R_1$ is a solution to the characteristic equation hence $y_1 = x^{R_1}$ is a solution of $L[y] = 0$. I claim you can argue that $L_1 = (xD - a_1)$ is a factor of $L$. Likewise, for the other zeros $a_2, a_3, \dots, a_n$ the linear differential operators $L_j = (xD - a_j)$ must somehow appear as a factor of $L$. Hence we have $n$-first order differential operators and since I wrote $L = x^nD^n + \cdots + a_n$ it follows that $L = L_1L_2 \cdots L_n$. From a DEqns perspective this discussion is not terribly useful as the process of factoring $L$ into a polynomial in $xD$ is not so intutive. Even the $n = 2$ case is tricky: $(xD - a)(xD - b)[f] = (x^2D^2 + (1 - a - b)xD + ab)[f] =$

$$= (xD - a)(xD - b)[f] = xD(xD[f] - b[f]) - axD[f] + ab[f] = (xD)^2 - (a + b)(xD) + ab)[f]$$

Notice the polynomials in $xD$ behave nicely but the $x^2D^2$ term does not translate simply into the $xD$ formulas. Let's see if we can derive some general formula to transform $x^nD^n$ into some polynomial in $xD$.

[14]theoretically it is always possible by the fundamental theorem of algebra applied to the characteristic equation and the scheme I'm about to outline

Calculate, for $f$ a suitably differentiable function,

$$(xD)^2[f] = xD[xD[f]] = xD[xf'] = xf' + x^2f'' = (xD + x^2D^2)[f] \quad \Rightarrow \quad \boxed{x^2D^2 = (xD)^2 - xD}$$

Next, order three, using Leibniz' product rule for second derivative of a product,

$$\begin{aligned}(xD)^3[f] = (xD + x^2D^2)[xf'] &= xf' + x^2f'' + x^2(x''f' + 2x'f'' + xf''') \\ &= (xD + x^2D^2 + 2x^2D^2 + x^3D^3)[f] \\ &= (xD + 3x^2D^2 + x^3D^3)[f] \\ &= (xD + 3(xD)^2 - 3xD + x^3D^3)[f] \\ &= (3(xD)^2 - 2xD + x^3D^3)[f] \\ &\Rightarrow \boxed{x^3D^3 = (xD)^3 - 3(xD)^2 + 2xD}.\end{aligned}$$

It should be fairly clear how to continue this to higher orders. Let's see how this might be useful[15] in the context of a particular third order cauchy-euler problem.

**Example 3.8.6.** *Solve $(x^3D^3 + 3x^2D^2 + 2xD)[y] = 0$. I'll use operator massage. By the calculations preceding this example:*

$$x^3D^3 + 3x^2D^2 + 2xD = (xD)^3 - 3(xD)^2 + 2xD + 3(xD)^2 - 3xD + 2xD = (xD)^3 + (xD)$$

*Now I can do algebra since $xD$ commutes with itself,*

$$(xD)^3 + (xD) = xD((xD)^2 + 1) = xD(xD - i)(xD + i)$$

*Hence $R = 0, R = \pm i$ are evidentally the characteristic values and we find real solution*

$$\boxed{y = c_1 + c_2\cos(\ln(x)) + c_3\sin(\ln(x))}$$

*Let's check this operator-based calculation against our characteristic equation method:*

$$R(R-1)(R-2) + 3R(R-1) + 2R = R^3 - 3R^2 + 2R + 3R^2 - 3R + 2R = R^3 + R.$$

*Which would then lead us to the same solution as we uncovered from the $xD$ factorization.*

There are a few loose ends here, I might ask a homework question to explore some ideas here further.

[15] my point in these calculations is not to find an optimal method to solve the cauchy euler problem, probably the characteristic equation is best, my point here is to explore the structure of operators and test our ability to differentiate and think!

## 3.9 applications

We explore two interesting applications in this section:

1. springs with friction
2. RLC circuits

We begin by studying the homogeneous case and then add external forces (1.) or a voltage source (2.). The mathematics is nearly the same for both applications. Finally we study resonance.

### 3.9.1 springs with and without damping

Suppose a mass $m$ undergoes one-dimensional motion under the influence of a spring force $F_s = -kx$ and a velocity dependent friction force $F_f = -\beta\dot{x}$. Newton's Second Law states $m\ddot{x} = -kx - \beta\dot{x}$. We find

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = 0}$$

The constants $m, \beta, k$ are non-negative and we assume $m \neq 0$ in all cases. Technically the value of $m$ should be assigned $kg$, that of $\beta$ should be assigned $kg/s$ and the spring constant $k$ should have a value with units of the form $N/m$. Please understand these are omitted in this section. When faced with a particular problem make sure you use quantities which have compatible units.

**Example 3.9.1. Problem: the over-damped spring:** *Suppose $m = 1, \beta = 3$ and $k = 2$. If the mass has velocity $v = -2$ and position $x = 1$ when $t = 0$ then what is the resulting equaiton of motion?*

**Solution:** *We are faced with $\ddot{x} + 3\dot{x} + 2x = 0$. This gives chararcteristic equation $\lambda^2 + 3\lambda + 2 = 0$ hence $(\lambda+1)(\lambda+2) = 0$ thus $\lambda_1 = -1$ and $\lambda_2 = -2$ and the general solution is*

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}$$

*Note that $\dot{x}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$. Apply the given initial conditions,*

$$x(0) = c_1 + c_2 = 1 \qquad \& \qquad \dot{x}(0) = -c_1 - 2c_2 = -2$$

*You can solve these equations to obtain $c_2 = 1$ and $c_1 = 0$. Therefore, $\boxed{x(t) = e^{-2t}}$.*

**Example 3.9.2. Problem: the critically-damped spring:** *Suppose $m = 1, \beta = 4$ and $k = 4$. If the mass has velocity $v = 1$ and position $x = 3$ when $t = 0$ then what is the resulting equaiton of motion?*

**Solution:** *We are faced with $\ddot{x} + 4\dot{x} + 4x = 0$. This gives chararcteristic equation $\lambda^2 + 4\lambda + 4 = 0$ hence $(\lambda+2)^2 = 0$ thus $\lambda_1 = \lambda_2 = -2$ and the general solution is*

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

*Note that $\dot{x}(t) = -2c_1 e^{-t} + c_2(e^{-2t} - 2te^{-2t})$. Apply the given initial conditions,*

$$x(0) = c_1 = 3 \qquad \& \qquad \dot{x}(0) = -2c_1 + c_2 = 1$$

*You can solve these equations to obtain $c_1 = 3$ and $c_2 = 7$. Therefore, $\boxed{x(t) = 3e^{-2t} + 7te^{-2t}}$.*

**Example 3.9.3. Problem: the under-damped spring:** *Suppose $m = 1, \beta = 2$ and $k = 6$. If the mass has velocity $v = 1$ and position $x = 1$ when $t = 0$ then what is the resulting equaiton of motion?*

**Solution:** *We are faced with $\ddot{x} + 2\dot{x} + 6x = 0$. This gives chararcteristic equation $\lambda^2 + 2\lambda + 6 = 0$ hence $(\lambda + 1)^2 + 5 = 0$ thus $\lambda = -1 \pm i\sqrt{5}$ and the general solution is*

$$x(t) = c_1 e^{-t} \cos(\sqrt{5}\, t) + c_2 e^{-t} \sin(\sqrt{5}\, t)$$

*Note that $\dot{x}(t) = c_1 e^{-t}(-\cos(\sqrt{5}\, t) - \sqrt{5} \sin(\sqrt{5}\, t) + c_2 e^{-t}(-\sin(\sqrt{5}\, t) + \sqrt{5} \cos(\sqrt{5}\, t)$. Apply the given initial conditions,*

$$x(0) = c_1 = 1 \qquad \& \qquad \dot{x}(0) = -c_1 + \sqrt{5} c_2 = 1$$

*You can solve these equations to obtain $c_1 = 1$ and $c_2 = 2/\sqrt{5}$. Therefore,*

$$\boxed{x(t) = e^{-t} \cos(\sqrt{5}\, t) + \frac{2}{\sqrt{5}} e^{-t} \sin(\sqrt{5}\, t)}.$$

**Example 3.9.4. Problem: spring without damping; simple harmonic oscillator:** *Suppose $\beta = 0$ and $m, k$ are nonzero. If the mass has velocity $v(0) = v_o$ and position $x(0) = x_o$ then find the resulting equaiton of motion.*

**Solution:** *We are faced with $m\ddot{x} + kx = 0$. This gives chararcteristic equation $m\lambda^2 + k = 0$ hence $\lambda = \pm i\sqrt{\frac{k}{m}}$ and the general solution is, using $\omega = \frac{k}{m}$,*

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega\, t)$$

*Note that*

$$\dot{x}(t) = -c_1 \omega \sin(\omega\, t) + c_2 \omega \cos(\omega\, t).$$

*Apply the given initial conditions,*

$$x(0) = c_1 = x_o \qquad \& \qquad \dot{x}(0) = c_2 \omega = v_o$$

*Therefore,*

$$\boxed{x(t) = x_o \cos(\omega t) + \frac{v_o}{\omega} \sin(\omega\, t)}.$$

### 3.9.2 the RLC-circuit

Now we turn to circuits. Suppose a resistor $R$, an inductor $L$ and a capacitor $C$ are placed in series then the we know that $V_R = IR$ by Ohm's Law for the resistor, whereas the voltage dropped on an inductor is proportional to the change in the current according to the definition of inductance paired with Faraday's Law: $V_L = L\frac{dI}{dt}$ for the inductor, the capacitor $C$ has charge $\pm Q$ on its plates when $V_C = Q/C$. We also know $I = \frac{dQ}{dt}$ since the capacitor is in series with $R$ and $L$. Finally, we apply Kirchoff's voltage law around the circuit to obtain $V_R + V_L + V_C = 0$, this yields:

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = 0 \quad \Rightarrow \quad \boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0}.$$

Obviously there is an analogy to be made here:

$$\boxed{m \leftrightarrows L \qquad \beta \leftrightarrows R \qquad k \leftrightarrows \frac{1}{C}}$$

I will exploit this analogy to construct the following examples.

**Example 3.9.5. Problem: the over-damped RLC circuit:** *Suppose $L = 1, R = 3$ and $C = 1/2$. If the circuit has current $I = -2$ and charge $Q = 1$ when $t = 0$ then what is the charge as a function of time? What is the current at a function of time?*

**Solution:** *We are faced with $\ddot{Q} + 3\dot{Q} + 2Q = 0$. This gives chararcteristic equation $\lambda^2 + 3\lambda + 2 = 0$ hence $(\lambda+1)(\lambda+2) = 0$ thus $\lambda_1 = -1$ and $\lambda_2 = -2$ and the general solution is*

$$Q(t) = c_1e^{-t} + c_2e^{-2t}$$

*Note that $\dot{Q}(t) = -c_1e^{-t} - 2c_2e^{-2t}$. Apply the given initial conditions,*

$$Q(0) = c_1 + c_2 = 1 \qquad \& \qquad \dot{Q}(0) = -c_1 - 2c_2 = -2$$

*You can solve these equations to obtain $c_2 = 1$ and $c_1 = 0$. Therefore, $\boxed{Q(t) = e^{-2t}}$. Differentiate to obtain the current $\boxed{I(t) = -2e^{-2t}}$.*

**Example 3.9.6. Problem: the critically-damped RLC circuit:** *Suppose $L = 1, R = 4$ and $C = 1/4$. If the circuit has current $I = 1$ and charge $Q = 3$ when $t = 0$ then what is the charge as a function of time? What is the current at a function of time?*

**Solution:** *We are faced with $\ddot{Q} + 4\dot{Q} + 4Q = 0$. This gives chararcteristic equation $\lambda^2 + 4\lambda + 4 = 0$ hence $(\lambda+2)^2 = 0$ thus $\lambda_1 = \lambda_2 = -2$ and the general solution is*

$$Q(t) = c_1e^{-2t} + c_2te^{-2t}$$

*Note that $\dot{Q}(t) = -2c_1e^{-t} + c_2(e^{-2t} - 2te^{-2t})$. Apply the given initial conditions,*

$$Q(0) = c_1 = 3 \qquad \& \qquad \dot{Q}(0) = -2c_1 + c_2 = 1$$

*You can solve these equations to obtain $c_1 = 3$ and $c_2 = 7$. Therefore, $\boxed{Q(t) = 3e^{-2t} + 7te^{-2t}}$. Differentiate the charge to find the current $\boxed{I(t) = e^{-2t} - 14te^{-2t}}$.*

**Example 3.9.7. Problem: the under-damped RLC circuit:** *Suppose $L = 1, R = 2$ and $C = 1/6$. If the circuit has current $I = 1$ and charge $x = 1$ when $t = 0$ then what is the charge as a function of time? What is the current at a function of time?*

**Solution:** *We are faced with $\ddot{Q} + 2\dot{Q} + 6Q = 0$. This gives chararcteristic equation $\lambda^2 + 2\lambda + 6 = 0$ hence $(\lambda+1)^2 + 5 = 0$ thus $\lambda = -1 \pm i\sqrt{5}$ and the general solution is*

$$Q(t) = c_1 e^{-t}\cos(\sqrt{5}\,t) + c_2 e^{-t}\sin(\sqrt{5}\,t)$$

*Note that $\dot{Q}(t) = c_1 e^{-t}(-\cos(\sqrt{5}\,t) - \sqrt{5}\sin(\sqrt{5}\,t) + c_2 e^{-t}(-\sin(\sqrt{5}\,t) + \sqrt{5}\cos(\sqrt{5}\,t)$. Apply the given initial conditions,*

$$Q(0) = c_1 = 1 \qquad \& \qquad \dot{Q}(0) = -c_1 + \sqrt{5}c_2 = 1$$

*You can solve these equations to obtain $c_1 = 1$ and $c_2 = 2/\sqrt{5}$. Therefore,*

$$\boxed{Q(t) = e^{-t}\cos(\sqrt{5}\,t) + \frac{2}{\sqrt{5}}e^{-t}\sin(\sqrt{5}\,t)}.$$

*Differentiate to find the current,*

$$\boxed{I(t) = e^{-t}\Big(-\cos(\sqrt{5}\,t) - \sqrt{5}\sin(\sqrt{5}\,t)\Big) + \frac{2}{\sqrt{5}}e^{-t}\Big(-\sin(\sqrt{5}\,t) + \sqrt{5}\cos(\sqrt{5}\,t)\Big)}.$$

**Example 3.9.8. Problem: the LC circuit or simple harmonic oscillator:** *Suppose $R = 0$ and $L, C$ are nonzero. If the circuit has current $I(0) = I_o$ and charge $Q(0) = Q_o$ then find the resulting equaitons for charge and current at time $t$.*

**Solution:** *We are faced with $L\ddot{Q} + \frac{1}{C}Q = 0$. This gives chararcteristic equation $\lambda^2 + \frac{1}{LC} = 0$ hence $\lambda = \pm i\sqrt{\frac{1}{LC}}$ and the general solution is, using $\omega = \sqrt{\frac{1}{LC}}$,*

$$Q(t) = c_1\cos(\omega t) + c_2\sin(\omega\, t)$$

*Note that*

$$\dot{Q}(t) = -c_1\omega\sin(\omega\, t) + c_2\omega\cos(\omega\, t).$$

*Apply the given initial conditions,*

$$Q(0) = c_1 = Q_o \qquad \& \qquad \dot{Q}(0) = c_2\omega = I_o$$

*Therefore,*

$$\boxed{Q(t) = Q_o\cos(\omega t) + \frac{I_o}{\omega}\sin(\omega\, t)}.$$

*Differentiate to find the current,*

$$\boxed{I(t) = -\omega Q_o\sin(\omega t) + I_o\cos(\omega\, t)}.$$

### 3.9.3 springs with external force

Suppose a mass $m$ undergoes one-dimensional motion under the influence of a spring force $F_s = -kx$ and a velocity dependent friction force $F_f = -\beta\dot{x}$ **and** some external force $f$. Newton's Second Law states $m\ddot{x} = -kx - \beta\dot{x} + f$. We find

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = f}$$

We have tools to solve this problem for many interesting forces.

**Example 3.9.9. Problem: constant force:** *Suppose $m \neq 0$ and $\beta, k > 0$. Suppose a constant force $f = F_o$ is placed on the spring. Describe the resulting motion.*

**Solution:** *We are faced with $m\ddot{x} + \beta\dot{x} + kx = F_o$. Notice that we can find $x_h$ to solve the homogeneous, force-free equation; $m\ddot{x_h} + \beta\dot{x_h} + kx_h = 0$. The particular solution is simply $x_p = F_o/k$ and it follows the general solution has the form:*

$$x(t) = x_h(t) + F_o/k$$

*We find motion that is almost identical to the problem with $F_o$ removed. If we change coordinates to $y = x - F_o/k$ then clearly $\dot{x} = \dot{y}$ and $\ddot{x} = \ddot{y}$ hence $m\ddot{y} + \beta\dot{y} + ky = 0$. An important example of a constant force is that of gravity on a srping hanging vertically. The net-effect of gravity is to reset the equilbrium position of the spring from $x = 0$ to $x = mg/k$. The frequency of any oscillations is not effected by gravity, moreover, the spring returns to the new equilbrium $x = mg/k$ in the same manner as it would with matching damping, mass and stiffness in a horizontal set-up. For example, to find the frequency of oscillation for shocks on a car is determined from the viscosity of the oil in the shock assembly, the stiffness of the springs and the mass of the car. Gravity doesn't enter the picture.*

**Example 3.9.10. Problem: sinusoidal, nonresonant, force on a simple harmonic oscillator** *Suppose $m = 1$ and $\beta = 0$ and $k = 1$. Suppose a sinusoidal force $f = F_o\cos(2t)$ is placed on the spring. Find the equations of motion given that $x(0) = 0$ and $\dot{x}(0) = 0$.*

**Solution:** *observe that $\ddot{x} + x = F_o\cos(2t)$ has homogeneous solution $x_h(t) = c_1\cos(t) + c_2\sin(t)$ and the method of annihilators can be used to indicate $x_p = A\cos(2t) + B\sin(2t)$. Calculate $\ddot{x}_p = -4x_p$ thus*

$$\ddot{x}_p + x_p = F_o\cos(2t) \quad \Rightarrow \quad -3A\cos(2t) - 3B\sin(2t) = F_o\cos(2t)$$

*Thus $A = -F_o/3$ and $B = 0$ which gives us the general solution,*

$$x(t) = c_1\cos(t) + c_2\sin(t) - \frac{F_o}{3}\cos(2t)$$

*We calculate $\dot{x}(t) = -c_1\sin(t) + c_2\cos(t) + \frac{2F_o}{3}\sin(2t)$. Apply initial conditions to the solution,*

$$c_1 - \frac{F_o}{3} = 0 \qquad \& c_2 = 0 \quad \Rightarrow \quad \boxed{x(t) = \tfrac{F_o}{3}\big[\cos(t) - \cos(2t)\big]}.$$

**Example 3.9.11. Problem: sinsoidal, resonant, force on a simple harmonic oscillator:** *Suppose $m = 1$ and $\beta = 0$ and $k = 1$. Suppose a sinusoidal force $f = 2\cos(t)$ is placed on the spring. Find the equations of motion given that $x(0) = 1$ and $\dot{x}(0) = 0$.*

**Solution:** *observe that $\ddot{x} + x = 2\cos(t)$ has homogeneous solution $x_h(t) = c_1\cos(t) + c_2\sin(t)$ and the method of annihilators can be used to indicate $x_p = At\cos(t) + Bt\sin(t)$. We calculate,*

$$\dot{x}_p = (A + Bt)\cos(t) + (B - At)\sin(t)$$

$$\ddot{x}_p = B\cos(t) - (A+Bt)\sin(t) - A\sin(t) + (B-At)\cos(t) = (2B-At)\cos(t) - (2A+Bt)\sin(t)$$

*Now plug these into $\ddot{x}_p + x_p = 2\cos(t)$ to obtain:*

$$At\cos(t) + Bt\sin(t) + (2B-At)\cos(t) - (2A+Bt)\sin(t) = 2\cos(t)$$

*notice the terms with coefficients $t$ cancel and we deduce $2B = 2$ and $-2A = 0$ thus $A = 0$ and $B = 1$. We find the general solution*

$$x(t) = c_1\cos(t) + c_2\sin(t) + t\sin(t)$$

*Note $\dot{x}(t) = -c_1\sin(t) + c_2\cos(t) + \sin(t) + t\cos(t)$. Apply the initial conditions, $x(0) = c_1 = 1$ and $\dot{x}(0) = c_2 = 0$. Therefore, the equation of motion is*

$$\boxed{x(t) = \cos(t) + t\sin(t)}.$$

Note that as $t \to \infty$ the equation above ceases to be physically reasonable. In the absence of damping it is possible for the energy injected from the external force to just build and build leading to infinite energy. Of course the spring cannot store infinite energy and it breaks. In this case without damping it is simple enough to judge the absence or presence of resonance. Resonance occurs iff the forcing function has the same frequency as the natural frequency $\omega = \sqrt{\frac{k}{m}}$. In the case that there is damping we say **resonance** is reached if for a given $m, \beta, k$ the applied force $F_o\cos(\gamma t)$ produces a particular solution of largest magnitude.

To keep it simple let us consider a damped spring in the arbitrary *underdamped* case where $\beta^2 - 4mk < 0$ with an external force $F_o\cos(\gamma t)$. We seek to study solutions of

$$m\ddot{x} + \beta\dot{x} + kx = F_o\cos(\gamma t)$$

Observe the characteristic equation is $m\lambda^2 + \beta\lambda + k = 0$ gives $\lambda^2 + \frac{\beta}{m}\lambda + \frac{k}{m} = 0$. Complete the square, or use the quadratic formula, whichever you prefer:

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} = \frac{-\beta \pm i\sqrt{4mk - \beta^2}}{2m}$$

It follows that the homogeneous (also called the **transient** solution since it goes away for $t >> 0$) is

$$x_h(t) = e^{\frac{-\beta t}{2m}}\big(c_1\cos(\omega t) + c_2\sin(\omega t)\big)$$

where I defined $\omega = \frac{\sqrt{4mk-\beta^2}}{2m}$ for convenience. The particular solution is also called the **steady-state** solution since it tends to dominate for $t >> 0$. Suppose $x_p = A\cos(\gamma t) + B\sin(\gamma t)$ calculate,

$$\dot{x}_p = -\gamma A\sin(\gamma t) + \gamma B\cos(\gamma t) \qquad \& \qquad \ddot{x}_p = -\gamma^2 A\cos(\gamma t) - \gamma^2 B\sin(\gamma t)$$

Subtitute into $m\ddot{x}_p + \beta\dot{x}_p + kx_p = F_o\cos(\gamma t)$ and find

$$-m\gamma^2 A\cos(\gamma t) - m\gamma^2 B\sin(\gamma t) - \beta\gamma A\sin(\gamma t) + \beta\gamma B\cos(\gamma t) + kA\cos(\gamma t) + kB\sin(\gamma t) = F_o\cos(\gamma t)$$

Hence,

$$\big[-m\gamma^2 A + \beta\gamma B + kA\big]\cos(\gamma t) + \big[-m\gamma^2 B - \beta\gamma A + kB\big]\sin(\gamma t) = F_o\cos(\gamma t)$$

Equating coefficients yield the conditions:

$$(k - m\gamma^2)A + \beta\gamma B = F_o \qquad \& \qquad (k - m\gamma^2)B - \beta\gamma A = 0$$

We solve the second equation for $B = \frac{\beta\gamma}{k-m\gamma^2}A$ and substitute this into the other equation,

$$(k - m\gamma^2)A + \frac{\beta^2\gamma^2}{k - m\gamma^2}A = F_o$$

Now make a common denominator,

$$\frac{(k - m\gamma^2)^2 + \beta^2\gamma^2}{k - m\gamma^2}A = F_o$$

We find,

$$A = \frac{(k - m\gamma^2)F_o}{(k - m\gamma^2)^2 + \beta^2\gamma^2} \qquad \& \qquad B = \frac{\beta\gamma F_o}{(k - m\gamma^2)^2 + \beta^2\gamma^2}$$

It follows that the particular solution has the form

$$x_p = \frac{F_o}{(k - m\gamma^2)^2 + \beta^2\gamma^2}\Big[(k - m\gamma^2)\cos(\gamma t) + \beta\gamma\sin(\gamma t)\Big]$$

You can show[16] that the amplitude of $A_1\cos(\gamma t) + A_2\sin(\gamma t)$ is given by $A = \sqrt{A_1^2 + A_2^2}$. Apply this lemma to the formula above to write the particular solution in the simplified form

$$\boxed{x_p = \frac{F_o}{\sqrt{(k - m\gamma^2)^2 + \beta^2\gamma^2}}\sin(\gamma t + \phi)}$$

where $\phi$ is a particular angle. We're mostly interested in the magnitude so let us focus our attention on the amplitude of the steady state solution[17].

Suppose $k, m, \beta$ are fixed and let us study $M(\gamma) = \frac{1}{\sqrt{(k-m\gamma^2)^2+\beta^2\gamma^2}}$. What choice of $\gamma$ maximizes this factor thus producing the resonant motion? Differentiate and seek the critical value:

$$\frac{dM}{d\gamma} = -\frac{1}{2}\cdot\frac{2(k - m\gamma^2)(-2m\gamma) + 2\beta^2\gamma}{[(k - m\gamma^2)^2 + \beta^2\gamma^2]^{3/2}} = 0$$

The critical value must arise from the vanishing of the numerator since the denominator is nonzero,

$$(k - m\gamma^2)(-2m\gamma) + \beta^2\gamma = 0 \quad \Rightarrow \quad (-2mk + 2m^2\gamma^2 + \beta^2)\gamma = 0$$

But, we already know $\gamma = 0$ is not the frequency we're looking for, thus

$$-2mk + 2m^2\gamma^2 + \beta^2 = 0 \quad \Rightarrow \quad \gamma = \pm\sqrt{\frac{2mk - \beta^2}{2m^2}}$$

Nothing is lost by choosing the + here and we can simplify to find

$$\boxed{\gamma_c = \sqrt{\frac{k}{m} - \frac{\beta^2}{2m^2}}}$$

It is nice to see that $\beta = 0$ returns us to the natural frequency $\omega = \sqrt{km}$ as we studied initially. Section 4.10 of Nagel Saff and Snider, or 6-3 of Ritger & Rose if you would like to see further analysis.

[16]the precalculus chapter in my calculus I notes has some of the ideas needed for this derivation

[17]see page 240-241 of Nagel Saff and Snider for a few comments beyond mine and a nice picture to see the difference between the transient and steady state solutions

## 3.10 RLC circuit with a voltage source

Suppose a resistor $R$, an inductor $L$ and a capacitor $C$ are placed in series with a voltage source $\mathcal{E}$. Kirchoff's Voltage Law reads

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}}$$

We can solve these problems in the same way as we have just explored for the spring force problem. I will jump straight to the resonance problem and change gears a bit to once more promote complex notation.

Suppose we have an underdamped $R, L, C$ circuit driven by a voltage source $\mathcal{E}(t) = V_o\cos(\gamma t)$. I propose we solve the related complex problem

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_o e^{i\gamma t}$$

We propose a complex particular solution: $Q_p = Ae^{i\gamma t}$ hence

$$Q_p' = i\gamma Ae^{i\gamma t} \qquad \& \qquad Q_p'' = -\gamma^2 Ae^{i\gamma t}$$

Subtitute into $LQ_p'' + RQ_p' + \frac{1}{C}Q_p = V_o e^{i\gamma t}$ and factor out the imaginary exponential

$$\big[-\gamma^2 L + i\gamma R + \frac{1}{C}\big]Ae^{i\gamma t} = V_o e^{i\gamma t}$$

Hence,

$$-\gamma^2 L + i\gamma R + \frac{1}{C} = \frac{V_o}{A}$$

Hence,

$$A = \frac{V_o}{1/C - \gamma^2 L + i\gamma R} \cdot \frac{1/C - \gamma^2 L - i\gamma R}{1/C - \gamma^2 L - i\gamma R} = \frac{V_o[1/C - \gamma^2 L - i\gamma R]}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}$$

Thus, using $e^{i\gamma t} = \cos(\gamma t) + i\sin(\gamma t)$, the complex particular solution is given by

$$Q_p(t) = \left[\frac{V_o[1/C - \gamma^2 L]}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2} - i\frac{V_o\gamma R}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}\right]\Big[\cos(\gamma t) + i\sin(\gamma t)\Big].$$

We can read solutions for particular solutions of any real linear combination of $V_o\cos(\gamma t)$ and $V_o\sin(\gamma t)$. For example, for $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_o\cos(\gamma t)$ we derive the particular solution

$$Q_{p_1}(t) = \frac{V_o[1/C - \gamma^2 L]}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}\cos(\gamma t) + \frac{V_o\gamma R}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}\sin(\gamma t)$$

Likewise, as $Im(e^{i\gamma t}) = \sin(\gamma t)$ the solution of $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_o\sin(\gamma t)$ is given by the $Im(Q_p) = Q_{p_2}$.

$$Q_{p_2}(t) = \frac{V_o[1/C - \gamma^2 L]}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}\sin(\gamma t) + \frac{V_o\gamma R}{(1/C - \gamma^2 L)^2 + \gamma^2 R^2}\cos(\gamma t)$$

To solve $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = B_1V_o\cos(\gamma t) + B_2V_o\sin(\gamma t)$ we use superposition to form the particular solution $Q_{p_3} = B_1Q_{p_1} + B_2Q_{p_2}$.

**Remark 3.10.1.**

Notice that $Q_{p_1}$ is analagous to the solution we found studying resonance for the underdamped spring. If we use the dictionary $m \leftrightarrows L$, $\beta \leftrightarrows R$, $k \leftrightarrows 1/C$, $F_o \leftrightarrows V_o$ then it ought to be obvious the solution above was already derived in real notation. However, the complex solution is quicker and cleaner. We also can deduce that resonance is reached at

$$\gamma_r = \sqrt{\frac{k}{m} - \frac{\beta^2}{2m^2}} \leftrightarrows \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

and note how $R = 0$ reduces the problem to the pure harmonic oscillation of the $LC$-tank.

# Chapter 4

# the series solution technique

Series techniques have been with us a long time now. Founders of calculus worked with series in a somewhat careless fashion and we will do the same here. The wisdom of nineteenth century analysis is more or less ignored in this work. In short, I'm not too worried about the interval of convergence in these notes. This is of course a dangerous game, but the density of math majors permits no other. I'll just make this comment: the series we find generally represent a function of interest only locally. Singularities prevent us from continuing the expansion past some particular point.

It doesn't concern this course too much, but perhaps it's worth mentioning: much of the work we see here arose from studying complex differential equations. The results for ordinary points were probably known by Euler and Lagrange even took analyticity as a starting point for what he thought of as a "function". The word "analytic" should be understood to mean that there exists a power series expansion representing the function near the point in question. There are functions which are not analytic and yet are smooth ($f(f(x) = \sin(x)$ defines such a function, see the math stack for more). Logically, functions need not be analytic. However, most nice formulas do impart analyticity at least locally.

Fuchs studied complex differential equations as did Weierstrauss, Cauchy, Riemann and most of the research math community of the nineteenth century. Fuchasian theory of DEqns dealt with the problem of singularities and there was (is) a theory of majorants due to Weierstrauss which was concerned with how singularities appear in solutions. In particular, the study of moveable singularities, the process of what we call *analytic continuation* was largely solved by Fuchs. However, his approach was more complicated than the methods we study. Frobenous proposed a method which clarified Fuch's work and we use it to this day. Read Hille's masterful text about differential equations in the complex plane for a more complete history. My point to you here is simply this: what we do here did not arise from the study of the real-valued problems we study alone. To really understand the genesis of this material you must study complex differential equations. We don't do this since complex variables are not a prequisite for this course.

The calculations in this chapter can be challenging. However, the power series approximation is one of our most flexible tools for mathematical modelling and it is most certainly worth understanding. If you compare these notes with Ritger & Rose then you'll notice that I have not covered too deeply the sections towards the end of Chapter 7; Bessel, Legendre, and the hypergeometric equations are interesting problems, but it would take several class periods to absorb the material and I think it better to spend our time on breadth. My philosophy is that once you've taken this course you ought to be ready to do further study on those sections.

## 4.1 calculus of series

I begin with a brief overview of terminology and general concepts about sequences and series. We will not need all of this, but I think it is best to at least review the terms as to recover as much as we can from your previous course work.
A sequence in $S$ is a function $a : \{n_o, n_o+1, n_o+2, \dots\} \to S$ where we usually denote $a(n) = a_n$ for all $n \in \mathbb{Z}$ with $n \geq n_o$. Typically $n_o = 0$ or $n_o = 1$, but certainly it is interesting to consider other initial points for the domain. If $a_n \in \mathbb{R}$ for all $n$ then we say $\{a_n\}$ is a sequence of real numbers. If $a_n \in \mathbb{C}$ for all $n$ then we say $\{a_n\}$ is a complex sequence. If $\mathcal{F}$ is a set of functions and $a_n \in \mathcal{F}$ for all $n$ then we say $\{a_n\}$ is sequence of functions. If the codomain for a sequence has an operation such as addition or multiplication then we can add or multiply such sequences by the usual pointwise defined rules; $(ab)_n = a_nb_n$ and $(a+b)_n = a_n + b_n$. In addition, we can define a **series** $s = a_{n_o} + a_{n_1} + \cdots$ in $S$ as follows:

$$s = \lim_{n\to\infty} \sum_{k=n_o}^{n} a_k$$

provided the limit above exists. In other words, the series above exists iff the sequence of partial sums $\{a_{n_o}, a_{n_o}+a_{n_1}, a_{n_o}+a_{n_1}+a_{n_2}, \dots\}$ converges. When the sequence of partial sums converges then the series is likewise said to converge and we can denote this by $s = \sum_{k=n_o}^{\infty} a_k$. You should remember studying the convergence of such series for a few weeks in your second calculus course. Perhaps you will be happy to hear that convergence is not the focus of our study in this chapter.

A **power function** is a function with formula $f(x) = x^n$ for some $n \in \mathbb{R}$. A **power series** is a series formed from adding together power functions. However, traditionally the term **power series** is reserved for series constructed with powers from $\mathbb{N} \cup \{0\}$. Equivalently we can say a **power series** is a function which is defined at each point by a series;

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k = c_o + c_1(x-a) + c_2(x-a)^2 + \cdots$$

The constants $c_o, c_1, c_2, \dots$ are fixed and essentially define $f$ uniquely once the center point $a$ is given. The domain of $f$ is understood to be the set of all real $x$ such that the series converges. Given that $f(x)$ is a power series it is a simple matter to compute that

$$c_o = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{1}{2}f''(a), \quad \dots, c_k = \frac{1}{k!}f^{(k)}(a).$$

Incidentially, the result above shows that if $\sum_{k=0}^{\infty} b_k(x-a)^k = \sum_{k=0}^{\infty} c_k(x-a)^k$ then $b_k = c_k$ for all $k \geq 0$ since both power series define the same derivatives and we know derivatives are single-valued when they exist. This result is called **equating coefficients** of power series, we will use it many times.

The domain of a power series is somewhat boring. Recall that there are three possibilities:

1. $dom(f) = \{a\}$
2. $dom(f) = \{x \in \mathbb{R} \mid |x-a| \leq R\}$ for some radius $R > 0$.
3. $dom(f) = (-\infty, \infty)$

The constant $R$ is called the **radius of convergence** and traditionally we extend it to all three cases above with the convention that for case (1.) $R = 0$ whereas for case (3.) $R = \infty$.

Given a function on $\mathbb{R}$ we can sometimes replace the given formula of the function with a power series. If it is possible to write the formula for the function $f$ as a power series centered at $x_o$ in some open set around $x_o$ then we say $f$ is **analytic at** $x_o$. When it is possible to write $f(x)$ as a single power series for all $x \in \mathbb{R}$ then we say $f$ is **entire**. A function is called **smooth** at $x_o$ if derivatives of arbitrary order exist for $f$ at $x_o$. Whenever a function is smooth at $x_o$ we can calculate $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$ which is called the **Taylor series** of $f$ centered at $x_o$. However, there are functions for which the series $T(x) \neq f(x)$ near $x_o$. Such a function is said to be **non-analytic**. If $f(x) = T(x)$ for all $x$ close to $x_o$ then we say $f$ is analytic at $x_o$. This question is not treated in too much depth in most calculus II courses. It is much harder to prove a function is analytic than it is to simply compute a Taylor series. We again set-aside the issue of analyticity for a later course where analysis is the focus. We now turn our focus to the computational aspects of series.

If $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is analytic at $x_o \in U$ then we can write

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{1}{2}f''(x_o)(x - x_o)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

We could write this in terms of the operator $D = \frac{d}{dt}$ and the evaluation of $t = x_o$

$$f(x) = \left[\sum_{n=0}^{\infty} \frac{1}{n!}(x - t)^n D^n f(t)\right]_{t=x_o} =$$

I remind the reader that a function is called **entire** if it is analytic on all of $\mathbb{R}$, for example $e^x, \cos(x)$ and $\sin(x)$ are all entire. In particular, you should know that:

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Since $e^x = \cosh(x) + \sinh(x)$ it also follows that

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$$

The geometric series is often useful, for $a, r \in \mathbb{R}$ with $|r| < 1$ it is known

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This generates a whole host of examples, for instance:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

$$\frac{x^3}{1-2x} = x^3(1 + 2x + (2x)^2 + \cdots) = x^3 + 2x^4 + 4x^5 + \cdots$$

Moreover, the term-by-term integration and differentiation theorems yield additional results in conjuction with the geometric series:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$$\ln(1-x) = \int \frac{d}{dx} \ln(1-x) dx = \int \frac{-1}{1-x} dx = -\int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}$$

Of course, these are just the basic building blocks. We also can twist things and make the student use algebra,

$$e^{x+2} = e^x e^2 = e^2(1 + x + \frac{1}{2}x^2 + \cdots)$$

or trigonmetric identities,

$$\sin(x) = \sin(x - 2 + 2) = \sin(x-2)\cos(2) + \cos(x-2)\sin(2)$$

$$\Rightarrow \quad \sin(x) = \cos(2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x-2)^{2n+1} + \sin(2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x-2)^{2n}.$$

Feel free to peruse my most recent calculus II materials to see a host of similarly sneaky calculations.

## 4.2 solutions at an ordinary point

An **ordinary point** for a differential equation is simply a point at which an analytic solution exists. I'll explain more carefully how to discern the nature of a given ODE in the next section. In this section we make the unfounded assumption that a power series solution exists in each example.

**Example 4.2.1. Problem:** *find the first four nontrivial terms in a series solution centered at $a = 0$ for $y' - y = 0$*

**Solution:** *propose that $y = c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \cdots$. Differentiating,*

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 \cdots$$

*We desire $y$ be a solution, therefore:*

$$y' - y = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 \cdots - (c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \cdots) = 0.$$

*Collect like terms:*

$$c_1 - c_o + x(2c_2 - c_1) + x^2(3c_3 - c_2) + x^3(4c_4 - c_3) + x^4(5c_5 - c_4) + \cdots = 0$$

*We find, by equating coefficients, that every coefficient on the l.h.s. of the expression above is zero thus:*

$$c_1 = c_o, \quad c_2 = \frac{1}{2}c_1, \quad c_3 = \frac{1}{3}c_2, \quad c_4 = \frac{1}{4}c_3$$

*Hence,*

$$c_1 = c_o, \quad c_2 = \frac{1}{2}c_o, \quad c_3 = \frac{1}{3}\frac{1}{2}c_o, \quad c_4 = \frac{1}{4}\frac{1}{3}\frac{1}{2}c_o$$

*Note that* $2 = 2!, 3 \cdot 2 = 3!, 4 \cdot 3 \cdot 2 = 4!$ *hence,*

$$y = c_o + c_o x + \frac{1}{2}c_o x^2 + \frac{1}{3!}c_o x^3 + \frac{1}{4!}c_o x^4 + \cdots$$

*Consequently,* $\boxed{y = c_o(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots)}$ *is the desired solution.*

Of course the example above is not surprising; $y' - y = 0$ has $\lambda - 1 = 0$ hence $y = c_o e^x$ is the solution. We just derived the first few terms in the power series expansion for $e^x$ centered at $a = 0$.

**Example 4.2.2. Problem:** *find the complete series solution centered at* $a = 0$ *for* $y'' + x^2 y = 0$.

**Solution:** *Suppose the solution is a power series and calculate,*

$$y = \sum_{k=0}^{\infty} c_k x^k, \quad y' = \sum_{k=0}^{\infty} k c_k x^{k-1}, \quad y'' = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2}$$

*Of course, the summations can be taken from* $k = 1$ *for* $y'$ *and* $k = 2$ *for* $y''$ *as the lower order terms vanish. Suppose* $y'' + x^2 y = 0$ *to find:*

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + x^2 \sum_{k=0}^{\infty} c_k x^k = 0$$

*Notice,*

$$x^2 \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^2 x^k = \sum_{k=0}^{\infty} c_k x^{k+2} = \sum_{j=2}^{\infty} c_{j-2} x^j$$

*where in the last step we set* $j = k + 2$ *hence* $k = 0$ *gives* $j = 2$*. Likewise, consider:*

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} = \sum_{j=0}^{\infty}(j+2)(j+1)c_{j+2} x^j.$$

*where we set* $k - 2 = j$ *hence* $k = 2$ *gives* $j = 0$*. Hence,*

$$\sum_{j=0}^{\infty}(j+2)(j+1)c_{j+2} x^j + \sum_{j=2}^{\infty} c_{j-2} x^j = 0.$$

*Sometimes we have to separate a few low order terms to clarify a pattern:*

$$2c_2 + 6c_3 x + \sum_{j=2}^{\infty}\big[(j+2)(j+1)c_{j+2} + c_{j-2}\big]x^j = 0$$

*It follows that* $c_2 = 0$ *and* $c_3 = 0$*. Moreover, for* $j = 2, 3, \dots$ *we have the recursive rule:*

$$c_{j+2} = \frac{-1}{(j+2)(j+1)}c_{j-2}$$

*Let us study the relations above to find a pattern if possible,*

$$c_4 = \frac{-1}{20}c_o, \quad c_5 = \frac{-1}{20}c_1, \quad c_6 = \frac{-1}{42}c_2, \quad c_7 = \frac{-1}{56}c_3, \quad c_8 = \frac{-1}{72}c_4, ...$$

*Notice that* $c_2 = 0$ *clearly implies* $c_{4k+2} = 0$ *for* $k \in \mathbb{N}$. *Likewise,* $c_3 = 0$ *clearly implies* $c_{4k+3} = 0$ *for* $k \in \mathbb{N}$. *However, the coefficients* $c_o, c_4, c_8, \dots$ *are linked as are* $c_1, c_5, c_9, \dots$. *In particular,*

$$c_{12} = \frac{-1}{(12)(11)}c_8 = \frac{-1}{(12)(11)} \cdot \frac{-1}{(8)(7)}c_4 = \frac{-1}{(12)(11)} \cdot \frac{-1}{(8)(7)} \cdot \frac{-1}{(4)(3)}c_o = c_{3(4)}$$

$$c_{16} = \frac{-1}{(16)(15)} \cdot \frac{-1}{(12)(11)} \cdot \frac{-1}{(8)(7)} \cdot \frac{-1}{(4)(3)}c_o = c_{4(4)}$$

*We find,*

$$c_{4k} = \frac{(-1)^k}{k!4^k(4k-1)(4k-5)\cdots 11 \cdot 7 \cdot 3}c_o$$

*Next, study* $c_1, c_5, c_9, ...$

$$c_9 = \frac{-1}{(9)(8)}c_5 = \frac{-1}{(9)(8)} \cdot \frac{-1}{(5)(4)}c_1 = c_{2(4)+1}$$

$$c_{13} = \frac{-1}{(13)(12)} \cdot \frac{-1}{(9)(8)} \cdot \frac{-1}{(5)(4)}c_1 = c_{3(4)+1}$$

*We find,*

$$c_{4k+1} = \frac{(-1)^k}{k!4^k(4k+1)(4k-3)\cdots 13 \cdot 9 \cdot 5}c_1$$

*We find the solution has two coefficients* $c_o, c_1$ *as we ought to expect for the general solution to a second order ODE.*

$$\boxed{y = c_o\sum_{k=0}^{\infty}\frac{(-1)^k}{k!4^k(4k-1)(4k-5)\cdots 11 \cdot 7 \cdot 3}x^{4k} + c_1\sum_{k=0}^{\infty}\frac{(-1)^k}{k!4^k(4k+1)(4k-3)\cdots 13 \cdot 9 \cdot 5}x^{4k+1}}$$

*If we just want the the solution up to 11-th order in* $x$ *then the following would have sufficed:*

$$y = c_o(1 - \tfrac{1}{12}x^4 + \tfrac{1}{672}x^8 + \cdots) + c_1(x - \tfrac{1}{20}x^5 + \tfrac{1}{1440}x^9 + \cdots).$$

**Remark 4.2.3.**

The formulas we derived for $c_{4k}$ and $c_{4k+1}$ are what entitle me to claim the solution is the **complete** solution. It is not always possible to find nice formulas for the general term in the solution. Usually if no "nice" formula can be found you might just be asked to find the first 6 nontrvial terms since this typically gives 3 terms in each fundamental solution to a second order problem. We tend to focus on second order problems in this chapter, but most of the techniques here apply equally well to arbitrary order.

**Example 4.2.4. Problem:** *find the complete series solution centered at $a = 0$ for $y'' + xy' + 3y = 0$.*

**Solution:** *Suppose the solution is a power series and calculate,*

$$y = \sum_{k=0}^{\infty} c_k x^k, \quad y' = \sum_{k=0}^{\infty} kc_k x^{k-1}, \quad y'' = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2}$$

*Suppose $y'' + xy' + 3y = 0$ to find:*

$$\sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} + x\sum_{k=0}^{\infty} kc_k x^{k-1} + 3\sum_{k=0}^{\infty} c_k x^k = 0.$$

*Hence, noting some terms vanish and $xx^{k-1} = x^k$:*

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} 3c_k x^k = 0$$

*Let $k-2 = j$ to relable $k(k-1)c_k x^{k-2} = (j+2)(j+1)c_{j+2}x^j$. It follows that:*

$$\sum_{j=0}^{\infty}(j+2)(j+1)c_{j+2}x^j + \sum_{j=1}^{\infty} jc_j x^j + \sum_{j=0}^{\infty} 3c_j x^j = 0$$

*We can combine all three sums for $j \geq 1$ however the constant terms break the pattern so list them seperately,*

$$2c_2 + 3c_o + \sum_{j=1}^{\infty}\Big[(j+2)(j+1)c_{j+2} + (3+j)c_j\Big]x^j = 0$$

*Equating coefficients yields, for $j = 1, 2, 3 \ldots$:*

$$2c_2 + 3c_o = 0, \quad (j+2)(j+1)c_{j+2} + (3+j)c_j = 0 \quad \Rightarrow c_2 = \frac{-2}{3}c_o, \quad c_{j+2} = \frac{-(j+3)}{(j+2)(j+1)}c_j.$$

*In this example the even and odd coefficients are linked. Let us study the recurrence relation above to find a general formula if possible.*

$$\begin{aligned}
(j=1): &\quad c_3 = \frac{-4}{(3)(2)}c_1 = \frac{(-1)^1(2^1)(2!)}{3!}c_1 \\
(j=3): &\quad c_5 = \frac{-6}{(5)(4)}c_3 = \frac{-6}{(5)(4)}\cdot\frac{-4}{(3)(2)}c_1 = \frac{(-1)^2(2^2)(3!)}{5!}c_1 \\
(j=5): &\quad c_7 = \frac{-8}{(7)(6)}c_5 = \frac{-8}{(7)(6)}\cdot\frac{-6}{(5)(4)}\cdot\frac{-4}{(3)(2)}c_1 = \frac{(-1)^3(2^3)(4!)}{7!}c_1 \\
(j=2k+1): &\quad c_{2k+1} = \frac{(-1)^k(2k+2)(2k)(2k-2)\cdots(6)(4)(2)}{(2k+1)!}c_1.
\end{aligned}$$

*Next, study the to even coefficients: we found* $c_2 = \frac{-2}{3}c_o$

$$\begin{aligned}
(j=2): &\quad c_4 = \frac{-5}{(4)(3)}c_2 = \frac{-5}{(4)(3)}\cdot\frac{-3}{2}c_o \\
(j=4): &\quad c_6 = \frac{-7}{(6)(5)}\cdot\frac{-5}{(4)(3)}\cdot\frac{-3}{2}c_o \\
(j=6): &\quad c_8 = \frac{-9}{(8)(7)}\cdot\frac{-7}{(6)(5)}\cdot\frac{-5}{(4)(3)}\cdot\frac{-3}{2}c_o \\
(j=2k+1): &\quad c_{2k} = \frac{(-1)^k(2k+1)(2k-1)(2k-3)\cdots(7)(5)(3)}{(2k)!}c_o.
\end{aligned}$$

*Therefore, the general solution is given by:*

$$\boxed{y = c_o\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)(2k-1)\cdots(7)(5)(3)}{(2k)!}x^{2k} + c_1\sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)(2k)\cdots(6)(4)(2)}{(2k+1)!}x^{2k+1}}.$$

*The first few terms in the solution are given by* $y = c_o(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 + \cdots) + c_1(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \cdots)$.

**Example 4.2.5. Problem:** *find the first few nontrivial terms in the series solution centered at* $a = 0$ *for* $y'' + \frac{1}{1-x}y' + e^x y = 0$. *Given that* $y(0) = 0$ *and* $y'(0) = 1$.

**Solution:** *Notice that* $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ *and* $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$ *hence:*

$$y'' + (1 + x + x^2 + \cdots)y' + (1 + x + \tfrac{1}{2}x^2 + \cdots)y = 0$$

*Suppose* $y = c_o + c_1x + c_2x^2 + \cdots$ *hence* $y' = c_1 + 2c_2x + 3c_3x^2 + \cdots$ *and* $y'' = 2c_2 + 6c_3x + 12c_4x^2 + \cdots$. *Put these into the differential equation, keep only terms up to quadratic order,*

$$2c_2 + 6c_3x + 12c_4x^2 + (1 + x + x^2)(c_1 + 2c_2x + 3c_3x^2) + (1 + x + \tfrac{1}{2}x^2)(c_o + c_1x + c_2x^2) + \cdots = 0$$

*The coefficients of* $1$ *in the equation above are*

$$2c_2 + c_1 + c_o = 0$$

*The coefficients of* $x$ *in the equation above are*

$$6c_3 + c_1 + 2c_2 + c_1 + c_o = 0$$

*The coefficients of* $x^2$ *in the equation above are*

$$12c_4 + c_1 + 2c_2 + 3c_3 + \frac{1}{2}c_o + c_1 + c_2 = 0$$

*I find these problems very challenging when no additional information is given. However, we were given* $y(0) = 0$ *and* $y'(0) = 1$ *hence*[1] $c_o = 0$ *whereas* $c_1 = 1$. *Thus* $c_2 = -1/2$ *and* $c_3 = \frac{-1}{6}(-2c_2 - 2c_1) = \frac{1}{6}$ *and* $c_4 = \frac{1}{12}(-2c_1 - 3c_2 - 3c_3) = \frac{1}{12}(-2 + 3/2 - 3/6) = \frac{-1}{12}$ *hence*

$$\boxed{y = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 + \cdots}.$$

[1] think about Taylor's theorem centered at zero

**Remark 4.2.6.**

When faced with a differential equation with variable coefficients we must expand the coefficient functions as power series when we seek a power series solution. Moreover, the center of the expansion ought to match the center of the desired solution. In this section we have only so far consider series centered at zero. Next we consider a nonzero center.

**Example 4.2.7. Problem:** *find the first few nontrivial terms in the series solution centered at $a = 1$ for $y' = \frac{\sin(x)}{1-(x-1)^2}$.*

**Solution:** *note that we can integrate to find an integral solution: $y = \int \frac{\sin(x)\,dx}{1-(x-1)^2}$. To derive the series solution we simply expand the integrand in powers of $(x-1)$. Note,*

$$\frac{1}{1-(x-1)^2} = 1 + (x-1)^2 + (x-1)^4 + (x-1)^6 + \cdots$$

*On the other hand, to expand sine, we should use the adding angles formula on $\sin(x) = \sin(x-1+1)$ to see*

$$\sin(x) = \cos(1)\sin(x-1) + \sin(1)\cos(x-1) = \sin(1) + \cos(1)(x-1) - \tfrac{\sin(1)}{2}(x-1)^2 + \cdots$$

*Consider the product of the power series above, up to quadratic order we find:*

$$\frac{\sin(x)}{1-(x-1)^2} = \sin(1) + \cos(1)(x-1) + \tfrac{\sin(1)}{2}(x-1)^2 + \cdots$$

*Therefore, integrating term-by-term, we find*

$$\boxed{y = c_1 + \sin(1)(x-1) + \tfrac{\cos(1)}{2}(x-1)^2 + \tfrac{\sin(1)}{6}(x-1)^3 + \cdots}.$$

**Remark 4.2.8.**

Taylor's formula $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots$ is one way we could compute the power series expansions for given functions, however, it is much faster to use algebra and known results when possible.

## 4.3 classification of singular points

We are primarily interested in real solutions to linear ODEs of first or second order in this chapter, however, the theory of singular points and the Frobenius method necessarily require us to consider singularities as having their residence in the complex plane. It would appear that our solutions are restrictions of complex solutions to the real axis in $\mathbb{C}$.

**Definition 4.3.1.** *singular points and ordinary points*

We say $x_o$ is a **ordinary point** of $y'' + Py' + Qy = 0$ iff $P$ and $Q$ are analytic at $x_o$. A point $x_o$ is a **singular point** of $y'' + Py' + Qy = 0$ if $x_o$ is not an ordinary point. A point $x_o$ is a **regular singular point** of $y''+Py'+Qy = 0$ if $x_o$ is a singular point however $(x-x_o)P(x)$ and $(x-x_o)^2Q(x)$ are analytic at $x_o$.

In the definition above we mean to consider the functions $(x-x_o)P(x)$ and $(x-x_o)^2Q(x)$ with any removable discontinuities removed. For example, while $f(x) = \frac{1}{x}$ has $xf(x)$ undefined at $x = 0$, we still insist that $xf(x)$ is an analytic function at $x = 0$. Another example, technically the expression $\sin(x)/x$ is not defined at $x = 0$, but it is an analytic expression $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots$ which is defined at $x = 0$. To be more careful, we could insist that the limit as $x \to x_o$ of $(x - x_o)P(x)$ and $(x - x_o)^2Q(x)$ exist. That would just be a careful way of insisting that the only divergence faced by $(x - x_o)P(x)$ and $(x - x_o)^2Q(x)$ are simple holes in the graph a.k.a removable discontinuities.

In addition, the singular point $x_o$ may be complex. This is of particular interest as we seek to determine the domain of solutions in the Frobenius method. I will illustrate by example:

**Example 4.3.2.** *For $b, c \in \mathbb{R}$, every point is an ordinary point for $y'' + by' + cy = 0$.*

**Example 4.3.3.** *Since $e^x$ and $\cos(x)$ are analytic it follows that the differential equation $y'' + e^xy' + \cos(x)y = 0$ has no singular point. Every point is an ordinary point.*

**Example 4.3.4.** *Consider $(x^2 + 1)y'' + y = 0$. We divide by $x^2 + 1$ and find $y'' + \frac{1}{x^2+1}y = 0$. Note:*

$$Q(x) = \frac{1}{x^2 + 1} = \frac{1}{(x + i)(x - i)}$$

*It follows that every $x \in \mathbb{R}$ is an ordinary point and the only singular points are found at $x_o = \pm i$. It turns out that the existence of these imaginary singular points limits the largest open domain of a solution centered at the ordinary point $x_o = 0$ to $(-1, 1)$.*

**Example 4.3.5.** *Consider $y'' + \frac{1}{x^2(x-1)}y' + \frac{1}{(x-1)^2(x^2+4x+5)}y = 0$. Consider,*

$$P(x) = \frac{1}{x^2(x-1)} \qquad \& \qquad Q(x) = \frac{1}{(x-1)^2(x-2+i)(x-2-i)}$$

*Observe that,*

$$xP(x) = \frac{x}{x^2(x-1)} = \frac{1}{x(x-1)}$$

*therefore $xP(x)$ is not analytic at $x = 0$ hence $x = 0$ is a singular point which is not regular; this is also called an* **irregular singular point**. *On the other hand, note:*

$$(x-1)P(x) = \frac{x-1}{x^2(x-1)} = \frac{1}{x^2} \qquad \& \qquad (x-1)^2Q(x) = \frac{(x-1)^2}{(x-1)^2(x^2+4x+5)} = \frac{1}{x^2+4x+5}$$

*are both analytic at $x = 1$ hence $x = 1$ is a* **regular singular point**. *Finally, note that the quadratic $x^2 + 4x + 5 = (x + 2 - i)(x + 2 + i)$ hence $x = -2 \pm i$ are singular points.*

It is true that $x = -2 \pm i$ are regular singular points, but this point does not interest us as we only seek solutions based at some real point.

**Theorem 4.3.6.** *ordinary points and frobenius' theorem*

A solution of $y'' + Py' + Qy = 0$ centered at an ordinary point $x_o$ can be extended to an open disk in the complex plane which reaches the closest singularity. A solution of $y'' + Py' + Qy = 0$ based at a regular singular point $x_o$ extends to an open interval with $x_o$ at one edge and $x_o \pm R$ on the other edge where $R$ is the distance to the next nearest singularity (besides $x_o$ of course)

See pages 477 and 494 for corresponding theorems in Nagel, Saff and Snider. It is also important to note that the series technique and the full method of Frobenius will provide a fundamental solution set on the domains indicated by the theorem above.

**Example 4.3.7.** *Consider $y'' + \frac{1}{x^2(x-1)}y' + \frac{1}{(x-1)^2(x^2+4x+5)}y = 0$. Recall we found singular points $x = 0, 1, -2+i, -2-i$. The point $x = 0$ is an irregular singular point hence we have nothing much to say. On the other hand, if we consider solutions on $(1, 1+R)$ we can make $R$ at most $R = 1$ the distance from $1$ to $0$. Likewise, we could find a solution on $(0,1)$ which puts the regular singular point on the right edge. A solution $\sum_{n=0}^{\infty} c_n(x+2)^n$ centered at $x = -2$ will extend to the open interval $(-3,-1)$ at most since the singularities $-2 \pm i$ are one-unit away from $-2$ in the complex plane. On the other hand, if we consider a solution of the form $\sum_{n=0}^{\infty} c_n(x+3)^n$ which is centered at $x = -3$ then the singularities $-2 \pm i$ are distance $\sqrt{2}$ away and we can be confident the domain of the series solution will extend to at least the open interval $(-3-\sqrt{2}, -3+\sqrt{2})$.*

You might notice I was intentionally vague about the regular singular point solutions in the example above. We extend our series techniques to the case of a regular singular point in the next section.

## 4.4 frobenius method

We consider the problem $y'' + Py' + Qy = 0$ with a regular singular point $x_o$. We can study the case $x_o = 0$ without loss of generality since the subsitution $x = t - a$ moves the regular singular point to $t = a$. For example:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \Leftrightarrow \quad (t-a)^2\frac{d^2z}{dt^2} + (t-a)\frac{dz}{dt} + z = 0$$

Where $z(t) = y(x+a)$ and $y(x) = z(t-a)$. Therefore, we focus our efforts on the problem

$$\boxed{y'' + Py' + Qy = 0 \ \text{ a singular DEqn at } x = 0 \text{ with } xP(x), x^2Q(x) \text{ analytic at } x = 0}$$

Let us make some standard notation for the taylor expansions of $xP(x)$ and $x^2Q(x)$. Suppose

$$P(x) = \frac{P_o}{x} + P_1 + P_2x^2 + \cdots \qquad \& \qquad Q(x) = \frac{Q_o}{x^2} + \frac{Q_1}{x} + Q_2 + Q_3x + \cdots$$

The extended Talyor series above are called **Laurent series**, they contain finitely many nontrivial reciprocal power terms. In the langauge of complex variables the pole $x = 0$ is removeable for $P$ and $Q$ where it is of order 1 and 2 respectively. Note we remove the singularity by multiplying by $x$ and $x^2$:

$$xP(x) = P_o + P_1x + xP_2x^3 + \cdots \qquad \& \qquad x^2Q(x) = Q_o + Q_1x + Q_2x^2 + Q_3x^3 + \cdots.$$

This must happen by the definition of a **regular singular point**.

**Theorem 4.4.1.** *frobenius solution at regular singular point*

There exists a number $r$ and coefficients $a_n$ such that $y'' + Py' + Qy = 0$ has solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

See Rabenstein for greater detail as to why this solution exists. We can denote $y(r,x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ if we wish to emphasize the dependence on $r$. Formally[2] it is clear that

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \ \ \& \ \ y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \ \ \& \ \ y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

Notice that we make no assumption that $r = 0, 1, 2, \dots$ hence $y(r,x)$ is not necessarily a power series. The frobenius solution is more general than a simple power series. Let us continue to plug in the formulas for $y, y', y''$ into $x^2y'' + x^2Py' + x^2Qy = 0$:

$$\begin{aligned} 0 = x^2 &\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} \\ &+ \Big(P_o + P_1x + xP_2x^3 + \cdots\Big)x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \\ &+ \Big(Q_o + Q_1x + Q_2x^2 + Q_3x^3 + \cdots\Big)\sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned}$$

[2] formal in the sense that we ignore questions of convergence

Hence, (call this $\star$ for future reference)

$$
\begin{aligned}
0 = \sum_{n=0}^{\infty} & a_n(n+r)(n+r-1)x^{n+r} \\
& + \Big(P_o + P_1 x + xP_2 x^3 + \cdots\Big)\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} \\
& + \Big(Q_o + Q_1 x + Q_2 x^2 + Q_3 x^3 + \cdots\Big)\sum_{n=0}^{\infty} a_n x^{n+r}
\end{aligned}
$$

You can prove that $\{x^r, x^{r+1}, x^{r+2}, \dots\}$ is a linearly independent set of functions on appropriate intervals. Therefore, $y(r,x)$ is a solution iff we make each coefficient vanish in the equation above. We begin by examining the $n = 0$ terms which are the coefficient of $x^r$:

$$a_o(0+r)(0+r-1) + P_o a_o(0+r) + Q_o a_o = 0$$

This gives no condition on $a_o$, but we see that $r$ must be chosen such that

$$\boxed{r(r-1) + rP_o + Q_o = 0} \qquad \text{the } \textbf{indicial} \text{ equation}$$

We find that we must begin the Frobenius problem by solving this equation. We are not free to just use any $r$, a particular pair of choices will be dictated from the zeroth coefficients of the $xP$ and $x^2Q$ Taylor expansions. Keeping in mind that $r$ is not free, let us go on to describe the next set of equations from the coefficient of $x^{r+1}$ of $\star$ $(n = 1)$,

$$a_1(1+r)r + (1+r)P_o a_1 + rP_1 a_o + Q_o a_1 + Q_1 a_o = 0$$

The equation above links $a_o$ to $a_1$. Next, for $x^{r+2}$ in $\star$ we need

$$a_2(2+r)(r+1) + (2+r)P_o a_2 + (1+r)P_1 a_1 + rP_2 a_o + Q_o a_2 + Q_1 a_1 + Q_o a_2 = 0$$

The equation above links $a_2$ to $a_1$ and $a_o$. In practice, for a given problem, the recurrence relations which define $a_k$ are best derived directly from $\star$. I merely wish to indicate the general pattern[3] with the remarks above.

**Example 4.4.2. Problem:** *solve* $3xy'' + y' - y = 0$.

**Solution:** *Observe that* $x_o = 0$ *is a regular singular point. Calculate,*

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

*and*

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \qquad \& \qquad 3xy'' = \sum_{n=0}^{\infty} 3a_n(n+r)(n+r-1)x^{n+r-1}$$

[3]if one wishes to gain a deeper calculational dexterity with this method I highly reccommend the sections in Rabenstein, he has a few techniques which are superior to the clumsy calculations I perform here

*Therefore,* $3xy'' + y' - y = 0$ *yields*

$$a_o[3r(r-1)+r]x^r + \sum_{n=1}^{\infty}\Big(3a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1}\Big)x^{n+r-1}$$

*Hence, for* $n = 1, 2, 3, \dots$ *we find:*

$$3r(r-1) + r = 0 \qquad \& \qquad \boxed{a_n = \frac{a_{n-1}}{(n+r)(3n+3r-2)}}. \ \star$$

*The indicial equation* $3r(r-1)+r = 3r^2 - 2r = r(3r-2) = 0$ *gives* $r_1 = 2/3$ *and* $r_2 = 0$. *Suppose* $r_1 = 2/3$ *and work out the recurrence relation* $\star$ *in this context:* $a_n = \frac{a_{n-1}}{n(3n+2)}$ *thus:*

$$a_1 = \frac{a_o}{5}, \quad a_2 = \frac{a_1}{8\cdot 2} = \frac{a_o}{8\cdot 5\cdot 2}, \quad a_3 = \frac{a_2}{11\cdot 3} = \frac{a_o}{11\cdot 8\cdot 5\cdot 3\cdot 2}$$

$$a_4 = \frac{a_3}{14\cdot 4} = \frac{a_o}{14\cdot 11\cdot 8\cdot 5\cdot 4\cdot 3\cdot 2} \quad \Rightarrow \quad a_n = \frac{a_o}{5\cdot 8\cdot 11\cdot 14\cdots(3n+2)n!} \quad (n = 1, 2, \dots)$$

*Therefore,* $y(2/3, x) = a_o\big(x^{2/3} + \sum_{n=1}^{\infty}\frac{x^{n+2/3}}{5\cdot 8\cdot 11\cdot 14\cdots(3n+2)n!}\big)$ *is a solution. Next, work out the recurrence relation* $\star$ *in the* $r_2 = 0$ *case:* $a_n = \frac{a_{n-1}}{n(3n-2)}$ *thus:*

$$a_1 = \frac{a_o}{1}, \quad a_2 = \frac{a_1}{2\cdot 4} = \frac{a_o}{2\cdot 4}, \quad a_3 = \frac{a_2}{3\cdot 7} = \frac{a_o}{7\cdot 4\cdot 3\cdot 2}$$

$$a_4 = \frac{a_3}{4\cdot 10} = \frac{a_o}{10\cdot 7\cdot 4\cdot 4\cdot 3\cdot 2} \quad \Rightarrow \quad a_n = \frac{a_o}{4\cdot 7\cdot 10\cdots(3n-2)n!} \quad (n = 2, 3, \dots)$$

*Consequently,* $y(0, x) = a_o\big(1 + x + \sum_{n=2}^{\infty}\frac{x^n}{4\cdot 7\cdot 10\cdots(3n-2)n!}\big)$. *We find the general solution*

$$\boxed{y = c_1\left(x^{2/3} + \sum_{n=1}^{\infty}\frac{x^{n+2/3}}{5\cdot 8\cdot 11\cdot 14\cdots(3n+2)n!}\right) + c_2\left(1 + x + \sum_{n=2}^{\infty}\frac{x^n}{4\cdot 7\cdot 10\cdots(3n-2)n!}\right)}.$$

**Remark 4.4.3.**

Before we try another proper example I let us apply the method of Frobenius to a Cauchy Euler problem. The Cauchy Euler problem $x^2y'' + Pxy' + Qy = 0$ has $P_o = P$ and $Q_o = Q$. Moreover, the characteristic equation $r(r-1) + rP_o + Q_o = 0$ is the indicial equation. In other words, the regular singular point problem is a generalization of the Cauchy Euler problem. In view of this you can see our discussion thus far is missing a couple cases: (1.) the repeated root case needs a natural log, (2.) the complex case needs the usual technique. It turns out there is another complication. When $r_1, r_2$ are the exponents with $Re(r_1) > Re(r_2)$ and $r_1 - r_2$ is a positive integer we sometimes need a natural log term.

**Example 4.4.4. Problem:** *solve* $x^2y'' + 3xy' + y = 0$.

**Solution:** *Observe that* $y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$ *has regular singular point* $x_o = 0$ *and* $P_o = 3$ *whereas* $Q_o = 1$. *The indicial equation* $r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0$ *gives* $r_1 = r_2 = -1$. *Suppose* $y = y(-1, x) = \sum_{n=0}^{\infty} a_n x^{n-1}$. *Plugging* $y(-1, x)$ *into* $x^2y'' + 3xy' + y = 0$ *yields:*

$$\sum_{n=0}^{\infty}(n-1)(n-2)a_n x^{n-1} + \sum_{n=0}^{\infty} 3(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n-1} = 0$$

*Collecting like powers is simple for the expression above, we find:*

$$\sum_{n=0}^{\infty}\Big((n-1)(n-2)a_n + 3(n-1)a_n + a_n\Big)x^{n-1} = 0$$

*Hence* $[(n-1)(n-2) + 3(n-1) + 1]a_n = 0$ *for* $n = 0, 1, 2, \dots$. *Put* $n = 0$ *to obtain* $0a_o = 0$ *hence no condition for* $a_o$ *is found. In contrast, for* $n \geq 1$ *the condition yields* $a_n = 0$. *Thus* $y(-1, x) = a_o x^{-1}$. *Of course, you should have expected this from the outset! This is a Cauchy Euler problem, we expect the general solution* $y = c_1\frac{1}{x} + c_2\frac{ln(x)}{x}$.

We examine a solution with imaginary exponents.

**Example 4.4.5. Problem:** *solve* $x^2y'' + xy' + (4-x)y = 0$.

**Solution:** *Observe that* $x_o = 0$ *is a regular singular point. Calculate, if* $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ *then*

$$(4-x)y = \sum_{n=0}^{\infty} 4a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=0}^{\infty} 4a_n x^{n+r} - \sum_{j=1}^{\infty} a_{j-1} x^{j+r}$$

*and*

$$xy' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} \qquad \& \qquad x^2y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r}$$

*Therefore,* $x^2y'' + xy' + (4-x)y = 0$ *yields*

$$a_o[r(r-1) + r + 4]x^r + \sum_{n=1}^{\infty}\Big(a_n(n+r)(n+r-1) + a_n(n+r) + 4a_n - a_{n-1}\Big)x^{n+r}$$

*Hence, for* $n = 1, 2, 3, \dots$ *we find:*

$$r^2 + 4 = 0 \qquad \& \qquad \boxed{a_n = \frac{a_{n-1}}{(n+r)^2 + 4}}. \quad \star$$

*The indicial equation* $r^2 + 4 = 0$ *gives* $r_1 = 2i$ *and* $r_2 = -2i$. *We study* $\star$ *in a few cases. Let me begin by choosing* $r = 2i$. *Let's reformulate* $\star$ *into a cartesian form:*

$$a_n = \frac{a_{n-1}}{(n+2i)^2 + 4} = \frac{a_{n-1}}{n^2 + 4ni - 4 + 4} = \frac{a_{n-1}}{n^2 + 4ni} \cdot \frac{n^2 - 4ni}{n^2 - 4ni} = \frac{a_{n-1}(n^2 - 4ni)}{n^2(n^2 + 16)} \quad \star^2$$

*Consider then, by* $\star^2$

$$a_1 = \frac{a_o(1-4i)}{17}, \qquad a_2 = \frac{a_1(4-8i)}{4(4+16)} = \frac{a_o(1-4i)}{17} \cdot \frac{4-8i}{80} = \frac{-a_o(28+12i)}{(17)(80)}$$

*Consequently, we find the complex solution are*

$$\begin{aligned} y &= a_o\left(x^{2i} + \frac{1-4i}{17}x^{1+2i} - \frac{28+12i}{(17)(80)}x^{2+2i} + \cdots\right) \\ &= a_o x^{2i}\Bigg(\underbrace{1 + \frac{1}{17}x - \frac{28}{(17)(80)}x^2 + \cdots}_{a(x)} + i\underbrace{\left[\frac{-4}{17}x - \frac{12}{(17)(80)}x^2 + \cdots\right]}_{b(x)}\Bigg) \end{aligned}$$

*Recall, for $x > 0$ we defined $x^{n+2i} = x^n[\cos(2\ln(x)) + i\sin(2\ln(x))]$. Therefore,*

$$y = a_o\Big[\cos(2\ln(x))a(x) - \sin(2\ln(x))b(x)\Big] + ia_o\Big[\sin(2\ln(x))a(x) + \cos(2\ln(x))b(x)\Big]$$

*forms the general complex solution. Set $a_o = 1$ to select the real fundmental solutions $y_1 = Re(y)$ and $y_2 = Im(y)$. The general real solution is $y = c_1y_1 + c_2y_2$. In particular,*

$$\boxed{y = c_1\Big[\cos(2\ln(x))a(x) - \sin(2\ln(x))b(x)\Big] + c_2\Big[\sin(2\ln(x))a(x) + \cos(2\ln(x))b(x)\Big]}$$

*We have made manifest the first few terms in $a$ and $b$, it should be clear how to find higher order terms through additional iteration on $\star^2$. The proof that these series converge can be found in more advanced sources (often Ince is cited by standard texts).*

**Remark 4.4.6.**

The calculation that follows differs from our initial example in one main aspect. I put in the exponents before I look for the recurrence relation. It turns out that the method of Example 4.4.2 is far more efficient a method of calculation. I leave this slightly clumsy calculation to show you the difference. You should use the approach of Example 4.4.2 for brevity's sake..

**Example 4.4.7. Problem:** *solve $xy'' + (3 + x^2)y' + 2xy = 0$.*

**Solution:** *Observe $y'' + (3/x + x)y' + 2y = 0$ thus identify that $P_o = 3$ whereas $Q_o = 0$. The indicial equation $r(r-1) + 3r = 0$ yields $r(r+2) = 0$ thus the* **exponents** *are $r_1 = 0, r_2 = -2$. In order to find the coefficients of $y(0, x) = y = \sum_{n=0}^{\infty} a_n x^n$ we must plug this into $xy'' + 3y' + x^2y' + 2xy = 0$,*

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 3na_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

*Examine these summations and note that $x^1, x^0, x^2, x^1$ are the lowest order terms respectively from left to right. To combine these we will need to start with $x^2$-terms.*

$$\begin{aligned} 0 &= 2a_2x + 3a_1 + 6a_2x + 2a_ox \\ &\quad + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=3}^{\infty} 3na_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=1}^{\infty} 2a_n x^{n+1} = 0 \end{aligned}$$

*Let $j = n - 1$ for the first two sums and let $j = n + 1$ for the next two sums.*

$$0 = 3a_1 + (2a_2 + 6a_2 + 2a_o)x + \sum_{j=2}^{\infty}\Big((j+1)ja_{j+1} + 3(j+1)a_{j+1} + (j-1)a_{j-1} + 2a_{j-1}\Big)x^j$$

*Collecting like terms we find:*

$$0 = 3a_1 + (8a_2 + 2a_o)x + \sum_{j=2}^{\infty}\Big((j+3)(j+1)a_{j+1} + (j+1)a_{j-1}\Big)x^j.$$

*Each power's coefficient must separately vanish, therefore:*

$$a_1 = 0, \qquad a_2 = -\frac{1}{4}a_o, \qquad a_{j+1} = \frac{-1}{j+3}a_{j-1}, \qquad \Rightarrow \qquad a_n = \frac{-1}{n+2}a_{n-2} \ \text{for } n \geq 2.$$

*It follows that* $a_{2k+1} = 0$ *for* $k = 0, 1, 2, 3, \ldots$. *However, the even coefficients are determined by the recurrence relation given above.*

$$a_2 = \frac{-1}{4}a_o$$

$$a_4 = \frac{-1}{6}a_2 = \frac{-1}{6}\cdot\frac{-1}{4}a_o$$

$$a_6 = \frac{-1}{8}a_4 = \frac{-1}{8}\cdot\frac{-1}{6}\cdot\frac{-1}{4}a_o$$

$$a_{2k} = \frac{-1}{2k+2}\cdot\frac{-1}{2k}\cdots\frac{-1}{6}\cdot\frac{-1}{4}a_o = \frac{(-1)^k}{2^k k!}a_o$$

*Therefore, we find the solution:*

$$\boxed{y(0,x) = a_o\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k(k+1)!}x^{2k}}$$

*We know from the theory we discussed in previous chapters the general solution should have the form* $y = c_1y_1 + c_2y_2$ *where* $\{y_1, y_2\}$ *is the fundamental solution set. We have found half of the solution at this point; identify* $y_1 = y(0,x)$. *In contrast to the series method, we found just one of the fundamental solutions.*

*To find* $y_2$ *we must turn our attention to the second solution of the indicial equation* $r_2 = -2$. *We find the coefficients of* $y(-2,x) = y = \sum_{n=0}^{\infty} a_n x^{n-2}$ *by plugging it into* $xy'' + 3y' + x^2y' + 2xy = 0$,

$$\sum_{n=0}^{\infty}(n-2)(n-3)a_nx^{n-3} + \sum_{n=0}^{\infty}3(n-2)a_nx^{n-3} + \sum_{n=0}^{\infty}(n-2)a_nx^{n-1} + \sum_{n=0}^{\infty}2a_nx^{n-1} = 0$$

1. $x^{-3}$ *has coefficient* $(-2)(-3)a_o + 3(-2)a_o = 0$ *(no condtion found)*
2. $x^{-2}$ *has coefficient* $(1-2)(1-3)a_1 + 3(1-2)a_1 = -a_1$ *hence* $a_1 = 0$
3. $x^{-1}$ *has coefficient* $(2-2)(2-3)a_2 + 3(2-2)a_2 + (0-2)a_o + 2a_o = 0$ *(no condtion found)*
4. $x^0$ *has coefficient* $(3-2)(3-3)a_3 + 3(3-2)a_3 + (1-2)a_1 + 2a_1 = 3a_3$. *Thus* $a_3 = 0$
5. $x^1$ *has coefficient* $(4-2)(4-3)a_4 + 3(4-2)a_4 + (2-2)a_2 + 2a_2 = 8a_4 + 2a_2$. *Thus* $a_4 = \frac{-1}{4}a_2$.
6. $x^2$ *has coefficient* $(5-2)(5-3)a_5 + 3(5-2)a_5 + (3-2)a_3 + 2a_3 = 15a_5 + 3a_3$. *Thus* $a_5 = \frac{-1}{5}a_3$. *We find* $a_{2k-1} = 0$ *for all* $k \in \mathbb{N}$.
7. $x^3$ *has coefficient* $(6-2)(6-3)a_6 + 3(6-2)a_6 + (4-2)a_4 + 2a_4 = 24a_6 + 4a_4$. *Thus* $a_6 = \frac{-1}{6}a_4$.

*This pattern should be recongnized from earlier in this problem. For $a_2, a_4, a_6, \dots$ we find terms*

$$a_2 - a_2\tfrac{1}{4}x^2 + a_2\tfrac{1}{4}\cdot\tfrac{1}{6}x^4 + \cdots = a_2(1 - \tfrac{1}{2^1 2!}x^2 + \tfrac{1}{2^2 3!}x^4 + \cdots)$$

*recognize this is simply a relabeled version of $y(0,x)$ hence we may set $a_2 = 0$ without loss of generality in the general solution. This means only $a_o$ remains nontrivial. Thus,*

$$\boxed{y(-2,x) = a_o x^{-2}}$$

*The general solution follows,*

$$\boxed{y = c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(k+1)!}x^{2k} + \frac{c_2}{x^2}}$$

**Remark 4.4.8.**

In the example above the exponents $r_1 = 0$ and $r_2 = -2$ have $r_1 - r_2 = 2$. It turns out that generally the solution $y(r_2, x)$ will not be a solution. A modification involving a logarithm is needed sometimes (but not in the example above!).

### 4.4.1 the repeated root technique

In the case that a characteristic root is repeated we have seen the need for special techniques to derive a second LI solution. I present a new idea, yet another way to search for such double-root solutions. Begin by observing that the double root solutions are connected to the first solution by differentiation of the characteristic value:

$$\frac{\partial}{\partial r}e^{rx} = te^{rx}, \qquad \& \qquad \frac{\partial}{\partial r}x^r = \ln(x)x^r.$$

Rabenstein gives a formal derivation of why $\frac{\partial x^r}{\partial r}\Big|_{r=r_1}$ solves a Cauchy Euler problem with repeated root $r_1$. I'll examine the corresponding argument for the repeated root case $(D^2 - 2\lambda_1 D + \lambda_1^2)[y] = L[y] = 0$. Suppose $y(\lambda, x) = e^{\lambda x}$. Note that:

$$L[e^{\lambda x}] = (\lambda^2 - 2\lambda_1\lambda + \lambda_1^2)e^{\lambda x} = (\lambda - \lambda_1)^2 e^{\lambda x}$$

Obviously $y_1 = y(\lambda_1, x)$ solves $L[y] = 0$. Consider $y_2 = \frac{\partial}{\partial\lambda}y(\lambda,x)\big|_{\lambda=\lambda_1}$

$$\begin{aligned} L[y_2] = L\left[\frac{\partial}{\partial\lambda}y(\lambda,x)\bigg|_{\lambda=\lambda_1}\right] &= \frac{\partial}{\partial\lambda}\Big[L[y(\lambda,x)]\Big]\bigg|_{\lambda=\lambda_1} \\ &= \frac{\partial}{\partial\lambda}\Big[(\lambda-\lambda_1)^2 e^{\lambda x}\Big]\bigg|_{\lambda=\lambda_1} \\ &= \Big[2(\lambda-\lambda_1)e^{\lambda x} + (\lambda-\lambda_1)^2 x e^{\lambda x}\Big]\bigg|_{\lambda=\lambda_1} \\ &= 0. \end{aligned}$$

Suppose that we face $x^2y'' + Px^2y' + x^2Q = 0$ which has an indicial equation with repeated root $r_1$. Suppose[4] $y(r,x) = x^r\sum_{n=0}^{\infty}a_n(r)x^n$ is a solution $x^2y'' + Px^2y' + x^2Q = 0$ when we set $r = r_1$. It can be shown[5] that

[4] I write the $x^r$ in front and emphasize the $r$-dependence of the $a_n$ coefficients as these are crucial to what follows, if you examine the previous calculations you will discover that $a_n$ does dependend on the choice of exponent

[5] see Rabenstein page 120

$y_2 = \frac{\partial y(r,x)}{\partial r}\Big|_{r=r_1}$ solves $x^2y'' + Px^2y' + x^2Q = 0$. Consider,

$$\frac{\partial y(r,x)}{\partial r} = \frac{\partial}{\partial r}\left[x^r \sum_{n=0}^{\infty} a_n(r)x^n\right] = \ln(x)x^r \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=0}^{\infty} a_n'(r)x^n$$

Setting $r = r_1$ and denoting $y_1(x) = y(r_1, x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$ we find the second solution

$$\boxed{y_2(x) = \ln(x)y_1(x) + x^{r_1} \sum_{n=0}^{\infty} a_n'(r_1)x^n.}$$

Compare this result to Theorem 7 of section 8.7 in Nagel Saff and Snider to appreciate the beauty of this formula. If we calculate the first solution then we find the second by a little differentiation and evaluation at $r_1$.

**Example 4.4.9.** *include example showing differentiation of $a_n(r)$ (to be given in lecture most likely)*

Turn now to the case $x^2y'' + Px^2y' + x^2Q = 0$ has exponents $r_1, r_2$ such that $r_1 - r_2 = N \in \mathbb{N}$. Following Rabenstein once more I examine the general form of the recurrence relation that formulates the coefficients in the Frobenius solution. We will find that for $r_1$ the coefficients exist, however for $r_2$ there exist $P, Q$ such that the recurrence relation is insolvable. We seek to understand these features.

**Remark 4.4.10.**

Sorry these notes are incomplete. I will likely add comments based on Rabenstein in lecture, his treatment of Frobenius was more generous than most texts at this level. In any event, you should remember these notes are a work in progress and you are welcome to ask questions about things which are not clear.

# Chapter 5

# systems of ordinary differential equations

A system of ordinary differential equations is precisely what is sounds like; a system of ODEs is several ODEs which share dependent variables and a single independent variable. In this chapter we learn the standard terminology for such problems and we study two strategies to solve such problems quantitatively. In a later chapter we will study the phase plane method which gives us a qualitative method which is readily tenable for the $n = 2$ problem.

The operator method has a natural implementation for systems with constant coefficients. We'll see how this approach allows us to extend the spring/mass problem to problems with several springs coupled together, or for RLC-circuits with several loops which likewise couple. Some of these examples are not in the typed notes, but I will present them in lecture.

The operator method is hard to beat in many respects, however, linear algebra offers another approach which is equally general and allows generalization to other fields of study. Focusing on the constant coefficient case it turns out the system of ODEs $\frac{d\vec{x}}{dt} = A\vec{x}$ has solution $\vec{x} = e^{tA}\vec{c}$. In the case the matrix $A$ is diagonalizable the method simplifies greatly and we begin in that simple case by discussing the e-vector-type solutions. As usual the solution is sometimes complex and in the event of that complex algebra we have to select real and imaginary parts to derive the real solution.

In the case $A$ is not diagonalizable we need a deeper magic. The chains of generalized e-vectors bind the solutions and force them to do our bidding via the magic formula.

Finally the nonhomogeneous case is once more solved by variation of parameters. On the other hand, we do not attempt to say much about systems with variable coefficients.

## 5.1 operator methods for systems

The method of this section applies primarily to systems of constant coefficient ODEs[1]. Generally the approach is as follows:

1. consider writing the given ODEs in operator notation
2. add, operate, substract in whatever combination will reduce the problem to one-dependent variable
3. solve by the usual characteristic equation method
4. find other dependent variable solutions as the algebra or operators demand.
5. resolve any excess constants by making use of the given differential relations and/or applying initial conditions.

It's best to illustrate this method by example.

**Example 5.1.1. Problem:** *Solve* $x' = -y$ *and* $y' = x$.

**Solution:** *note* $Dx = -y$ *and* $Dy = x$ *thus* $D^2x = -Dy = -x$. *Therefore,* $(D^2+1)[x] = 0$ *and we find the solution* $\boxed{x(t) = c_1\cos(t) + c_2\sin(t)}$. *Note that* $y = -Dx$ *thus the remaining solution is* $\boxed{y = c_1\sin(t) - c_2\cos(t)}$.

**Example 5.1.2. Problem:** *Solve* $x' = x - y$ *and* $y' = x + y$.

**Solution:** *note* $Dx = x - y$ *and* $Dy = x + y$. *Notice this gives*

$$(D-1)x = -y \qquad \& \qquad (D-1)y = x$$

*Operate by* $D-1$ *to obtain* $(D-1)^2x = -(D-1)y = -x$. *Thus,*

$$(D^2 - 2D + 1)[x] = -x \quad \Rightarrow \quad (D^2 - 2D + 2)[x] = 0 \quad \Rightarrow \quad ((D-1)^2 + 1)[x] = 0$$

*Therefore,* $\boxed{x(t) = c_1e^t\cos(t) + c_2e^t\sin(t)}$. *Calculate,*

$$Dx = c_1e^t(\cos(t) - \sin(t)) + c_2e^t(\sin(t) + \cos(t))$$

*Consqeuently,* $y = -(D-1)x = x - Dx$ *yields*

$$\boxed{y(t) = c_1e^t\sin(t) - c_2e^t\cos(t).}$$

**Example 5.1.3. Problem:** *Solve* $\ddot{x} + 5x - 4y = 0$ *and* $\ddot{y} + 4y - x = 0$.

**Solution:** *As operator equations we face*

$$(D^2 + 3)x - 2y = 0$$

$$(D^2 + 2)y - x = 0$$

[1] if you could express the system as a polynomials in a particular smooth differential operator then the idea would generalize to that case

*Operate by $D^2+3$ on the second equation to derive $(D^2+3)x = (D^2+3)(D^2+2)y$. Substituting into the first equation gives a fourth order ODE for $y$,*

$$(D^2+3)(D^2+2)y - 2y = 0.$$

*Hence, $(D^4+5D^2+4)y = 0$ which gives*

$$(D^2+1)(D^2+4)y = 0$$

*Therefore,*

$$\boxed{y(t) = c_1\cos(t) + c_2\sin(t) + c_3\cos(2t) + c_4\sin(2t)}$$

*Note that $D^2y = -c_1\cos(t) - c_2\sin(t) - 4c_3\cos(2t) - 4c_4\sin(2t)$. Note $x = (D^2+2)y$ thus we find*

$$\boxed{x(t) = c_1\cos(t) + c_2\sin(t) - 2c_3\cos(2t) - 2c_4\sin(2t).}$$

We can also cast the solutions above in a more physically useful notation:

$$y(t) = A_1\cos(t+\phi_1) + A_2\cos(2t+\phi_2) \qquad \& \qquad x(t) = A_1\cos(t+\phi_1) - 2A_2\cos(2t+\phi_2)$$

You can see there are two modes in the solution above. One mode has angular frequency $\omega_1 = 1$ whereas the second has angular frequency $\omega_2 = 2$. Motions of either frequency are possible ($A_1 \neq 0$ and $A_2 = 0$ or vice-versa) however, more generally the motion is a superposition of those two motions. This type of system can arise from a system of coupled springs without damping or in a coupled pair of LC circuits. Naturally, those are just the examples we've already discussed, the reader is invited to find other applications.

**Example 5.1.4. Problem:** *Solve $x' + y' = 2t$ and $y'' - x' = 0$.*

**Solution:** *We have $Dx + Dy = 2t$ and $D^2y - Dx = 0$. Operate by $D^2$ to obtain:*

$$D^3x + D^3y = D^2[2t] = 0.$$

*Note that $Dx = D^2y$ hence we find by substitution:*

$$D^2D^2y + D^3y = 0 \quad \Rightarrow \quad D^3(D+1)[y] = 0.$$

*Therefore, $y = c_1 + c_2t + c_3t^2 + c_4e^{-t}$. To find $x$ we should solve $Dx = D^2y$:*

$$Dx = 2c_3 + c_4e^{-t} \quad \Rightarrow \quad x(t) = 2c_3t - c_4e^{-t} + c_5$$

*Let us apply the given nonhomogeneous DEqn to refine these solutions:*

$$x' + y' = (c_2 + 2c_3t - c_4e^t) + (2c_3 + c_4e^t) = 2t$$

*Equating coefficients yield $c_2 + 2c_3 = 0$ and $2c_3 = 2$ thus $c_3 = 1$ and $c_2 = -2$. We find,*

$$x(t) = 2t - c_4e^{-t} + c_5, \qquad \& \qquad y(t) = c_1 - 2t + t^2 + c_4e^{-t}.$$

*Finally, we should check that $y'' - x' = 0$*

$$(c_1 - 2t + t^2 + c_4e^{-t})'' - (2t - c_4e^{-t} + c_5)' = 2 + c_4e^{-t} - 2 - c_4e^{-t} = 0$$

*Thus,*

$$\boxed{x(t) = 2t + 2e^{-t} + c_5, \qquad \& \qquad y(t) = c_1 - 2t + t^2 - 2e^{-t}}.$$

**Example 5.1.5. Problem:** *Solve $x'' + y = \cos(t)$ and $y'' + x = \sin(t)$.*

**Solution:** *sometimes a problem allows immediate substitution. Here $x = \sin(t) - y''$ hence*

$$(\sin(t) - y'')'' + y = \cos(t)$$

*Twice differentiating* $\sin(t)$ *yields* $-\sin(t)$ *thus* $-\sin(t) - y^{(4)} + y = \cos(t)$ *hence*

$$(D^4 - 1)[y] = -\sin(t) - \cos(t)$$

*Note that $D^4 - 1 = (D^2 + 1)(D^2 - 1)$ and the operator $D^2 + 1$ also annihilates the forcing term above. We find:*

$$(D^2 + 1)^2(D^2 - 1)[y] = 0$$

*Hence, $y(t) = (c_1 + tc_2)\cos(t) + (c_3 + tc_4)\sin(t) + c_5\cosh(t) + c_6\sinh(t)$. Calculate $y''$;*

$$y'' = (c_2 + c_3 + tc_4)\cos(t) + (c_4 - c_1 - tc_2)\sin(t) + c_5\cosh(t) + c_6\sinh(t)$$

*Consequently, $x = \sin(t) - y''$ gives*

$$x(t) = (-c_2 - c_3 - tc_4)\cos(t) + (1 - c_4 + c_1 + tc_2)\sin(t) - c_5\cosh(t) - c_6\sinh(t)$$

*Differentiate this to find*

$$x' = (-2c_4 + 1 + c_1 + tc_2)\cos(t) + (2c_2 + c_3 + tc_4)\sin(t) - c_5\sinh(t) - c_6\cosh(t)$$

*Differentiate once more,*

$$x'' = (3c_2 + c_3 + tc_4)\cos(t) + (3c_4 - 1 - c_1 - tc_2)\sin(t) - c_5\sinh(t) - c_6\cosh(t)$$

*Add $y(t) = (c_1 + tc_2)\cos(t) + (c_3 + tc_4)\sin(t) + c_5\cosh(t) + c_6\sinh(t)$ and set $x'' + y = \cos(t)$,*

$$(3c_2 + c_3 + tc_4 + c_1 + tc_2)\cos(t) + (3c_4 - 1 - c_1 - tc_2 + c_3 + tc_4)\sin(t) = \cos(t)$$

*It follows that $1 + 2c_1 = 0$ and $c_2 = c_4 = c_3 = 0$ hence $c_1 = -1/2$ and (***ERROR HERE***)*

$$\boxed{x(t) = \frac{1}{2}\sin(t) - c_5\cosh(t) - c_6\sinh(t) \qquad \& \qquad y(t) = -\frac{1}{2}\cos(t) + c_5\cosh(t) + c_6\sinh(t)}.$$

I will not seek to offer general advice on this method. If you would like a little more structure on this topic I invite the reader to consult Nagel Saff and Snider section 5.2 (pages 263-270 in the 5th edition).

## 5.2 calculus and matrices

The construction of matrices and the operations thereof are designed to simplify arguments about algebraic systems of linear equations. We will see that the matrix is also of great utitility for the solution of systems of linear differential equations. We've already seen how matrix calculations unify and simplify with the theory of the Wronskian and the technique of variation of parameters. I now pause to introduce and define explicitly the algebra and construction of matrices and we also derive some important theorems about their calculus.

A $p \times q$ **matrix** over $R$ is an ordered array of $pq$-objects from $R$ which has $p$-**rows** and $q$-**columns**. The objects in the matrix are called its **components**. In particular, if matrix $A$ has components $A_{ij} \in R$ for $i, j$ with $1 \leq i \leq p$ and $1 \leq j \leq q$ then we denote the array by:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \cdots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix} = [A_{ij}]$$

We also view a matrix as columns or rows glued together:

$$A = \big[ col_1(A) | col_2(A) | \cdots | col_q(A) \big] = \left[ \begin{array}{c} row_1(A) \\ \hline row_2(A) \\ \hline \vdots \\ \hline row_p(A) \end{array} \right]$$

where we define $col_j(A) = [A_{1j}, A_{2j}, \dots, A_{pj}]^T$ and $row_i(A) = [A_{i1}, A_{i2}, \dots, A_{iq}]$. The set of all $p \times q$ matrices assembled from objects in $R$ is denoted $R^{p \times q}$. Notice that if $A, B \in R^{p \times q}$ then $A = B$ iff $A_{ij} = B_{ij}$ for all $i, j$ with $1 \leq i \leq p$ and $1 \leq j \leq q$. In other words, two matrices are **equal** iff all the matching components are equal. We use this principle in many definitions, for example: if $A \in R^{p \times q}$ then the **transpose** $A^T \in R^{q \times p}$ is defined by $A^T_{ij} = A_{ji}$ for all $i, j$.

We are primarily interested in the cases $R = \mathbb{R}, \mathbb{C}$ or some suitable set of functions. All of these spaces allow for addition and multiplication of the components. It is therefore logical to define the sum, difference, scalar multiple and product of matrices as follows:

**Definition 5.2.1.** *If $A, B \in R^{p \times q}$ and $C \in R^{q \times r}$ and $c \in R$ then define*

$$(A + B)_{ij} = A_{ij} + B_{ij} \qquad (A - B)_{ij} = A_{ij} - B_{ij} \qquad (cA)_{ij} = cA_{ij} \qquad (BC)_{ik} = \sum_{j=1}^{q} B_{ik} C_{kj}.$$

This means that $(A + B), (A - B), cA \in R^{p \times q}$ whereas $BC \in R^{p \times r}$. The **matrix product** of a $p \times q$ and $q \times r$ matrix is a $p \times r$ matrix. In order for the product $BC$ to be defined we must have the rows in $B$ be the same size as the columns in $C$. We can express the product in terms of dot-products:

$$\boxed{(BC)_{ik} = row_i(B) \bullet col_k(C)}$$

Let me give a few examples to help you understand these formulas.

**Example 5.2.2.** *The product of a $3 \times 2$ and $2 \times 3$ is a $3 \times 3$*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} [1,0] \bullet [4,7] & [1,0] \bullet [5,8] & [1,0] \bullet [6,9] \\ [0,1] \bullet [4,7] & [0,1] \bullet [5,8] & [0,1] \bullet [6,9] \\ [0,0] \bullet [4,7] & [0,0] \bullet [5,8] & [0,0] \bullet [6,9] \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

**Example 5.2.3.** *The product of a $3 \times 1$ and $1 \times 3$ is a $3 \times 3$*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 5 \cdot 1 & 6 \cdot 1 \\ 4 \cdot 2 & 5 \cdot 2 & 6 \cdot 2 \\ 4 \cdot 3 & 5 \cdot 3 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

**Example 5.2.4.** *Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. We calculate*

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} [1,2] \bullet [5,7] & [1,2] \bullet [6,8] \\ [3,4] \bullet [5,7] & [3,4] \bullet [6,8] \end{bmatrix} \\ &= \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

*Notice the product of square matrices is square. For numbers $a, b \in \mathbb{R}$ it we know the product of $a$ and $b$ is commutative ($ab = ba$). Let's calculate the product of $A$ and $B$ in the opposite order,*

$$\begin{aligned} BA &= \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} [5,6] \bullet [1,3] & [5,6] \bullet [2,4] \\ [7,8] \bullet [1,3] & [7,8] \bullet [2,4] \end{bmatrix} \\ &= \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \end{aligned}$$

*Clearly $AB \neq BA$ thus matrix multiplication is* **noncommutative** *or* **nonabelian**.

When we say that matrix multiplication is noncommuative that indicates that the product of two matrices does not *generally* commute. However, there are special matrices which commute with other matrices.

**Example 5.2.5.** *Let $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We calculate*

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*Likewise calculate,*

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*Since the matrix $A$ was arbitrary we conclude that $IA = AI$ for all $A \in \mathbb{R}^{2\times 2}$.*

The Kronecker delta $\delta_{ij}$ is defined to be zero if $i \neq j$ and $\delta_{ii} = 1$. The **identity matrix** is the matrix $I$ such that $I_{ij} = \delta_{ij}$. It is simple to show that $AI = A$ and $IA = A$ for all matrices.

**Definition 5.2.6.**

Let $A \in \mathbb{R}^{\,n\times n}$. If there exists $B \in \mathbb{R}^{\,n\times n}$ such that $AB = I$ and $BA = I$ then we say that $A$ is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix.

**Example 5.2.7.** *In the case of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a nice formula to find the inverse is known provided $det(A) = ad - bc \neq 0$:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*It's not hard to show this formula works,*

$$\begin{aligned} \tfrac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+da \end{bmatrix} \\ &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

*This formula is worth memorizing for future use.*

The problem of inverting an $n \times n$ matrix for $n > 0$ is more challenging. However, it is generally true[2] that $A^{-1}$ exists iff $det(A) \neq 0$. Recall our discussion of Cramer's rule in the variation of parameters section, we divided by the determinant for form the solution. If the determinant is zero then we cannot use Cramer's rule and we must seek other methods of solution. In particular, the methods of Gaussian elimination or back subsitution are general and we will need to use those techniques to solve the eigenvector problem in the later part of this chapter. But, don't let me get too ahead of the story. Let's finish our tour of matrix algebra.

**Proposition 5.2.8.**

If $A, B$ are invertible square matrices and $c$ is nonzero then

1. $(AB)^{-1} = B^{-1}A^{-1}$,
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$,

**Proof:** property (1.) is called the **socks-shoes** property because in the same way you first put on your socks and then your shoes to invert the process you first take off your shoes then your socks. The proof is just a calculation:

$$(AB)B^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.$$

[2] the formula is simply $A^{-1} = \frac{1}{det(A)}ad(A)^T$ where $ad(A)$ is the adjoint of $A$, see my linear notes where I give the explicit calculation for an arbitrary $3 \times 3$ case

The proof of (2.) is similar $\square$

The power of a matrix is defined in the natural way. Notice we need for $A$ to be square in order for the product $AA$ to be defined.

**Definition 5.2.9.**

Let $A \in \mathbb{R}^{n\times n}$. We define $A^0 = I$, $A^1 = A$ and $A^m = AA^{m-1}$ for all $m \geq 1$. If $A$ is invertible then $A^{-p} = (A^{-1})^p$.

As you would expect, $A^3 = AA^2 = AAA$.

**Proposition 5.2.10.**

Let $A, B \in \mathbb{R}^{n\times n}$ and $p, q \in \mathbb{N} \cup \{0\}$

1. $(A^p)^q = A^{pq}$.
2. $A^p A^q = A^{p+q}$.
3. If $A$ is invertible, $(A^{-1})^{-1} = A$.

You should notice that $(AB)^p \neq A^p B^p$ for matrices. Instead,

$$(AB)^2 = ABAB, \qquad (AB)^3 = ABABAB, etc...$$

This means the binomial theorem will not hold for matrices. For example,

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB$$

hence $(A+B)^2 \neq A^2 + 2AB + B^2$ as the matrix product is not generally commutative. If we have $A$ and $B$ commute then $AB = BA$ and we can prove that $(AB)^p = A^p B^p$ and the binomial theorem holds true.

**Example 5.2.11.** *A square matrix $A$ is said to be* **idempotent** *of order $k$ if there exists $k \in \mathbb{N}$ such that $A^{k-1} \neq I$ and $A^k = I$. On the other hand, a square matrix $B$ is said to be* **nilpotent** *of order $k$ if there exists $k \in \mathbb{N}$ such that $B^{k-1} \neq 0$ and $B^k = 0$. Suppose $B$ is idempotent of order 2; $B^2 = 0$ and $B \neq 0$. Let $X = I + B$ and calculate,*

$$X^2 = (I+B)(I+B) = II + IB + BI + B^2 = I + 2B$$

$$X^3 = (I+B)(I+2B) = II + I2B + BI + B2B = I + 3B$$

*You can show by induction that $X^k = I + kB$. (neat, that is all I have to say for now)*

**Example 5.2.12.** *A square matrix which only has zero entries in all components except possibly the diagonal is called a* **diagonal matrix***. We say $D \in R^{n\times n}$ is diagonal iff $D_{ij} = 0$ for $i \neq j$. Consider, if $X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ then we find*

$$XY = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & 0 \\ 0 & x_2y_2 \end{bmatrix} = \begin{bmatrix} y_1x_1 & 0 \\ 0 & y_2x_2 \end{bmatrix} = YX.$$

*These results extend beyond the $2 \times 2$ case. If $X, Y$ are diagonal $n \times n$ matrices then $XY = YX$. You can also show that if $X$ is diagonal and $A$ is any other square matrix then $AX = XA$. We will later need the formula below:*

$$\begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & D_n \end{bmatrix}^k = \begin{bmatrix} D_1^k & 0 & \cdots & 0 \\ 0 & D_2^k & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & D_n^k \end{bmatrix}.$$

**Example 5.2.13.** *The product of a $2 \times 2$ and $2 \times 1$ is a $2 \times 1$. Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and let $v = \left[\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right]$,*

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} [1,2] \bullet [5,7] \\ [3,4] \bullet [5,7] \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

*Likewise, define $w = \left[\begin{smallmatrix} 6 \\ 8 \end{smallmatrix}\right]$ and calculate*

$$Aw = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} [1,2] \bullet [6,8] \\ [3,4] \bullet [6,8] \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

*Something interesting to observe here, recall that in Example 5.2.4 we calculated $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. But these are the same numbers we just found from the two matrix-vector products calculated above. We identify that $B$ is just the* **concatenation** *of the vectors $v$ and $w$; $B = [v|w] = \left[\begin{array}{c|c} 5 & 6 \\ 7 & 8 \end{array}\right]$. Observe that:*

$$AB = A[v|w] = [Av|Aw].$$

The term **concatenate** is sometimes replaced with the word **adjoin**. I think of the process as gluing matrices together. This is an important operation since it allows us to lump together many solutions into a single matrix of solutions.

**Proposition 5.2.14.**

Let $A \in \mathbb{R}^{\,m \times n}$ and $B \in \mathbb{R}^{\,n \times p}$ then we can understand the matrix multiplication of $A$ and $B$ as the concatenation of several matrix-vector products,

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

The proof is left to the reader. Finally, to conclude our brief tour of matrix algebra, I collect all my favorite properties for matrix multiplication in the theorem below. To summarize, matrix math works as you would expect with the exception that matrix multiplication is not commutative. We must be careful about the order of letters in matrix expressions.

**Example 5.2.15.** *Suppose $Ax = b$ has solution $x_1$ and $Ax = c$ has solution $x_2$ then note that $X_o = [x_1|x_2]$ is a* **solution matrix** *of the matrix equation $AX = [b|c]$. In particular, observe:*

$$AX_o = A[x_1|x_2] = [Ax_2|Ax_2] = [b|c].$$

For the sake of completeness and perhaps to satisfy the curiousity of the inquisitive student I pause to give a brief synopsis of how we solve systems of equations with matrix techniques. We will not need technology to solve most problems we confront, but I think it is useful to be aware of just how you can use the "rref" command to solve any linear system.

**Remark 5.2.16.** *summary of how to solve linear equations*

(1.) Write the system of equations in matrix notation $Au = b$

(2.) Perform Gaussian eliminate to reduce the augmented coefficient matrx $[A|b]$ to its reduced-row echelon form $rref[A|b]$ (usually I use a computer for complicated examples)

(3.) Read the solution from $rref[A|b]$. There are three cases:

- (a.) there are no solutions
- (b.) there is a unique solution
- (c.) there are infinitely many solutions

The nuts and bolts of gaussian elimination is the process of adding, subtracting and multiplying equations by a nonzero constant towards the goal of eliminating as many variables as possible.
Let us illustrate the remark above.

**Example 5.2.17.** *Suppose $u_1+u_2 = 3$ and $u_1-u_2 = -1$. Then $Au = b$ for coefficient matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $b = [3,-1]^T$. By gaussian elimination,*

$$rref\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

*It follows that $u_1 = 1$ and $u_2 = 2$. This is the* **unique solution**. *The solution set $\{(1,2)\}$ contains a single solution.*

Set aside matrix techniques, you can solve the system above by adding equations to obtain $2u_1 = 2$ hence $u_1 = 1$ and $u_2 = 3 - 1 = 2$.

**Example 5.2.18.** *Suppose $u_1 + u_2 + u_3 = 1$ and $2u_1 + 2u_2 + 2u_3 = 4$. Then $Au = b$ for coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ and $b = [1,2]^T$. By gaussian elimination,*

$$rref\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

*The second row suggests that $0u_1 + 0u_2 + 0u_3 = 1$ or $0 = 1$ which is clearly false hence the system is* **inconsistent** *and the solution set in this case is the empty set.*

Set aside matrix techniques, you can solve the system above by dividing the second equation by 2 to reveal $u_1 + u_2 + u_3 = 2$. Thus insisting both equations are simultaneously true amounts to insisting that $1 = 2$. For this reason the system has no solutions.

**Example 5.2.19.** *Suppose* $u_1+u_2+u_3=0$ *and* $2u_1+2u_2+2u_3=0$. *Then* $Au=b$ *for coefficient matrix* $A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ *and* $b=[0,0]^T$. *By gaussian elimination,*

$$rref\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

*The second row suggests that* $0u_1+0u_2+0u_3=0$ *or* $0=0$ *which is clearly true. This system is* **consistent** *and the solutions have* $u_1+u_2+u_3=0$. *It follows that the solution set is infinite*

$$\{[-u_2-u_3,u_2,u_3]^T \mid u_2,u_3\in\mathbb{R}\}.$$

*Any solution can be written as* $u_2[-1,1,0]^T+u_3[-1,0,1]^T$ *for particular constants* $u_2,u_3$.

It turns out that the last example is the type of matrix algebra problem we wil face with the eigenvector method. The theorem that follows summarizes the algebra of matrices.

**Theorem 5.2.20.**

If $A,B,C\in\mathbb{R}^{m\times n}$, $X,Y\in\mathbb{R}^{n\times p}$, $Z\in\mathbb{R}^{p\times q}$ and $c_1,c_2\in\mathbb{R}$ then

1. $(A+B)+C=A+(B+C)$,
2. $(AX)Z=A(XZ)$,
3. $A+B=B+A$,
4. $c_1(A+B)=c_1A+c_2B$,
5. $(c_1+c_2)A=c_1A+c_2A$,
6. $(c_1c_2)A=c_1(c_2A)$,
7. $(c_1A)X=c_1(AX)=A(c_1X)=(AX)c_1$,
8. $1A=A$,
9. $I_mA=A=AI_n$,
10. $A(X+Y)=AX+AY$,
11. $A(c_1X+c_2Y)=c_1AX+c_2AY$,
12. $(A+B)X=AX+BX$,

The proof of the theorem above follows easily from the definitions of matrix operation. I give some explicit proof in my linear algebra notes. In fact, all of the examples thus far are all taken from my linear algebra notes where I discuss not just these formulas, but also their motivation from many avenues of logic. The example that follows would not be something I would commonly include in the linear algebra course.

**Example 5.2.21.** *Suppose* $R=C^\infty(\mathbb{R})$ *be the set of all smooth functions on* $\mathbb{R}$. *For example,*

$$A=\begin{bmatrix} \cos(t) & t^3 \\ 3^t & \ln(t^2+1) \end{bmatrix}\in R^{2\times 2}.$$

*We can multiply* $A$ *above by* $3^{-t}$ *by*

$$3^{-t}A=\begin{bmatrix} 3^{-t}\cos(t) & t^3 3^{-t} \\ 1 & \ln(t^2+1)3^{-t} \end{bmatrix}\in R^{2\times 2}.$$

*We can subtract the identity matrix to form $A - I$:*

$$A - I = \begin{bmatrix} \cos(t) & t^3 \\ 3^t & \ln(t^2+1) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(t) - 1 & t^3 \\ 3^t & \ln(t^2+1) - 1 \end{bmatrix}$$

Another way of looking at $A$ in the example above is that it is a **matrix-valued** function of a real variable $t$. In other words, $A : \mathbb{R} \to \mathbb{R}^{2\times 2}$; this means for each $t \in \mathbb{R}$ we assign a single matrix $A(t) \in \mathbb{R}^{2\times 2}$. We can similarly consider $p \times q$-matrix valued functions of a real variable[3] . We now turn to the calculus of such matrices.

**Definition 5.2.22.**

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{\ m\times n}$. Suppose $A : I \subseteq \mathbb{R} \to \mathbb{R}^{\ m\times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for each $i, j$ then we define

$$\tfrac{dA}{dt} = [\tfrac{dA_{ij}}{dt}]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int A dt = [\int A_{ij} dt]$ for $A$ with integrable components. Definite integrals and higher derivatives are also defined component-wise.

**Example 5.2.23.** *Suppose* $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. *I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:*

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \qquad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \qquad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

*Integrate by integrating each component:*

$$\int A(t)dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \qquad \int_0^2 A(t)dt = \begin{bmatrix} t^2\big|_0^2 & t^3\big|_0^2 \\ & \\ t^4\big|_0^2 & t^5\big|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

**Example 5.2.24.** *Suppose* $A = \begin{bmatrix} t & 1 \\ 0 & t^2 \end{bmatrix}$. *Calculate* $A^2 = \begin{bmatrix} t & 1 \\ 0 & t^2 \end{bmatrix} \begin{bmatrix} t & 1 \\ 0 & t^2 \end{bmatrix}$ *hence,*

$$A^2 = \begin{bmatrix} t^2 & t + t^2 \\ 0 & t^4 \end{bmatrix}$$

*Clearly* $\frac{d}{dt}[A^2] = \begin{bmatrix} 2t & 1 + 2t \\ 0 & 4t^3 \end{bmatrix}$. *On the other hand, calculate*

$$2A\frac{dA}{dt} = 2\begin{bmatrix} t & 1 \\ 0 & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix} = 2\begin{bmatrix} t & 2t \\ 0 & 2t^3 \end{bmatrix} = \begin{bmatrix} 2t & 4t \\ 0 & 4t^3 \end{bmatrix} \neq \frac{d}{dt}[A^2]$$

*The naive chain-rule fails.*

[3]for those of you who have (or are) taking linear algebra, the space $R^{p\times q}$ is not necessarily a vector space since $R$ is not a field in some examples. The space of smooth functions forms what is called a ring and the set of matrices over a ring can be understood as a "module". A module is like a vector space where the scalar multiplication is taken from a ring rather than a field. Every vector space is a module but some modules are not vector spaces.

**Theorem 5.2.25.**

Suppose $A, B$ are matrix-valued functions of a real variable, $f$ is a function of a real variable, $c$ is a constant, and $C$ is a constant matrix then

1. $(AB)' = A'B + AB'$ (product rule for matrices)
2. $(AC)' = A'C$
3. $(CA)' = CA'$
4. $(fA)' = f'A + fA'$
5. $(cA)' = cA'$
6. $(A + B)' = A' + B'$

where each of the functions is evaluated at the same time $t$ and I assume that the functions and matrices are differentiable at that value of $t$ and of course the matrices $A, B, C$ are such that the multiplications are well-defined.

**Proof:** Suppose $A(t) \in \mathbb{R}^{\,m\times n}$ and $B(t) \in \mathbb{R}^{\,n\times p}$ consider,

$$\begin{aligned}
{(AB)'}_{ij} &= \tfrac{d}{dt}((AB)_{ij}) && \text{defn. derivative of matrix} \\
&= \tfrac{d}{dt}(\textstyle\sum_k A_{ik}B_{kj}) && \text{defn. of matrix multiplication} \\
&= \textstyle\sum_k \tfrac{d}{dt}(A_{ik}B_{kj}) && \text{linearity of derivative} \\
&= \textstyle\sum_k \big[\tfrac{dA_{ik}}{dt}B_{kj} + A_{ik}\tfrac{dB_{kj}}{dt}\big] && \text{ordinary product rules} \\
&= \textstyle\sum_k \tfrac{dA_{ik}}{dt}B_{kj} + \sum_k A_{ik}\tfrac{dB_{kj}}{dt} && \text{algebra} \\
&= (A'B)_{ij} + (AB')_{ij} && \text{defn. of matrix multiplication} \\
&= (A'B + AB')_{ij} && \text{defn. matrix addition}
\end{aligned}$$

this proves (1.) as $i, j$ were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since $C$ constant means $C' = 0$. Proof of (4.) is similar to (1.):

$$\begin{aligned}
{(fA)'}_{ij} &= \tfrac{d}{dt}((fA)_{ij}) && \text{defn. derivative of matrix} \\
&= \tfrac{d}{dt}(fA_{ij}) && \text{defn. of scalar multiplication} \\
&= \tfrac{df}{dt}A_{ij} + f\tfrac{dA_{ij}}{dt} && \text{ordinary product rule} \\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. matrix addition} \\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. scalar multiplication.}
\end{aligned}$$

The proof of (5.) follows from taking $f(t) = c$ which has $f' = 0$. I leave the proof of (6.) as an exercise for the reader. $\square$.

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

Let's investigate, just for the sake of some practice mostly, what the non-naive chain rule for the square of matrix function.

**Example 5.2.26.** *Let $A : \mathbb{R} \to \mathbb{R}^{\,n\times n}$ be a square-matrix-valued differentiable function of a real variable $t$. Calculate, use the product rule:*

$$\frac{d}{dt}[A^2] = \frac{d}{dt}[AA] = \frac{dA}{dt}A + A\frac{dA}{dt}.$$

In retrospect, it must be the case that the matrix $A$ does not commute with $\frac{dA}{dt}$ in Example 5.2.24. The noncommutative nature of the matrix multiplication is the source of the naive chain-rule not working in the current context. In contrast, we have seen that the chain-rule for complex-valued functions of a real variable does often work. For example, $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ or $\frac{d}{dx}x^{\lambda} = \lambda x^{\lambda-1}$. It is possible to show that if $f(z)$ is analytic and $g(t)$ is differentiable from $\mathbb{R}$ to $\mathbb{C}$ then $\frac{d}{dt}f(g(t)) = \frac{df}{dz}(g(t))\frac{dg}{dt}$ where $\frac{df}{dz}$ is the derivative of $f$ with respect to the complex variable $z$. However, you probably will not discuss this in complex variables since it's not terribly interesting in the big-scheme of that course. I find it interesting to contrast to the matrix case here. You might wonder if there is a concept of differentiation with respect to a matrix, or differentiation with respect to a vector. The answer is yes. However, I leave that for some other course.

**Example 5.2.27.** *Another example for fun. The set of* **orthogonal matrices** *is denoted $O(n)$ and is defined to be the set of $n \times n$ matrices $A$ such that $A^TA = I$. These matrices correspond to changes of coordinate which do not change the length of vectors; $A\vec{x} \bullet A\vec{y} = \vec{x} \bullet \vec{y}$. It turns out that $O(n)$ is made from rotations and relections.*

*Suppose we have a curve of orthogonal matrices; $A : \mathbb{R} \to O(n)$ then we know that $A^T(t)A(t) = I$ for all $t \in \mathbb{R}$. If the component functions are differentiable then we can differentiate this equation to learn about the structure that the tangent vector to an orthogonal matrix must possess. Observe:*

$$\frac{d}{dt}[A^T(t)A(t)] = \frac{d}{dt}[I] \quad \Rightarrow \quad \frac{dA^T}{dt}A(t) + A^T(t)\frac{dA}{dt} = 0$$

*Suppose the curve we considered passed through the identity matrix $I$ (which is in $O(n)$ as $I^TI = I$) and suppose this happened at $t = 0$ then we have*

$$\frac{dA^T}{dt}(0) + \frac{dA}{dt}(0) = 0$$

*Let $B = \frac{dA}{dt}(0)$ then we see that $B^T = -B$ is a necessary condition for tangent vectors to the orthogonal matrices at the identity matrix. A matrix with $B^T = -B$ is said to be* **antisymmetric** *or* **skew-symmetric***. The space of all such skew matrices is called $o(n)$. The set $O(n)$ paired with matrix multiplication is called a* **Lie Group** *whereas the set $o(n)$ paired with the matrix commutator is called a* **Lie Algebra**[4]*. These are concepts of considerable interest in modern studies of differential equations.*

[4] it's is pronounced "Lee" not as you might expect

## 5.3 the normal form and theory for systems

A system of ODEs in normal form is a finite collection of first order ODEs which share dependent variables and a single independent variable.

1. $(n=1)$ $\frac{dx}{dt} = A_{11}x + f$
2. $(n=2)$ $\frac{dx}{dt} = A_{11}x + A_{12}y + f_1$ and $\frac{dy}{dt} = A_{21}x + A_{22}y + f_2$ we can express this in **matrix normal form** as follows, use $x = x_1$ and $y = x_2$,

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

   This is nicely abbreviated by writing $d\vec{x}/dt = A\vec{x} + \vec{f}$ where $\vec{x} = (x_1, x_2)$ and $\vec{f} = (f_1, f_2)$ whereas the $2 \times 2$ matrix $A$ is called the **coefficient matrix** of ths system.

3. $(n=3)$ The matrix normal form is simply

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

   Expanded into **scalar normal form** we have $\frac{dx_1}{dt} = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + f_1$ and $\frac{dx_2}{dt} = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + f_2$ and $\frac{dx_3}{dt} = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + f_3$.

Generally an $n$-th order system of ODEs in normal form on an interval $I \subseteq \mathbb{R}$ can be written as $\frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij}x_j + f_i$ for **coefficient functions** $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ and **forcing functions** $f_i : I \subseteq \mathbb{R} \to \mathbb{R}$. You might consider the problem of solving a system of $k$-first order differential equations in $n$-dependent variables where $n \neq k$, however, we do not discuss such over or underdetermined problems in these notes. That said, the concept of a system of differential equations in normal form is perhaps more general than you expect. Let me illustrate this by example. I'll start with a single second order ODE:

**Example 5.3.1.** *Consider $ay'' + by' + cy = f$. We define $x_1 = y$ and $x_2 = y'$. Observe that*

$$x_1' = x_2 \qquad \& \qquad x_2' = y'' = -\frac{1}{a}(f - by' - cy) = \frac{1}{a}(f - bx_2 - cx_1)$$

*Thus,*

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f/a \end{bmatrix}$$

*The matrix* $\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$ *is called the* **companion matrix** *of the second order ODE $ay'' + by' + cy = f$.*

The example above nicely generalizes to the general $n$-th order linear ODE.

**Example 5.3.2.** *Consider $a_o y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f$. Introduce variables to reduce the order:*

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \qquad \dots \qquad x_n = y^{(n-1)}$$

*From which is is clear that $x_1' = x_2$ and $x_2' = x_3$ continuing up to $x_{n-1}' = x_n$ and $x_n' = y^{(n)}$. Hence,*

$$x_n' = -\frac{a_1}{a_o}x_n - \cdots - \frac{a_{n-1}}{a_o}x_2 - \frac{a_n}{a_o}x_1 + f$$

*Once again the matrix below is called the* **companion matrix** *of the given $n$-th order ODE.*

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_n}{a_o} & -\frac{a_{n-1}}{a_o} & -\frac{a_{n-2}}{a_o} & \cdots & -\frac{a_2}{a_o} & -\frac{a_1}{a_o} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{f}{a_o} \end{bmatrix}$$

The problem of many higher order ODEs is likewise confronted by introducing variables to reduce the order.

**Example 5.3.3.** *Consider $y'' + 3x' = \sin(t)$ and $x'' + 6y' - x = e^t$. We begin with a system of two second order differential equations. Introduce new variables:*

$$x_1 = x, \quad x_2 = y, \quad x_3 = x', \quad x_4 = y'$$

*It follows that $x_3' = x''$ and $x_4' = y''$ whereas $x_1' = x_3$ and $x_2' = x_4$. We convert the given differential equations to first order ODEs:*

$$x_4' + 3x_3 = \sin(t) \quad \& \quad x_3' + 6x_4 - x_1 = e^t$$

*Let us collect these results as a matrix problem:*

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \\ \sin(t) \end{bmatrix}$$

Generally speaking the order of the normal form corresponding to a system of higher order ODE will simply be the sum of the orders of the systems (assuming the given system has no reundancies; for example $x'' + y'' = x$ and $x'' - x = -y''$ are redundant). I will not prove the following assertion, however, it should be fairly clear why it is true given the examples thus far discussed:

**Proposition 5.3.4.** *linear systems have a normal form.*

A given systems of linear ODEs may be converted to an equivalent system of first order ODEs in normal form.

For this reason the first order problem will occupy the majority of our time. That said, the method of the next section is applicable to any order.

Since normal forms are essentially general it is worthwhile to state the theory which will guide our work. I do not offer all the proof here, but you can find proof in many texts. For example, in Nagel Saff and Snider these theorems are given in §9.4 and are proved in Chapter 13.

**Definition 5.3.5.** *linear independence of vector-valued functions*

Suppose $\vec{v}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a function for $j = 1, 2, \dots, k$ then we say that $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent on $I$ iff $\sum_{j=1}^k c_j \vec{v}_j(t) = 0$ for all $t \in I$ implies $c_j = 0$ for $j = 1, 2, \dots, k$.

We can use the determinant to test LI of a set of $n$-vectors which are all $n$-dimensional vectors. It is true that $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is LI on $I$ iff $det[\vec{v}_1(t)|\vec{v}_2(t)|\cdots|\vec{v}_n(t)| \neq 0$ for all $t \in I$.

**Definition 5.3.6.** *wronskian for vector-valued functions of a real variable.*

Suppose $\vec{v}_j : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$ is differentiable for $j = 1, 2, \dots, n$. The **Wronskian** is defined by $W(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n; t) = det[\vec{v}_1|\vec{v}_2|\dots|\vec{v}_n]$ for each $t \in I$.

Theorems for wronskians of solutions sets mirror those already discussed for the $n$-th order problem.

**Definition 5.3.7.** *solution and homogeneous solutions of $d\vec{x}/dt = A\vec{x} + \vec{f}$*

Let $A : I \rightarrow \mathbb{R}^{n \times n}$ and $\vec{f} : I \rightarrow \mathbb{R}^n$ be continuous. A **solution of** $d\vec{v}/dt = A\vec{v} + \vec{f}$ **on** $I \subseteq \mathbb{R}$ is a vector-valued function $\vec{x} : I \rightarrow \mathbb{R}^n$ such that $d\vec{x}/dt = A\vec{x} + \vec{f}$ for all $t \in I$. A **homogeneous solution** on $I \subseteq \mathbb{R}$ is a solution of $d\vec{v}/dt = A\vec{v}$.

In the example below we see three LI homogeneous solutions and a single particular solution.

**Example 5.3.8.** *Suppose $x' = x - 1$, $y' = 2y - 2$ and $z' = 3z - 3$. In matrix normal form we face:*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

*It is easy to show by separately solving the the DEqns that $x = c_1e^t + 1$, $y = c_2e^{2t} + 2$ and $z = c_3e^{3t} + 3$. In vector notation the solution is*

$$\vec{x}(t) = \begin{bmatrix} c_1e^t + 1 \\ c_2e^{2t} + 2 \\ c_3e^{3t} + 3 \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} c_2 \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} c_3 \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

*I invite the reader to show that $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is LI on $\mathbb{R}$ where $\vec{x}_1(t) = \langle e^t, 0, 0\rangle$, $\vec{x}_2(t) = \langle 0, e^{2t}, 0\rangle$ and $\vec{x}_3(t) = \langle 0, 0, e^{3t}\rangle$. On the other hand, $\vec{x}_p = \langle 1, 2, 3\rangle$ is a particular solution to the given problem.*

In truth, any choice of $c_1, c_2, c_3$ with at least one nonzero constant will produce a homogeneous solution. To obtain the solutions I pointed out in the example you can choose $c_1 = 1, c_2 = 0, c_3 = 0$ to obtain $\vec{x}_1(t) = \langle e^t, 0, 0\rangle$ or $c_1 = 0, c_2 = 1, c_3 = 0$ to obtain $\vec{x}_2(t) = \langle 0, e^{2t}, 0\rangle$ or $c_1 = 0, c_2 = 0, c_3 = 1$ to obtain $\vec{x}_3(t) = \langle 0, 0, e^{3t}\rangle$.

**Definition 5.3.9.** *fundamental solution set of a linear system $d\vec{x}/dt = A\vec{x} + \vec{f}$*

Let $A : I \rightarrow \mathbb{R}^{n \times n}$ and $\vec{f} : I \rightarrow \mathbb{R}^n$ be continuous. A **fundmental solution set** on $I \subseteq \mathbb{R}$ is a set of $n$-homogeneous solutions of $d\vec{v}/dt = A\vec{v} + \vec{f}$ for which $\{\vec{x}_1, \vec{x}_1, \dots, \vec{x}_n\}$ is a LI set on $I$. A **solution matrix** on $I \subseteq \mathbb{R}$ is a matrix $X$ is a matrix for which each column is a homogeneous solution on $I$. A **fundamental matrix** on $I \subseteq \mathbb{R}$ is an invertible solution matrix.

**Example 5.3.10.** *Continue Example 5.3.8. Note that $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a fundamental solution set. The fundamental solution matrix is found by concatenating $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$:*

$$X = [\vec{x}_1|\vec{x}_2|\vec{x}_3] = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

*Observe $det(X) = e^te^{2t}e^{3t} = e^{6t} \neq 0$ on $\mathbb{R}$ hence $X$ is invertible on $\mathbb{R}$.*

**Example 5.3.11.** *Let* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *define the system of DEqns* $\frac{d\vec{x}}{dt} = A\vec{x}$*. I claim that the matrix* $X(t) = \begin{bmatrix} 0 & -e^{4t} & -2e^{2t} \\ e^{4t} & 0 & e^{2t} \\ 0 & e^{4t} & e^{2t} \end{bmatrix}$ *is a solution matrix. Calculate,*

$$AX = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 & -e^{4t} & -2e^{2t} \\ e^{4t} & 0 & e^{2t} \\ 0 & e^{4t} & e^{2t} \end{bmatrix} = \begin{bmatrix} 0 & -4e^{4t} & -4e^{2t} \\ 4e^{4t} & 0 & 2e^{2t} \\ 0 & 4e^{4t} & 2e^{2t} \end{bmatrix}.$$

*On the other hand, differentiation yields* $X' = \begin{bmatrix} 0 & -4e^{4t} & -4e^{2t} \\ 4e^{4t} & 0 & 2e^{2t} \\ 0 & 4e^{4t} & 2e^{2t} \end{bmatrix}$*. Therefore* $X' = AX$*. Notice that if we express* $X$ *in terms of its columns* $X = [\vec{x}_1|\vec{x}_2|\vec{x}_3]$ *then it follows that* $X' = [\vec{x}_1{}'|\vec{x}_2{}'|\vec{x}_3{}']$ *and* $AX = A[\vec{x}_1|\vec{x}_2|\vec{x}_3] = [A\vec{x}_1|A\vec{x}_2|A\vec{x}_3]$ *hence*

$$\vec{x}_1{}' = A\vec{x}_1 \quad \& \quad \vec{x}_2{}' = A\vec{x}_2 \quad \& \quad \vec{x}_3{}' = A\vec{x}_3$$

*We find that* $\vec{x}_1(t) = \langle 0, e^{4t}, 0\rangle$*,* $\vec{x}_2(t) = \langle -e^{4t}, 0, e^{4t}\rangle$ *and* $\vec{x}_3(t) = \langle -2e^{2t}, e^{2t}, e^{2t}\rangle$ *form a fundamental solution set for the given system of DEqns.*

**Theorem 5.3.12.** *Let* $A : I \to \mathbb{R}^{n\times n}$ *and* $\vec{f} : I \to \mathbb{R}^n$ *be continuous.*

1. there exists a fundamental solution set $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ on $I$
2. if $t_o \in I$ and $\vec{x}_o$ is a given initial condition vector then there exists a unique solution $\vec{x}$ on $I$ such that $\vec{x}(t_o) = \vec{x}_o$
3. the **general solution** has the form $\vec{x} = \vec{x}_h + \vec{x}_p$ where $\vec{x}_p$ is a **particular solution** and $\vec{x}_h$ is the **homogeneous solution** is formed by a real linear combination of the fundamental solution set ($\vec{x}_h = c_1\vec{x}_1 + c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$)

The term *general solution* is intended to indicate that the formula given includes all possible solutions to the problem. Part (2.) of the theorem indicates that there must be some 1-1 correspondance between a given initial condition and the choice of the constants $c_1, c_2, \dots, c_n$ with respect to a given fundamental solution set. Observe that if we define $\vec{c} = [c_1, c_2, \dots, c_n]^T$ and the fundamental matrix $X = [\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n]$ we can express the homogeneous solution via a matrix-vector product:

$$\vec{x}_h = X\vec{c} = c_1\vec{x}_1 + c_1\vec{x}_1 + \cdots + c_n\vec{x}_n \qquad \Rightarrow \qquad \boxed{\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t)}$$

Further suppose that we wish to set $\vec{x}(t_o) = \vec{x}_o$. We need to solve for $\vec{c}$:

$$\vec{x}_o = X(t_o)\vec{c} + \vec{x}_p(t_o) \quad \Rightarrow \quad X(t_o)\vec{c} = \vec{x}_o - \vec{x}_p(t_o)$$

Since $X^{-1}(t_o)$ exists we can multiply by the inverse on the right and find

$$\vec{c} = X^{-1}(t_o)\big[\vec{x}_o - \vec{x}_p(t_o)\big]$$

Next, place the result above back in the general solution to derive

$$\boxed{\vec{x}(t) = X(t)X^{-1}(t_o)\big[\vec{x}_o - \vec{x}_p(t_o)\big] + \vec{x}_p(t)}$$

We can further simplify this general formula in the constant coefficient case, or in the study of variation of parameters for systems. Note that in the homogeneous case this gives us a clean formula to calculate the constants to fit initial data:

$$\boxed{\vec{x}(t) = X(t)X^{-1}(t_o)\vec{x}_o} \qquad \text{(homogeneous case)}$$

**Example 5.3.13.** *We found $x' = -y$ and $y' = x$ had solutions $x(t) = c_1\cos(t) + c_2\sin(t)$ and $y(t) = c_1\sin(t) - c_2\cos(t)$. It follows that $X(t) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{array}\right]$. Calculate that $det(X) = -1$ to see that $X^{-1}(t) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{array}\right]$. Suppose we want the solution through $(a, b)$ at time $t_o$ then the solution is given by*

$$\vec{x}(t) = X(t)X^{-1}(t_o)\vec{x}_o = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{array}\right]\left[\begin{array}{cc} \cos(t_o) & \sin(t_o) \\ \sin(t_o) & -\cos(t_o) \end{array}\right]\left[\begin{array}{c} a \\ b \end{array}\right].$$

This concludes our brief tour of the theory for systems of ODEs. Clearly we have two main goals past this point (1.) find the fundamental solution set (2.) find the particular solution.

## 5.4 solutions by eigenvector

We narrow our focus at this point: our goal is to find nontrivial[5] solutions to the homogeneous constant coefficient problem $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A \in \mathbb{R}^{\,n\times n}$. A reasonable ansatz for this problem is that the solution should have the form $\vec{x} = e^{\lambda t}\vec{u}$ for some constant scalar $\lambda$ and some constant vector $\vec{u}$. If such solutions exist then what conditions must we place on $\lambda$ and $\vec{u}$? To begin clearly $\vec{u} \neq 0$ since we are seeking nontrivial solutions. Differentiate,

$$\frac{d}{dt}\big[e^{\lambda t}\vec{u}\big] = \big[e^{\lambda t}\big]\vec{u} = \lambda e^{\lambda t}\vec{u}$$

Hence $\frac{d\vec{x}}{dt} = A\vec{x}$ implies $\lambda e^{\lambda t}\vec{u} = Ae^{\lambda t}\vec{u}$. However, $e^{\lambda t} \neq 0$ hence we find $\lambda\vec{u} = A\vec{u}$. We can write the vector $\lambda\vec{u}$ as a matrix product with identity matrix $I$; $\lambda\vec{u} = \lambda I\vec{u}$. Therefore, we find

$$\boxed{(A-\lambda I)\vec{u} = 0}$$

to be a necessary condition for the solution. Note that the system of linear equations defined by $(A-\lambda I)\vec{u} = 0$ is consistent since 0 is a solution. It follows that for $\vec{u} \neq 0$ to be a solution we must have that the matrix $(A - \lambda I)$ is singular. It follows that we find

$$\boxed{det(A-\lambda I) = 0}$$

a necessary condition for our solution. Moreover, for a given matrix $A$ this is nothing more than an $n$-th order polynomial in $\lambda$ hence there are at most $n$-distinct solutions for $\lambda$. The equation $det(A - \lambda I) = 0$ is called the **characteristic equation** of $A$ and the solutions are called **eigenvalues**. The nontrivial vector $\vec{u}$ such that $(A - \lambda I)\vec{u} = 0$ is called the **eigenvector** with **eigenvalue** $\lambda$. We often abbreviate these by referring to "e-vectors" or "e-values". Many interesting theorems are known for eigenvectors, see a linear algebra text or my linear notes for elaboration on this point.

**Definition 5.4.1.** *eigenvalues and eigenvectors*

Suppose $A$ is an $n \times n$ matrix then we say $\lambda \in \mathbb{C}$ which is a solution of $det(A - \lambda I) = 0$ is an **eigenvalue of** $A$. Given such an eigenvalue $\lambda$ a nonzero vector $\vec{u}$ such that $(A - \lambda I)\vec{u} = 0$ is called an **eigenvector** with eigenvalue $\lambda$.

**Example 5.4.2. Problem:** *find the fundamental solutions of the system $x' = -4x - y$ and $y' = 5x + 2y$*

**Solution:** *we seek to solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} -4 & -1 \\ 5 & 2 \end{bmatrix}$. Consider the characteristic equation:*

$$\begin{aligned} det(A-\lambda I) &= det\begin{bmatrix} -4-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix} \\ &= (-4-\lambda)(2-\lambda) + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda+3)(\lambda-1) \\ &= 0 \end{aligned}$$

[5] nontrivial simply means the solution is not identically zero. The zero solution does exist, but it is not the solution we are looking for...

*We find* $\lambda_1 = 1$ *and* $\lambda_2 = -3$. *Next calculate the e-vectors for each e-value. We seek* $\vec{u}_1 = [u,v]^T$ *such that* $(A-I)\vec{u}_1 = 0$ *thus solve:*

$$\begin{bmatrix} -5 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -5u - v = 0 \quad \Rightarrow \quad v = -5u, \ u \neq 0 \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ -5u \end{bmatrix}$$

*Naturally we can write* $\vec{u}_1 = u[1,-5]^T$ *and for convenience we set* $u = 1$ *and find* $\vec{u}_1 = [1,-5]^T$ *which gives us the fundamental solution* $\boxed{\vec{x}_1(t) = e^t[1,-5]^T}$. *Continue*[6] *to the next e-value* $\lambda_2 = -3$ *we seek* $\vec{u}_2 = [u,v]^T$ *such that* $(A+3I)\vec{u}_2 = 0$.

$$\begin{bmatrix} -1 & -1 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -u - v = 0 \quad \Rightarrow \quad v = -u, \ u \neq 0 \quad \Rightarrow \quad \vec{u}_2 = \begin{bmatrix} u \\ -u \end{bmatrix}$$

*Naturally we can write* $\vec{u}_2 = u[1,-1]^T$ *and for convenience we set* $u = 1$ *and find* $\vec{u}_2 = [1,-1]^T$ *which gives us the fundamental solution* $\boxed{\vec{x}_2(t) = e^{-3t}[1,-1]^T}$. *The fundmental solution set is given by* $\{\vec{x}_1, \vec{x}_2\}$ *and the domains of these solution clearly extend to all of* $\mathbb{R}$.

We can assemble the general solution as a linear combination of the fundamental solutions $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$. In particular this yields

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1e^t\begin{bmatrix} 1 \\ -5 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2e^{-3t} \\ -5c_1e^t - c_2e^{-3t} \end{bmatrix}$$

Thus the system $x' = -4x - y$ and $y' = 5x + 2y$ has **scalar** solutions $x(t)c_1e^t + c_2e^{-3t}$ and $y(t) = -5c_1e^t - c_2e^{-3t}$. Finally, a fundamental matrix for this problem is given by

$$X = [\vec{x}_1|\vec{x}_2] = \begin{bmatrix} e^t & e^{-3t} \\ -5e^t & -e^{-3t} \end{bmatrix}.$$

**Example 5.4.3. Problem:** *find the fundamental solutions of the system* $x' = -3x$ *and* $y' = -3y$

**Solution:** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$. *Consider the characteristic equation:*

$$det(A - \lambda I) = det\begin{bmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} = (\lambda+3)^2 = 0$$

*We find* $\lambda_1 = -3$ *and* $\lambda_2 = -3$. *Finding the eigenvectors here offers an unusual algebra problem; to find* $\vec{u}$ *with e-value* $\lambda = -3$ *we should find nontrivial solutions of* $(A+3I)\vec{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$. *We find no condition on* $\vec{u}$. *It follows that* **any** *nonzero vector is an eigenvector of* $A$. *Indeed, note that* $A = -3I$ *and* $A\vec{u} = -3I\vec{u}$ *hence* $(A+3I)\vec{u} = 0$. *Convenient choices for* $\vec{u}$ *are* $[1,0]^T$ *and* $[0,1]^T$ *hence we find fundamental solutions:*

$$\vec{x}_1(t) = e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix} \qquad \& \qquad \vec{x}_2(t) = e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}.$$

[6] the upcoming $u, v$ are not the same as those I just worked out, I call these letters disposable variables because I like to reuse them in several ways in a particular example where we repeat the e-vector calculation over several e-values.

We can assemble the general solution as a linear combination of the fundamental solutions $\vec{x}(t) = c_1 \begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$. Thus the system $x' = -3x$ and $y' = -3y$ has **scalar** solutions $x(t) = c_1 e^{-3t}$ and $y(t) = c_2 e^{-3t}$. Finally, a fundamental matrix for this problem is given by

$$X = [\vec{x}_1 | \vec{x}_2] = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}.$$

**Example 5.4.4. Problem:** *find the fundamental solutions of the system $x' = 3x + y$ and $y' = -4x - y$*

**Solution:** *we seek to solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$. Consider the characteristic equation:*

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} \\ &= (\lambda - 3)(\lambda + 1) + 4 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2 \\ &= 0 \end{aligned}$$

*We find $\lambda_1 = 1$ and $\lambda_2 = 1$. Let us find the e-vector $\vec{u}_1 = [u, v]^T$ such that $(A - I)\vec{u}_1 = 0$*

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 2u + v = 0 \quad \Rightarrow \quad v = -2u, \ u \neq 0 \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ -2u \end{bmatrix}$$

*We choose $u = 1$ for convenience and thus find the fundamental solution $\vec{x}_1(t) = e^t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.*

**Remark 5.4.5.**

In the previous example the **algebraic multiplicity** of the e-value $\lambda = 1$ was 2. However, we found only one LI e-vector. This means the **geometric multiplicity** for $\lambda = 1$ is only 1. This means we are missing a vector and hence a fundamental solution. Where is $\vec{x}_2$ which is LI from the $\vec{x}_1$ we just found? This question is ultimately answered via the matrix exponential.

**Example 5.4.6. Problem:** *find the fundamental solutions of the system $x' = -y$ and $y' = 4x$*

**Solution:** *we seek to solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$. Consider the characteristic equation:*

$$det(A - \lambda I) = det \begin{bmatrix} -\lambda & -1 \\ 4 & -\lambda \end{bmatrix} = \lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i.$$

*This e-value is a* **pure imaginary** *number which is a special type of* **complex number** *where there is no real part. Careful review of the arguments that framed the e-vector solution reveal that the same calculations apply when either $\lambda$ or $\vec{u}$ are complex. With this in mind we seek the e-vector for $\lambda = 2i$: let us find the e-vector $\vec{u}_1 = [u, v]^T$ such that $(A - 2iI)\vec{u}_1 = 0$*

$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 2iu - v = 0 \quad \Rightarrow \quad v = 2iu, \ u \neq 0 \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ 2iu \end{bmatrix}$$

*Let $u = 1$ for convenience and find $\vec{u}_1 = [1, 2i]^T$. We find the* **fundamental complex solution** *$\vec{x}$:*

$$\vec{x} = e^{2it}\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \big(\cos(2t) + i\sin(2t)\big)\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \begin{bmatrix} \cos(2t) + i\sin(2t) \\ 2i\cos(2t) - 2\sin(2t) \end{bmatrix}$$

*Note: if $\vec{x} = Re(\vec{x}) + iIm(\vec{x})$ then it follows that the real and imaginary parts of the complex solution are themselves real solutions. Why? Because differentiation with respect to $t$ is defined such that:*

$$\frac{d\vec{x}}{dt} = \frac{dRe(\vec{x})}{dt} + i\frac{dIm(\vec{x})}{dt}$$

*and $A\vec{x} = A[Re(\vec{x}) + iIm(\vec{x})] = A\,Re(\vec{x}) + iA\,Im(\vec{x})$. However, we know $d\vec{x}/dt = A\vec{x}$ hence we find, equating real parts and imaginary parts separately that:*

$$\frac{dRe(\vec{x})}{dt} = ARe(\vec{x}) \qquad \& \qquad \frac{dIm(\vec{x})}{dt} = AIm(\vec{x})$$

*Hence $\vec{x}_1 = Re(\vec{x})$ and $\vec{x}_2 = Im(\vec{x})$ give a solution set for the given system. In particular we find the fundamental solution set*

$$\vec{x}_1(t) = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} \qquad \& \qquad \vec{x}_2(t) = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}.$$

We can assemble the general solution as a linear combination of the fundamental solutions $\vec{x}(t) = c_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$. Thus the system $x' = -y$ and $y' = 4x$ has **scalar** solutions $x(t) = c_1\cos(2t) + c_2\sin(2t)$ and $y(t) = -2c_1\sin(2t) + 2c_2\cos(2t)$. Finally, a fundamental matrix for this problem is given by

$$X = [\vec{x}_1|\vec{x}_2] = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}.$$

**Example 5.4.7. Problem:** *find the fundamental solutions of the system $x' = 2x - y$ and $y' = 9x + 2y$*

**Solution:** *we seek to solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 2 & -1 \\ 9 & 2 \end{bmatrix}$. Consider the characteristic equation:*

$$det(A - \lambda I) = det\begin{bmatrix} 2-\lambda & -1 \\ 9 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 + 9 = 0.$$

*Thus $\lambda = 2 \pm 3i$. Consider $\lambda = 2 + 3i$, we seek the e-vector subject to $(A - (2+3i)I)\vec{u} = 0$. Solve:*

$$\begin{bmatrix} -3i & -1 \\ 9 & -3i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -3iu - v = 0 \quad \Rightarrow \quad v = -3iu, \; u \neq 0 \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ -3iu \end{bmatrix}$$

*We choose $u = 1$ for convenience and thus find the fundamental complex solution*

$$\vec{x}(t) = e^{(2+3i)t}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = e^{2t}(\cos(3t) + i\sin(3t))\begin{bmatrix} 1 \\ -3i \end{bmatrix} = e^{2t}\begin{bmatrix} \cos(3t) + i\sin(3t) \\ -3i\cos(3t) + 3\sin(3t) \end{bmatrix}$$

*Therefore, using the discussion of the last example, we find fundamental real solutions of the system by selecting real and imaginary parts of the complex solution above:*

$$\vec{x}_1(t) = \begin{bmatrix} e^{2t}\cos(3t) \\ 3e^{2t}\sin(3t) \end{bmatrix} \qquad \& \qquad \vec{x}_2(t) = \begin{bmatrix} e^{2t}\sin(3t) \\ -3e^{2t}\cos(3t) \end{bmatrix}.$$

We can assemble the general solution as a linear combination of the fundamental solutions $\vec{x}(t) = c_1 \begin{bmatrix} e^{2t}\cos(3t) \\ 3e^{2t}\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}\sin(3t) \\ -3e^{2t}\cos(3t) \end{bmatrix}$. Thus the system $x' = 2x - y$ and $y' = 9x + 2y$ has **scalar** solutions $x(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$ and $y(t) = 3c_1 e^{2t}\sin(3t) - 3c_2 e^{2t}\cos(3t)$. Finally, a fundamental matrix for this problem is given by

$$X = [\vec{x}_1|\vec{x}_2] = \begin{bmatrix} e^{2t}\cos(3t) & e^{2t}\sin(3t) \\ 3e^{2t}\sin(3t) & -3e^{2t}\cos(3t) \end{bmatrix}.$$

**Example 5.4.8. Problem:** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -4 & -1 \\ 0 & 5 & 2 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$\begin{aligned} 0 = det(A - \lambda I) &= det \begin{bmatrix} 2-\lambda & 0 & 0 \\ -1 & -4-\lambda & -1 \\ 0 & 5 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)\big[(\lambda-2)(\lambda+4)+5\big] \\ &= (2-\lambda)(\lambda-1)(\lambda+3). \end{aligned}$$

*Thus* $\lambda_1 = 1$, $\lambda_2 = 2$ *and* $\lambda_3 = -3$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A - I)\vec{u}_1 = 0$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} u = 0 \\ 5v + w = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} u = 0 \\ w = -5v \end{array} \quad \Rightarrow \quad \vec{u}_1 = v \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

*Choose* $v = 1$ *for convenience and find* $\vec{u}_1 = [0, 1, -5]^T$. *Next, seek* $\vec{u}_2 = [u, v, w]^T$ *such that* $(A - 2I)\vec{u}_2 = 0$:

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -6 & -1 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} -u - 6v - w = 0 \\ v = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} v = 0 \\ w = -u \end{array} \quad \Rightarrow \quad \vec{u}_2 = u \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

*Choose* $u = 1$ *for convenience and find* $\vec{u}_2 = [1, 0, -1]^T$. *Last, seek* $\vec{u}_3 = [u, v, w]^T$ *such that* $(A + 3I)\vec{u}_3 = 0$:

$$\begin{bmatrix} 5 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} 5u = 0 \\ 5v + 5w = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} u = 0 \\ w = -v \end{array} \quad \Rightarrow \quad \vec{u}_3 = v \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

*Choose* $v = 1$ *for convenience and find* $\vec{u}_3 = [0, 1, -1]^T$. *The general solution follows:*

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.}$$

The fundamental solutions and the fundamental matrix for the system above are given as follows:

$$\vec{x_1}(t) = \begin{bmatrix} 0 \\ e^{t} \\ -5e^{t} \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}, \quad \vec{x}_3(t) = \begin{bmatrix} 0 \\ e^{-3t} \\ -e^{-3t} \end{bmatrix}, \qquad X(t) = \begin{bmatrix} 0 & e^{2t} & 0 \\ e^{t} & 0 & e^{-3t} \\ -5e^{t} & -e^{2t} & -e^{-3t} \end{bmatrix}.$$

**Example 5.4.9.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)(3-\lambda) = 0.$$

*Thus* $\lambda_1 = 2$, $\lambda_2 = 2$ *and* $\lambda_3 = 3$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A - 2I)\vec{u}_1 = 0$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} u + w = 0 \\ v \ \ free \end{matrix} \quad \Rightarrow \quad \begin{matrix} v \ \ free \\ w = -u \end{matrix} \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ v \\ -u \end{bmatrix}.$$

*There are two free variables in the solution above and it follows we find two e-vectors. A convenient choice is* $u = 1$ *and* $v = 0$ *or* $u = 0$ *and* $v = 1$; $\vec{u_1} = [1, 0, -1]^T$ *and* $\vec{u_2} = [0, 1, 0]^T$. *Next, seek* $\vec{u}_3 = [u, v, w]^T$ *such that* $(A - 3I)\vec{u}_3 = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} u = 0 \\ v = 0 \\ w \ \ free \end{matrix} \quad \Rightarrow \quad \vec{u}_3 = w \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

*Choose* $w = 1$ *for convenience to find* $\vec{u}_3 = [0, 0, 1]^T$. *The general solution follows:*

$$\boxed{\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.}$$

The fundamental solutions and the fundamental matrix for the system above are given as follows:

$$\vec{x_1}(t) = \begin{bmatrix} e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix}, \quad \vec{x}_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{-3t} \end{bmatrix}, \qquad X(t) = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ -e^{2t} & 0 & e^{-3t} \end{bmatrix}.$$

**Example 5.4.10.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 3-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda) = 0.$$

*Thus* $\lambda_1 = 2$, $\lambda_2 = 2$ *and* $\lambda_3 = 3$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A - 2I)\vec{u}_1 = 0$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} v = 0 \\ u + w = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} v = 0 \\ w = -u \end{matrix} \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ 0 \\ -u \end{bmatrix}.$$

*Choose* $u = 1$ *to select* $\vec{u}_1 = [1, 0, -1]^T$. *Next find* $\vec{u}_2$ *such that* $(A - 3I)\vec{u}_2 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} -u + v = 0 \\ -v = 0 \\ w \;\; free \end{matrix} \quad \Rightarrow \quad \begin{matrix} u = 0 \\ v = 0 \\ w \;\; free \end{matrix} \quad \Rightarrow \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}.$$

*Choose* $w = 1$ *to find* $\vec{u}_2 = [0, 0, 1]^T$. *We find two fundamental solutions from the e-vector method:*

$$\vec{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \& \qquad \vec{x}_1(t) = e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

*We cannot solve the system at this juncture since we are missing the third fundamental solution* $\vec{x}_3$. *In the next section we will find the missing solution via the generalized e-vector/ matrix exponential method.*

**Example 5.4.11.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$det \begin{bmatrix} 7-\lambda & 0 & 0 \\ 0 & 7-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{bmatrix} = (7-\lambda)^3 = 0.$$

*Thus* $\lambda_1 = 7$, $\lambda_2 = 7$ *and* $\lambda_3 = 7$. *The e-vector equation in this case is easy to solve; since* $A - 7I = 7I - 7I = 0$ *it follows that* $(A - 7I)\vec{u} = 0$ *for all* $\vec{u} \in \mathbb{R}^3$. *Therefore, any nontrivial vector is an eigenvector with e-value 7. A natural choice is* $\vec{u}_1 = [1, 0, 0]^T$, $\vec{u}_2 = [0, 1, 0]^T$ *and* $\vec{u}_3 = [0, 0, 1]^T$. *Thus,*

$$\boxed{\vec{x}_(t) = c_1 e^{7t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{7t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{7t} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.}$$

**Example 5.4.12.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$det(A - \lambda I) = det \begin{bmatrix} -2-\lambda & 0 & 0 \\ 4 & -2-\lambda & 0 \\ 1 & 0 & -2-\lambda \end{bmatrix} = -(\lambda+2)^3 = 0.$$

*Thus* $\lambda_1 = -2$, $\lambda_2 = -2$ *and* $\lambda_3 = -2$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A + 2I)\vec{u}_1 = 0$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} u = 0 \\ v \ \ free \\ w \ \ free \end{matrix} \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} 0 \\ v \\ w \end{bmatrix}.$$

*Choose* $v = 1, w = 0$ *to select* $\vec{u}_1 = [0, 1, 0]^T$ *and* $v = 0, w = 1$ *to select* $\vec{u}_2 = [0, 0, 1]^T$. *Thus we find fundamental solutions:*

$$\vec{x}_1(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \& \qquad \vec{x}_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

*We cannot solve the system at this juncture since we are missing the third fundamental solution* $\vec{x}_3$. *In the next section we will find the missing solution via the generalized e-vector/ matrix exponential method.*

**Example 5.4.13.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 2-\lambda & 1 & -1 \\ -3 & -1-\lambda & 1 \\ 9 & 3 & -4-\lambda \end{bmatrix} \\ &= (2-\lambda)[(\lambda+1)(\lambda+4) - 3] - [3(\lambda+4) - 9] - [-9 + 9(\lambda+1)] \\ &= (2-\lambda)[\lambda^2 + 5\lambda + 1] - 3\lambda - 3 - 9\lambda \\ &= -\lambda^3 - 5\lambda^2 - \lambda + 2\lambda^2 + 10\lambda + 2 - 12\lambda - 3 \\ &= -\lambda^3 - 3\lambda^2 - 3\lambda - 1 \\ &= -(\lambda+1)^3 \end{aligned}$$

*Thus* $\lambda_1 = -1$, $\lambda_2 = -1$ *and* $\lambda_3 = -1$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A + I)\vec{u}_1 = 0$:

$$\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} 3u + v - w = 0 \\ -3u + w = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} w = 3u \\ v = w - 3u = 0 \end{matrix} \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ 0 \\ 3u \end{bmatrix}.$$

*Choose* $u = 1$ *to select* $\vec{u}_1 = [1, 0, 3]^T$. *We find just one fundamental solution:* $\boxed{\vec{x}_1 = e^{-t}[1, 0, 3]^T}$. *We cannot solve the problem in it's entirety with our current methods. In the section that follows we find the missing pair of solutions.*

**Example 5.4.14.** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

**Solution:** *Begin by solving the characteristic equation:*

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix} \\ &= -\lambda[\lambda(\lambda - 1) + 1] + 1 \\ &= -\lambda^3 + \lambda^2 - \lambda + 1 \\ &= -\lambda^2(\lambda - 1) - (\lambda - 1) \\ &= (1 - \lambda)(\lambda^2 + 1) \end{aligned}$$

*Thus* $\lambda_1 = 1$, $\lambda_2 = i$ *and* $\lambda_3 = -i$. *We seek* $\vec{u}_1 = [u, v, w]^T$ *such that* $(A - I)\vec{u}_1 = 0$*:*

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} -u + v = 0 \\ -v + w = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} v = u \\ w = v \end{matrix} \quad \Rightarrow \quad \vec{u}_1 = \begin{bmatrix} u \\ u \\ u \end{bmatrix}.$$

*Choose* $u = 1$ *thus select* $\vec{u}_1 = [1, 1, 1]^T$. *Now seek* $\vec{u}_2$ *such that* $(A - iI)\vec{u}_2 = 0$

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ 1 & -1 & 1-i \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} v = iu \\ w = iv = i(iu) = -u \\ (i-1)w = u - v \end{matrix} \quad \Rightarrow \quad \vec{u}_2 = \begin{bmatrix} u \\ iu \\ -u \end{bmatrix}.$$

*Set* $u = 1$ *to select the following complex solution:*

$$\vec{x}(t) = e^{it} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = \begin{bmatrix} e^{it} \\ ie^{it} \\ -e^{it} \end{bmatrix} = \begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \\ -\cos(t) - i\sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ -\cos(t) \end{bmatrix} + i \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}.$$

*We select the second and third solutions by taking the real and imaginary parts of the above complex solution;* $\vec{x}_2(t) = Re(\vec{x}(t))$ *and* $\vec{x}_3(t) = Im(\vec{x}(t))$. *The general solution follows:*

$$\boxed{\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ -\sin(t) \\ -\cos(t) \end{bmatrix} + c_3 \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}.}$$

The fundamental solution set and fundamental matrix of the example above are simply:

$$\vec{x}_1 = \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ -\cos(t) \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix} \qquad \& \qquad X = \begin{bmatrix} e^t & \cos(t) & \sin(t) \\ e^t & -\sin(t) & \cos(t) \\ e^t & -\cos(t) & -\sin(t) \end{bmatrix}$$

## 5.5 solutions by matrix exponential

Recall the Maclaurin series for the exponential is given by:

$$e^t = \sum_{j=o}^{\infty} \frac{t^j}{j!} = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots$$

This provided the inspiration for the definition given below[7]

**Definition 5.5.1.** *matrix exponential*

Suppose $A$ is an $n \times n$ matrix then we define the **matrix exponential of** $A$ by:

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

Suppose $A = 0$ is the zero matrix. Note that

$$e^0 = I + 0 + \frac{1}{2}0^2 + \cdots = I.$$

Furthermore, as $(-A)^j = (-1)^j A^j$ it follows that $e^{-A} = I - A + \frac{1}{2}A^2 - \frac{1}{6}A^3 + \cdots$. Hence,

$$\begin{aligned}
e^A e^{-A} &= \left(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots\right)\left(I - A + \frac{1}{2}A^2 - \frac{1}{6}A^3 + \cdots\right) \\
&= I - A + \frac{1}{2}A^2 - \frac{1}{6}A^3 + \cdots + A\left(I - A + \frac{1}{2}A^2 + \cdots\right) + \frac{1}{2}A^2\left(I - A + \cdots\right) + \frac{1}{6}A^3 I + \cdots \\
&= I + A - A + \frac{1}{2}A^2 - A^2 + \frac{1}{2}A^2 - \frac{1}{6}A^3 + \frac{1}{2}A^3 - \frac{1}{2}A^3 + \frac{1}{6}A^3 + \cdots \\
&= I.
\end{aligned}$$

I have only shown the result up to the third-order in $A$, but you can verify higher orders if you wish. We find an interesting result:

$$\boxed{(e^A)^{-1} = e^{-A} \qquad \Rightarrow \qquad det(e^A) \neq 0 \qquad \Rightarrow \qquad \text{columns of } A \text{ are LI.}}$$

[7] the concept of an exponential actually extends in much more generality than this, we could derive this from more basic and general principles, but that has little to do with this course so we behave. In addition, the reason the series of matrices below converges is not immediately obvious, see my linear notes for a sketch of the analysis needed here

Noncommutativity of matrix multiplication spoils the usual law of exponents. Let's examine how this happens. Suppose $A, B$ are square matrices. Calculate $e^{A+B}$ to second order in $A, B$:

$$e^{A+B} = I + (A+B) + \frac{1}{2}(A+B)^2 + \cdots = I + A + B + \frac{1}{2}\big(A^2 + AB + BA + B^2\big) + \cdots$$

On the other hand, calculate the product $e^A e^B$ to second order in $A, B$,

$$e^A e^B = (I + A + \frac{1}{2}A^2 + \cdots)(I + B + \frac{1}{2}B^2 + \cdots) = I + A + B + \frac{1}{2}\big(A^2 + 2AB + B^2\big) + \cdots$$

We find that, to second order, $e^A e^B - e^{A+B} = \frac{1}{2}(AB - BA)$. Define the **commutator** $[A, B] = AB - BA$ and note (after a short calculation)

$$\boxed{e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}}$$

When $A, B$ are **commuting** matrices the commutator $[A, B] = AB - BA = AB - AB = 0$ hence the usual algebra $e^A e^B = e^{A+B}$ applies. It turns out that the higher-order terms in the boxed formula above can be written as nested-commutators of $A$ and $B$. This formula is known as the Baker-Campbell-Hausdorff, it is the essential calculation in the theory of matrix Lie groups (which is the math used to formulate important symmetry aspects of modern physics).

Let me pause[8] to give a better proof that $AB = BA$ implies $e^A e^B = e^{A+B}$. The heart of the argument follows from the fact the binomial theorem holds for $(A + B)^k$ in this context. I seek to prove by mathematical induction on $k$ that $(A + B)^k = \sum_{n=0}^{k} \binom{k}{n} A^{k-n} B^n$. Note $k = 1$ is clearly true as $\binom{1}{0} = \binom{1}{1} = 1$ and $(A + B)^1 = A + B$. Assume inductively the binomial theorem holds for $k$ and seek to prove $k + 1$ true:

$$\begin{aligned}
(A+B)^{k+1} &= (A+B)^k(A+B) \\
&= \left(\sum_{n=0}^{k}\binom{k}{n}A^{k-n}B^n\right)(A+B) && : \text{by induction hypothesis} \\
&= \sum_{n=0}^{k}\binom{k}{n}A^{k-n}B^nA + \sum_{n=0}^{k}\binom{k}{n}A^{k-n}B^nB \\
&= \sum_{n=0}^{k}\binom{k}{n}A^{k-n}AB^n + \sum_{n=0}^{k}\binom{k}{n}A^{k-n}B^{n+1} && : AB = BA \text{ implies } B^nA = AB^n \\
&= \sum_{n=0}^{k}\binom{k}{n}A^{k+1-n}B^n + \sum_{n=0}^{k}\binom{k}{n}A^{k-n}B^{n+1}
\end{aligned}$$

[8]you may skip ahead if you are not interested in how to make arguments precise, in fact, even this argument has gaps, but I include it to give the reader some idea about what is missing when we resort to $+\cdots$-style induction

Continuing,

$$
\begin{aligned}
(A+B)^{k+1} &= A^{k+1} + \sum_{n=1}^{k} \binom{k}{n} A^{k+1-n}B^n + \sum_{n=0}^{k-1} \binom{k}{n} A^{k-n}B^{n+1} + B^{k+1} \\
&= A^{k+1} + \sum_{n=1}^{k} \binom{k}{n} A^{k+1-n}B^n + \sum_{n=1}^{k} \binom{k}{n-1} A^{k+1-n}B^{n} + B^{k+1} \\
&= A^{k+1} + \sum_{n=1}^{k} \left[\binom{k}{n} + \binom{k}{n-1}\right] A^{k+1-n}B^n + B^{k+1} \\
&= A^{k+1} + \sum_{n=1}^{k} \binom{k+1}{n} A^{k+1-n}B^n + B^{k+1} \qquad : \text{by Pascal's Triangle} \\
&= \sum_{n=0}^{k+1} \binom{k+1}{n} A^{k+1-n}B^n
\end{aligned}
$$

Which completes the induction step and we find by mathematical induction the binomial theorem for commuting matrices holds for all $k \in \mathbb{N}$ . Consider the matrix expontial formula in light of the binomial theorem, also recall $\binom{k+1}{n} = \frac{k!}{n!(k-n)!}$,

$$
\begin{aligned}
e^{A+B} &= \sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{1}{k!}\frac{k!}{n!(k-n)!} A^{k-n}B^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{1}{n!}\frac{1}{(k-n)!} A^{k-n}B^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{1}{n!}\frac{1}{(k-n)!} A^{k-n}B^n
\end{aligned}
$$

On the other hand, if we compute the product of $e^A$ with $e^B$ we find:

$$
e^A e^B = \sum_{j=0}^{\infty} \frac{1}{j!}A^j \sum_{n=0}^{\infty} \frac{1}{n!}B^n = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{j!}A^jB^n
$$

It follows[9] that $e^Ae^B = e^{A+B}$. We use this result implicitly in much of what follows in this section.

[9] after some analytical arguments beyond this course; what is missing is an explicit examination of the infinite limits at play here, the doubly infinite limits seem to reach the same terms but the structure of the sums differs

Suppose $A$ is a constant $n \times n$ matrix. Calculate[10]

$$\begin{aligned}
\tfrac{d}{dt}\left[exp(tA)\right] &= \tfrac{d}{dt}\left[\sum_{k=0}^{\infty} \tfrac{1}{k!}t^k A^k\right] && \text{defn. of matrix exponential} \\
&= \sum_{k=0}^{\infty} \tfrac{d}{dt}\left[\tfrac{1}{k!}t^k A^k\right] && \text{since matrix exp. uniformly conv.} \\
&= \sum_{k=0}^{\infty} \tfrac{k}{k!}t^{k-1} A^k && A^k \text{ constant and } \tfrac{d}{dt}(t^k) = kt^{k-1} \\
&= A\sum_{k=1}^{\infty} \tfrac{1}{(k-1)!}t^{k-1} A^{k-1} && \text{since } k! = k(k-1)! \text{ and } A^k = AA^{k-1}. \\
&= Aexp(tA) && \text{defn. of matrix exponential.}
\end{aligned}$$

I suspect the following argument is easier to follow:

$$\begin{aligned}
\tfrac{d}{dt}(exp(tA)) &= \tfrac{d}{dt}(I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots) \\
&= \tfrac{d}{dt}(I) + \tfrac{d}{dt}(tA) + \tfrac{1}{2}\tfrac{d}{dt}(t^2A^2) + \tfrac{1}{3\cdot 2}\tfrac{d}{dt}(t^3A^3) + \cdots \\
&= A + tA^2 + \tfrac{1}{2}t^2A^3 + \cdots \\
&= A(I + tA + \tfrac{1}{2}t^2A^2 + \cdots) \\
&= Aexp(tA). \qquad \square
\end{aligned}$$

Whichever notation you prefer, the calculation above completes the proof of the following central theorem for this section:

**Theorem 5.5.2.**

Suppose $A \in \mathbb{R}^{n\times n}$. The matrix exponential $e^{tA}$ gives a fundamental matrix for $\frac{d\vec{x}}{dt} = A\vec{x}$.

**Proof:** we have already shown that (1.) $e^{tA}$ is a solution matrix ($\frac{d}{dt}[e^{tA}] = Ae^{tA}$) and (2.) $(e^{tA})^{-1} = e^{-tA}$ thus the columns of $e^{tA}$ are LI. $\square$

It follows that the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is simply $\vec{x}(t) = e^{tA}\vec{c}$ where $\vec{c} = [c_1, c_2, \dots, c_n]^T$ determines the initial conditions of the solution. In theory this is a great formula, we've solved most of the problems we set-out to solve. However, more careful examination reveals this result is much like the result from calculus; any continuous function is integrable. Ok, so $f$ continuous on an interval $I$ implies $F$ exists on $I$ and $F' = f$, but... how do you actually calculate the antiderivative $F$? It's possible in principle, but in practice the computation may fall outside the computation scope of the techniques covered in calculus[11].

**Example 5.5.3.** *Suppose $x' = x, y' = 2y, z' = 3z$ then in matrix form we have:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

[10] the term-by-term differentiation theorem for power series extends to a matrix power series, the proof of this involves real analysis

[11] for example, $\int \frac{\sin(x)dx}{x}$ or $\int e^{-x^2}dx$ are known to by incalculable in terms of elementary functions

*The coefficient matrix is* diagonal *which makes the $k$-th power particularly easy to calculate,*

$$A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \sum_{k=0}^{\infty} \tfrac{t^k}{k!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}1^k & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}2^k & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

*Thus we find three solutions to $x' = Ax$,*

$$x_1(t) = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} \qquad x_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

*In turn these vector solutions amount to the solutions $x = e^t, y = 0, z = 0$ or $x = 0, y = e^{2t}, z = 0$ or $x = 0, y = 0, z = e^{3t}$. It is easy to check these solutions.*

Of course the example above is very special. In order to unravel the mystery of just how to calculate the matrix exponential for less trivial matrices we return to the construction of the previous section. Let's see what happens when we calculate $e^{tA}\vec{u}$ for $\vec{u}$ and e-vector with e-value $\lambda$.

$$\begin{aligned} e^{tA}\vec{u} &= e^{t(A-\lambda I+\lambda I)}\vec{u} && : \text{added zero anticipating use of } (A-\lambda I)\vec{u} = 0, \\ &= e^{t\lambda I + t(A-\lambda I)}\vec{u} \\ &= e^{t\lambda I}e^{t(A-\lambda I)}\vec{u} && : \text{noted that } t\lambda I \text{ commutes with } t(A-\lambda I), \\ &= e^{t\lambda}Ie^{t(A-\lambda I)}\vec{u} && : \text{a short exercise shows } e^{t\lambda I} = e^{t\lambda}I. \\ &= e^{t\lambda}\left(I + t(A-\lambda I) + \frac{t^2}{2}t(A-\lambda I)^2 + \cdots\right)\vec{u} \\ &= e^{t\lambda}\big(I\vec{u} + t(A-\lambda I)\vec{u} + \frac{t^2}{2}t(A-\lambda I)^2\vec{u} + \cdots\big) \\ &= e^{t\lambda}\vec{u} && : \text{as it was given } (A-\lambda I)\vec{u} = 0 \text{ hence all but the first term vanishes.} \end{aligned}$$

The fact that this is a a solution of $\vec{x}\,' = A\vec{x}$ was already known to us, however, it is nice to see it arise from the matrix exponential. Moreover the calculation above reveals the central formula that guides the technique of this section. The **magic formula**. For any square matrix and possibly constant $\lambda$ we find:

$$\boxed{e^{tA} = e^{t\lambda}\left(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \cdots\right) = e^{t\lambda}\sum_{k=0}^{\infty}\frac{t^k}{k!}(A-\lambda I)^k.}$$

When we choose $\lambda$ as an e-value and multiply this formula by the corresponding e-vector then this infinite series truncates nicely to reveal $e^{\lambda t}\vec{u}$. It follows that we should define vectors which truncate the series at higher order, this is the natural next step:

**Definition 5.5.4.** *generalized eigenvectors and chains of generalized e-vectors*

Given an eigenvalue $\lambda$ a nonzero vector $\vec{u}$ such that $(A-\lambda I)^p\vec{u} = 0$ is called an **generalized eigenvector of order** $p$ with eigenvalue $\lambda$. If $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p$ are nonzero vectors such that $(A-\lambda I)\vec{u}_j = \vec{u}_{j-1}$ for $j = 2, 3, \dots, p$ and $\vec{u}_1$ is an e-vector with e-value $\lambda$ then we say $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ forms a **chain of generalized e-vectors of length** $p$.

In the notation of the definition above, it is true that $\vec{u}_k$ is a generalized e-vector of order $k$ with e-value $\lambda$. Let's examine $k = 2$,

$$(A-\lambda I)\vec{u}_2 = \vec{u}_1 \qquad \Rightarrow \qquad (A-\lambda I)^2\vec{u}_2 = (A-\lambda I)\vec{u}_1 = 0.$$

Then suppose inductively the claim is true for $k$ which means $(A-\lambda I)^k\vec{u}_k = 0$, consider $k+1$

$$(A-\lambda I)\vec{u}_{k+1} = \vec{u}_k \qquad \Rightarrow \qquad (A-\lambda I)^{k+1}\vec{u}_{k+1} = (A-\lambda I)^k\vec{u}_k = 0.$$

Hence, in terms of the notation in the definition above, we have shown by mathematical induction that $\vec{u}_k$ is a generalized e-vector of order $k$ with e-value $\lambda$.

I do not mean to claim this is true for all $k \in \mathbb{N}$. In practice for an $n \times n$ matrix we cannot find a chain longer than length $n$. However, up to that bound such chains are possible for an arbitrary matrix.

**Example 5.5.5.** *The matrices below are in* **Jordan form** *which means the vectors* $e_1 = [1,0,0,0,0]^T$ *etc...* $e_5 = [0,0,0,0,1]^T$ *are (generalized)-e-vectors:*

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad \& \qquad B = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

*You can easily calculate* $(A-2I)e_1 = 0$, $(A-2I)e_2 = e_1$, $(A-2I)e_3 = e_2$ *or* $(A-3I)e_4 = 0$, $(A-2I)e_5 = e_4$. *On the other hand,* $(B-4I)e_1 = 0$, $(B-4I)e_2 = e_1$ *and* $(A-5I)e_3 = 0$ *and* $(A-6I)e_4 = 0$, $(A-6I)e_5 = 0$. *The matrix B needs only one generalized e-vector whereas the matrix A has* 3 *generalized e-vectors.*

Let's examine why chains are nice for the magic formula:

**Example 5.5.6. Problem:** *Suppose A is a* $3\times 3$ *matrix with a chain of generalized e-vector* $\vec{u}_1, \vec{u}_2, \vec{u}_3$ *with respect to e-value* $\lambda = 2$. *Solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *in view of these facts.*

**Solution:** *we are given* $(A-2I)\vec{u}_1 = 0$ *and* $(A-2I)\vec{u}_2 = \vec{u}_1$ *and* $(A-2I)\vec{u}_3 = \vec{u}_2$. *It is easily shown that* $(A-2I)^2\vec{u}_2 = 0$ *and* $(A-2I)^3\vec{u}_3 = 0$. *It is also possible to prove* $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ *is a LI set. Apply the magic formula with* $\lambda = 2$ *to derive the following results:*

1. $\vec{x}_1(t) = e^{tA}\vec{u}_1 = e^{2t}\vec{u}_1$ *(we've already shown this in general earlier in this section)*
2. $\vec{x}_2(t) = e^{tA}\vec{u}_2 = e^{2t}\big(I\vec{u}_2 + t(A-2I)\vec{u}_2 + \frac{t^2}{2}(A-2I)^2\vec{u}_2 + \cdots\big) = e^{2t}(\vec{u}_2 + t\vec{u}_1)$.
3. *note that* $(A-2I)^2\vec{u}_3 = (A-2I)\vec{u}_2 = \vec{u}_1$ *hence:*

$$\vec{x}_3(t) = e^{tA}\vec{u}_3 = e^{2t}\big(I\vec{u}_3 + t(A-2I)\vec{u}_3 + \frac{t^2}{2}(A-2I)^2\vec{u}_3 + \cdots\big) = e^{2t}(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_1).$$

*Therefore,* $\boxed{\vec{x}(t) = c_1e^{2t}\vec{u}_1 + c_2e^{2t}(\vec{u}_2 + t\vec{u}_1) + c_3e^{2t}(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_1)}$ *is the general solution.*

Perhaps it is interesting to calculate $e^{tA}[\vec{u}_1|\vec{u}_2|\vec{u}_3]$ in view of the calculations in the example above. Observe:

$$e^{tA}[\vec{u}_1|\vec{u}_2|\vec{u}_3] = [e^{tA}\vec{u}_1|e^{tA}\vec{u}_2|e^{tA}\vec{u}_3] = e^{2t}\left[\vec{u}_1\middle|\vec{u}_2 + t\vec{u}_1\middle|\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_1\right].$$

I suppose we could say more about this formula, but let's get back on task: we seek to complete the solution of the unsolved problems of the previous section. It is our hope that we can find generalized e-vector solutions to complete the fundamental solution sets in Examples 5.4.4, 5.4.10,5.4.12 and 5.4.13.

**Example 5.5.7. Problem: (returning to Example 5.4.4)** *solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$

**Solution:** *we found* $\lambda_1 = 1$ *and* $\lambda_2 = 1$ *and a single e-vector* $\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. *Now seek a generalized e-vector* $\vec{u}_2 = [u, v]^T$ *such that* $(A - I)\vec{u}_2 = \vec{u}_1$,

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad\Rightarrow\quad 2u + v = 1 \quad\Rightarrow\quad v = 1 - 2u,\ u \neq 0 \quad\Rightarrow\quad \vec{u}_1 = \begin{bmatrix} u \\ 1 - 2u \end{bmatrix}$$

*We choose* $u = 0$ *for convenience and thus find* $\vec{u}_2 = [0, 1]^T$ *hence the fundamental solution*

$$\vec{x}_2(t) = e^{tA}\vec{u}_2 = e^t(I + t(A - I) + \cdots)\vec{u}_2 = e^t(\vec{u}_2 + t\vec{u}_1) = e^t\begin{bmatrix} t \\ 1 - 2t \end{bmatrix}.$$

*Therefore, we find* $\boxed{\vec{x}(t) = c_1e^t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2e^t\begin{bmatrix} t \\ 1 - 2t \end{bmatrix}}$.

**Example 5.5.8. Problem: (returning to Example 5.4.10)** *solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$.

**Solution:** *we found* $\lambda_1 = 2$, $\lambda_2 = 2$ *and* $\lambda_3 = 3$ *and we also found e-vector* $\vec{u}_1 = [1, 0, -1]^T$ *with e-value* 2 *and e-vector* $\vec{u}_2 = [0, 0, 1]^T$. *Seek* $\vec{u}_3$ *such that* $(A - 2I)\vec{u}_3 = \vec{u}_1$ *since we are missing a solution paired with* $\lambda_2 = 2$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad\Rightarrow\quad \begin{matrix} v = 1 \\ u - 1 + w = -1 \end{matrix} \quad\Rightarrow\quad \begin{matrix} v = 1 \\ w = -u \end{matrix} \quad\Rightarrow\quad \vec{u}_1 = \begin{bmatrix} u \\ 1 \\ -u \end{bmatrix}.$$

*Choose* $u = 0$ *to select* $\vec{u}_1 = [0, 1, 0]^T$. *It follows from the magic formula that* $\vec{x}_3(t) = e^{tA}\vec{u}_3 = e^{2t}(\vec{u}_3 + t\vec{u}_1)$. *Hence, the general solution is*

$$\boxed{\vec{x}(t) = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3e^{2t}\begin{bmatrix} t \\ 1 \\ -t \end{bmatrix}}.$$

Once more we found a generalized e-vector of order two to complete the solution set and find $\vec{x}_3$ in the example above. You might notice that had we replaced the choice $u = 0$ in both of the last examples with some nonzero $u$ then we would have added a copy of $\vec{x}_1$ to the generalized e-vector solution. This is permissable since the sum of solutions to the system $\vec{x}\,' = A\vec{x}$ is once more a solution. This freedom works hand-in-hand with the ambiguity of the generalized e-vector problem.

**Example 5.5.9. Problem: (returning to Example 5.4.12)** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$.

**Solution:** *We already found* $\lambda_1 = -2$, $\lambda_2 = -2$ *and* $\lambda_3 = -2$ *and a pair of e-vectors* $\vec{u}_1 = [0,1,0]^T$ *and* $v = 0, w = 1$ *to select* $\vec{u}_2 = [0,0,1]^T$. *We face a dilemma, should we look for a chain that ends with* $\vec{u}_1 = [0,1,0]^T$ *or* $\vec{u}_2 = [0,0,1]^T$*? Generally it may not be possible to do either. Thus, we set aside the chain condition and instead look for directly for solutions of* $(A+2I)^2\vec{u}_3 = 0$.

$$(A+2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

*Since we seek* $\vec{u}_3$ *which forms a LI set with* $\vec{u}_1, \vec{u}_2$ *it is natural to choose* $\vec{u}_3 = [1,0,0]^T$. *Calculate,*

$$\begin{aligned} \vec{x}_3(t) = e^{tA}\vec{u}_3 &= e^{-2t}\Big(I\vec{u}_3 + t(A+2I)\vec{u}_3 + \frac{t^2}{2}(A+2I)^2\vec{u}_3 + \cdots\Big) \qquad (5.1) \\ &= e^{-2t}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right] \\ &= e^{-2t}\begin{bmatrix} 1 \\ 4t \\ t \end{bmatrix} \end{aligned}$$

*Thus we find the general solution:*

$$\boxed{\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} 1 \\ 4t \\ t \end{bmatrix}}.$$

I leave the complete discussion of the chains in the subtle case above for the second course on linear algebra. See Insel Spence and Friedberg's *Linear Algebra* text for an accessible treatment aimed at advanced undergraduates.

**Example 5.5.10. Problem: (returning to Example 5.4.13)** *we seek to solve* $\frac{d\vec{x}}{dt} = A\vec{x}$ *where* $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$.

**Solution:** *we found* $\lambda_1 = -1$, $\lambda_2 = -1$ *and* $\lambda_3 = -1$ *and a single e-vector* $\vec{u}_1 = [1,0,3]^T$. *Seek* $\vec{u}_2$ *such that* $(A+I)\vec{u}_2 = \vec{u}_1$,

$$\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \;\Rightarrow\; \begin{matrix} 3u+v-w=1 \\ -3u+w=0 \end{matrix} \;\Rightarrow\; \begin{matrix} w = 3u \\ v = w-3u+1 \end{matrix} \;\Rightarrow\; \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

*where we set $u = 0$ for convenience. Continuing, we seek $\vec{u}_3$ where $(A+I)\vec{u}_3 = \vec{u}_2$,*

$$\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} 3u + v - w = 0 \\ -3u + w = 1 \end{array} \Rightarrow \begin{array}{c} w = 1 + 3u \\ v = w - 3u \end{array} \Rightarrow \begin{array}{c} w = 1 + 3u \\ v = 1 \end{array}$$

*Choose $u = 0$ to select $\vec{u}_3 = [0,1,1]^T$. Given the algebra we've completed we know that*

$$(A+I)\vec{u}_1 = (A+I)^2\vec{u}_2 = (A+I)^3\vec{u}_3 = 0, \quad (A+I)\vec{u}_2 = \vec{u}_1, \quad (A+I)\vec{u}_3 = \vec{u}_2, \quad (A+I)^2\vec{u}_3 = \vec{u}_1$$

*These identities paired with the magic formula with $\lambda = -1$ yield:*

$$e^{tA}\vec{u}_1 = e^{-t}\vec{u}_1 \qquad \& \qquad e^{tA}\vec{u}_2 = e^{-t}(\vec{u}_2 + t\vec{u}_1) \qquad \& \qquad e^{tA}\vec{u}_3 = e^{-t}(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_1)$$

*Therefore, we find general solution:*

$$\boxed{\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} t \\ 1 \\ 3t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 1 + t + \frac{t^2}{2} \\ 1 + \frac{t^2}{2} \end{bmatrix}}.$$

The method we've illustrated extends naturally to the case of repeated complex e-values where there are insufficient e-vectors to form the general solution.

**Example 5.5.11. Problem:** *Suppose $A$ is a $6 \times 6$ matrix with a chain of generalized e-vector $\vec{u}_1, \vec{u}_2, \vec{u}_3$ with respect to e-value $\lambda = 2 + i$. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in view of these facts.*

**Solution:** *we are given $(A - (2+i)I)\vec{u}_1 = 0$ and $(A - (2+i)I)\vec{u}_2 = \vec{u}_1$ and $(A - (2+i)I)\vec{u}_3 = \vec{u}_2$. It is easily shown that $(A-(2+i)I)^2\vec{u}_2 = 0$ and $(A-(2+i)I)^3\vec{u}_3 = 0$. It is also possible to prove $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a LI set. Apply the magic formula with $\lambda = (2+i)$ to derive the following results:*

1. $\vec{x}_1(t) = e^{tA}\vec{u}_1 = e^{(2+i)t}\vec{u}_1$ *(we've already shown this in general earlier in this section)*
2. $\vec{x}_2(t) = e^{tA}\vec{u}_2 = e^{(2+i)t}\big(I\vec{u}_2 + t(A-(2+i)I)\vec{u}_2 + \frac{t^2}{2}(A-(2+i)I)^2\vec{u}_2 + \cdots\big) = e^{(2+i)t}(\vec{u}_2 + t\vec{u}_1)$.
3. *note that $(A-(2+i)I)^2\vec{u}_3 = (A-(2+i)I)\vec{u}_2 = \vec{u}_1$ hence:*

$$\vec{x}_3(t) = e^{tA}\vec{u}_3 = e^{(2+i)t}\big(I\vec{u}_3 + t(A-(2+i)I)\vec{u}_3 + \frac{t^2}{2}(A-(2+i)I)^2\vec{u}_3 + \cdots\big) = e^{(2+i)t}(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_1).$$

*The solutions $\vec{x}_1(t)$, $\vec{x}_2(t)$ and $\vec{x}_3(t)$ are complex-valued solutions. To find the real solutions we select the real and imaginary parts to form the fundamental solution set*

$$\{Re(\vec{x}_1), Im(\vec{x}_1), Re(\vec{x}_2), Im(\vec{x}_2), Re(\vec{x}_3), Im(\vec{x}_3)\}$$

*I leave the explicit formulas to the reader, it is very similar to the case we treated in the last section for the complex e-vector problem.*

Suppose $A$ is idempotent or order $k$ then $A^{k-1} \neq I$ and $A^k = I$. In this case the matrix exponential simplifies:

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1} + \left(\frac{t^k}{k!} + \frac{t^{k+1}}{(k+1)!} + \cdots\right)I$$

However, $\frac{t^k}{k!} + \frac{t^{k+1}}{(k+1)!} + \cdots = e^t - 1 - t - \frac{t^2}{2} - \cdots - \frac{t^{k-1}}{(k-1)!}$ hence we can calculate $e^{tA}$ nicely in such a case. On the other hand, if the matrix $A$ is nilpotent of order $k$ then $A^{k-1} \neq 0$ and $A^k = 0$. Once again, the matrix exponential simplifies:

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}$$

Therefore, if $A$ is nilpotent then we can calculate the matrix exponential directly without too much trouble... of course this means we can solve $\vec{x}\,' = A\vec{x}$ without use of the generalized e-vector method.

Finally, I conclude this section with a few comments about direct computation via the Cayley Hamilton Theorem (this is proved in an advanced linear algebra course)

**Theorem 5.5.12.**

If $A \in \mathbb{R}^{n\times n}$ and $p(\lambda) = det(A - \lambda I) = 0$ is the characteristic equation then $p(A) = 0$.

Note that if $p(x) = x^2 + 3$ then $p(A) = A^2 + 3I$.

**Example 5.5.13. Problem: solve the system given in Example 5.4.13)** *by applying the Cayley Hamilton Theorem to* $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$.

**Solution:** *we found* $p(\lambda) = -(\lambda - 1)^3 = 0$ *hence* $-(A - I)^3 = 0$. *Consider the magic formula:*

$$e^{tA} = e^t(I + t(A - I) + \frac{t^2}{2}(A - I)^2 + \frac{t^3}{3!}(A - I)^3 + \cdots) = e^t(I + t(A - I) + \frac{t^2}{2}(A - I)^2)$$

*Calculate,*

$$A - I = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -2 & 1 \\ 9 & 3 & -5 \end{bmatrix} \qquad \& \qquad (A - I)^2 = \begin{bmatrix} -11 & -4 & 5 \\ 12 & 2 & -4 \\ -45 & -12 & 19 \end{bmatrix}$$

*Therefore,*

$$e^{tA} = e^t \begin{bmatrix} 1 + t - \frac{11t^2}{2} & t - 2t^2 & -t + \frac{5t^2}{2} \\ -3t + 6t^2 & 1 - 2t + t^2 & t - 2t^2 \\ 9t - \frac{45t^2}{2} & 3t - 6t^2 & 1 - 5t - \frac{19t^2}{2} \end{bmatrix}$$

*The general solution is given by* $\vec{x}(t) = e^{tA}\vec{c}$.

There are certainly additional short-cuts and deeper understanding that stem from a working knowledge of full-fledged linear algebra, but, I hope I have shown you more than enough in these notes to solve any constant-coefficient system $\vec{x}\,' = A\vec{x}$. It turns out there are always enough generalized e-vectors to complete the solution. The existence of the basis made of generalized e-vectors (called a **Jordan basis**) is a deep theorem of linear algebra. It is often, sadly, omitted from undergraduate linear algebra texts. The pair of examples below illustrate some of the geometry behind the calculations of this section.

**Example 5.5.14.** *Consider for example, the system*

$$x' = x + y, \qquad y' = 3x - y$$

*We can write this as the matrix problem*

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{d\vec{x}/dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

*It is easily calculated that $A$ has eigenvalue $\lambda_1 = -2$ with e-vector $\vec{u}_1 = (-1, 3)$ and $\lambda_2 = 2$ with e-vectors $\vec{u}_2 = (1, 1)$. The general solution of $d\vec{x}/dt = A\vec{x}$ is thus*

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{2t} \\ 3c_1 e^{-2t} + c_2 e^{2t} \end{bmatrix}$$

*So, the* **scalar solutions** *are simply* $\boxed{x(t) = -c_1 e^{-2t} + c_2 e^{2t}}$ *and* $\boxed{y(t) = 3c_1 e^{-2t} + c_2 e^{2t}}$.

Thus far I have simply told you how to solve the system $d\vec{x}/dt = A\vec{x}$ with e-vectors, it is interesting to see what this means geometrically. For the sake of simplicity we'll continue to think about the preceding example. In it's given form the DEqn is **coupled** which means the equations for the derivatives of the dependent variables $x, y$ cannot be solved one at a time. We have to solve both at once. In the next example I solve the same problem we just solved but this time using a change of variables approach.

**Example 5.5.15.** *Suppose we change variables using the diagonalization idea: introduce new variables $\bar{x}, \bar{y}$ by $P(\bar{x}, \bar{y}) = (x, y)$ where $P = [\vec{u}_1|\vec{u}_2]$. Note $(\bar{x}, \bar{y}) = P^{-1}(x, y)$. We can diagonalize $A$ by the similarity transformation by $P$; $D = P^{-1}AP$ where $Diag(D) = (-2, 2)$. Note that $A = PDP^{-1}$ hence $d\vec{x}/dt = A\vec{x} = PDP^{-1}\vec{x}$. Multiply both sides by $P^{-1}$:*

$$P^{-1}\frac{d\vec{x}}{dt} = P^{-1}PDP^{-1}\vec{x} \quad \Rightarrow \quad \frac{d(P^{-1}\vec{x})}{dt} = D(P^{-1}\vec{x}).$$

*You might not recognize it but the equation above is decoupled. In particular, using the notation $(\bar{x}, \bar{y}) = P^{-1}(x, y)$ we read from the matrix equation above that*

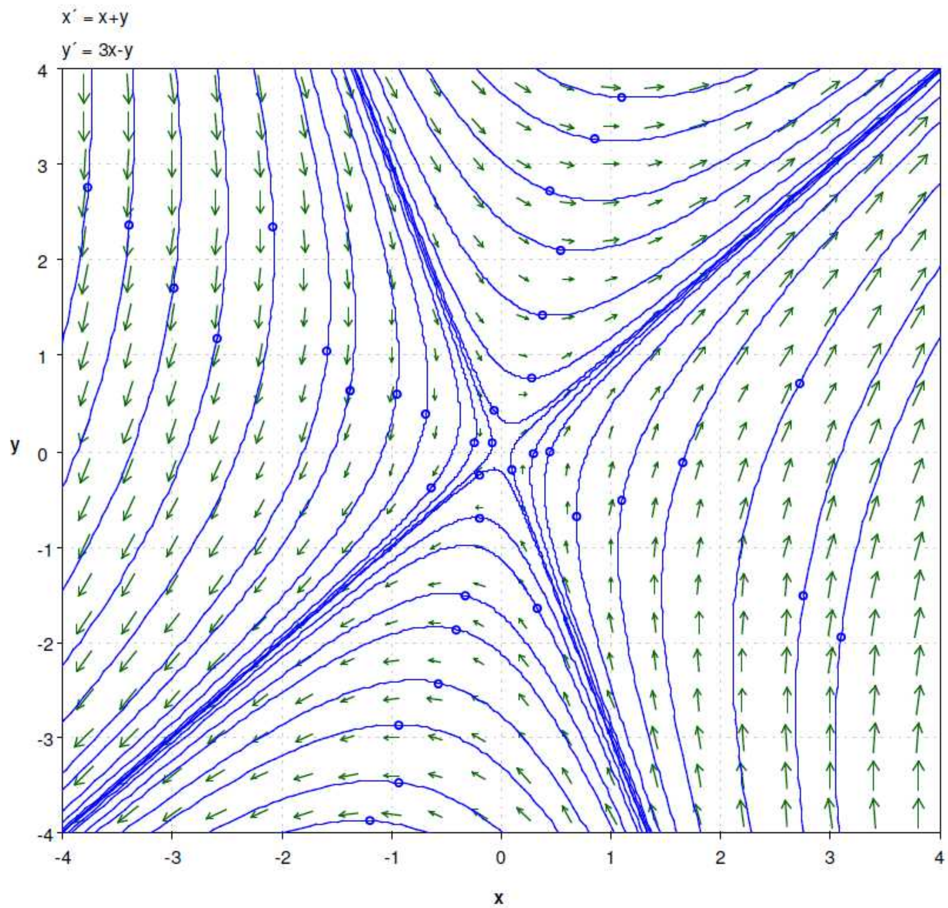
$$\frac{d\bar{x}}{dt} = -2\bar{x}, \qquad \frac{d\bar{y}}{dt} = 2\bar{y}.$$

*Separation of variables and a little algebra yields that $\bar{x}(t) = c_1 e^{-2t}$ and $\bar{y}(t) = c_2 e^{2t}$. Finally, to find the solution back in the original coordinate system we multiply $P^{-1}\vec{x} = (c_1 e^{-2t}, c_2 e^{2t})$ by $P$ to isolate $\vec{x}$,*

$$\vec{x}(t) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{2t} \\ 3c_1 e^{-2t} + c_2 e^{2t} \end{bmatrix}.$$

*This is the same solution we found in the last example. Usually linear algebra texts present this solution because it shows more interesting linear algebra, however, from a pragmatic viewpoint the first method is clearly faster.*

Finally, we can better appreciate the solutions we found if we plot the direction field $(x', y') = (x+y, 3x-y)$ via the "pplane" tool in Matlab. I have clicked on the plot to show a few representative trajectories (solutions):



## 5.6 nonhomogeneous problem

**Theorem 5.6.1.**

The nonhomogeneous case $\vec{x}\,' = A\vec{x} + \vec{f}$ the general solution is $\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t)$ where $\vec{c}$ is a vector of constants, $X$ is a fundamental matrix for the corresponding homogeneous system and $\vec{x}_p$ is a particular solution to the nonhomogeneous system. We can calculate $\vec{x}_p(t) = X(t)\int X^{-1}\vec{f}dt$.

**Proof:** suppose that $\vec{x}_p = X\vec{v}$ for $X$ a fundamental matrix of $\vec{x}\,' = A\vec{x}$ and some vector of unknown functions $\vec{v}$. We seek conditions on $\vec{v}$ which make $\vec{x}_p$ satisfy $\vec{x}_p\,' = A\vec{x}_p + \vec{f}$. Consider,

$$(\vec{x}_p)' = (X\vec{v})' = X'\vec{v} + X\vec{v}' = AX\vec{v} + X\vec{v}'$$

But, $\vec{x}_p\,' = A\vec{X}_p + \vec{f} = AX\vec{v} + \vec{f}$ hence

$$X\tfrac{d\vec{v}}{dt} = \vec{f} \quad \Rightarrow \quad \tfrac{d\vec{v}}{dt} = X^{-1}\vec{f}$$

Integrate to find $\vec{v} = \int X^{-1}\vec{f}dt$ therefore $x_p(t) = X(t)\int X^{-1}\vec{f}dt$. $\square$

If you ever work through variation of parameters for higher order ODEqns then you should appreciate the calculation above. In fact, we can derive $n$-th order variation of parameters from converting the $n$-th order ODE by reduction of order to a system of $n$ first order linear ODEs. You can show that the so-called Wronskian of the fundamental solution set is precisely the determinant of the fundamental matrix for the system $\vec{x}\,' = A\vec{x}$ where $A$ is the companion matrix.

**Example 5.6.2. Problem:** *Suppose that* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *and* $\vec{f} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$*, find the general solution of the nonhomogenous DEqn* $\vec{x}\,' = A\vec{x} + \vec{f}$*.*

**Solution:** *you can easily show* $\vec{x}\,' = A\vec{x}$ *has fundamental matrix* $X = \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}$*. Use variation of parameters for systems of ODEs to constuct* $\vec{x}_p$*. First calculate the inverse of the fundamental matrix, for a* $2 \times 2$ *we know a formula:*

$$X^{-1}(t) = \tfrac{1}{e^{4t}-(-3)e^{4t}}\begin{bmatrix} e^{4t} & -e^{4t} \\ 3 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t} \end{bmatrix}$$

*Thus,*

$$\begin{aligned}
x_p(t) = X(t)\int \tfrac{1}{4}\begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t} \end{bmatrix}\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt &= \tfrac{1}{4}X(t)\int \begin{bmatrix} e^t - e^{-t} \\ 3e^{-3t} + e^{-5t} \end{bmatrix} dt \\
&= \tfrac{1}{4}\begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}\begin{bmatrix} e^t + e^{-t} \\ -e^{-3t} - \frac{1}{5}e^{-5t} \end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} 1(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \\ -3(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} e^t + e^{-t} - e^t - \frac{1}{5}e^{-t} \\ -3e^t - 3e^{-t} - e^t - \frac{1}{5}e^{-t} \end{bmatrix} \\
&= \tfrac{1}{4}\begin{bmatrix} \frac{4}{5}e^{-t} \\ -4e^t - \frac{16}{5}e^{-t} \end{bmatrix}
\end{aligned}$$

*Therefore, the general solution is*

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} e^{-t} \\ -e^{t} - 4e^{-t} \end{bmatrix}.$$

*The general scalar solutions implicit within the general vector solution* $\vec{x}(t) = [x(t), y(t)]^T$ *are*

$$x(t) = c_1 + c_2 e^{4t} + \tfrac{1}{5}e^{-t} \qquad y(t) = -3c_1 + c_2 e^{4t} - \tfrac{1}{5}e^{t} - \tfrac{4}{5}e^{-t}.$$

I might ask you to solve a $3 \times 3$ system in the homework. The calculation is nearly the same as the preceding example with the small inconvenience that finding the inverse of a $3 \times 3$ requires some calculation.

**Remark 5.6.3.**

You might wonder how would you solve a system of ODEs $x' = Ax$ such that the coefficients $A_{ij}$ are not constant. The theory we've discussed holds true with appropriate modification of the interval of applicability. In the constant coefficient case $I = \mathbb{R}$ so we have had no need to discuss it. In order to solve non-constant coefficient problems we will need to find a method to solve the homogeneous problem to locate the fundamental matrix. Once that task is accomplished the technique of this section applies to solve any associated nonhomogeneous problem.

# Chapter 6

# the Laplace transform technique

I roughly follow Chapter 9 of Ritger and Rose. The section on linear systems analysis is deeper than many texts. For the most part the material can be found in any introductory DEqns text. I add discussion of periodic functions, that seems to be missing from Ritger and Rose. It is somewhat unlikely for me to have time to type these notes for the Spring 2013 semester. I will probably post some pdf's and link to somewhich are already posted on my website when we approach this material in lecture. It is important you take complete notes in lecture.

## 6.1 history and overview

## 6.2 definition and existence

## 6.3 basic techniques

## 6.4 the inverse transform

## 6.5 discontinuous functions

## 6.6 transform of periodic functions

## 6.7 the dirac delta device

## 6.8 convolution

## 6.9 linear system theory

# Chapter 7

# energy analysis and the phase plane approach

This chapter collects our thoughts on how to use energy to study problems in Newtonian mechanics. In particular we explain how to plot possible motions in the Poincare plane ($x, \dot{x}$ plane, or the one-dimensional tangent bundle if you're interested). A simple method allows us to create plots in the Poincare plane from corresponding data for the plot of the potential energy function. Nonconservative examples can be included as modifications of corresponding conservative systems.

All of this said, there are mathematical techniques which extend past physical examples. We begin by discussing such generic features. In particular, the nature of critical points for autonomous linear ODEs have structure which is revealed from the spectrum (list of eigenvalues from smallest to largest) of the coefficient matrix. In fact, such observations are easily made for $n$-dimensional problems. Of course our graphical methods are mainly of use for two-dimensional problems. We discuss almost linear systems and some of the deeper results due to Poincare for breadth. We omit discussion of Liapunov exponents, however the interested reader would be well-advised to study that topic along side what is discussed here (chapter 10 of Ritger & Rose has much to add to these notes).

Time-permitting we may exhibit the linearization of a non-linear system of ODEs and study how successful our approximation of the system is relative to the numerical data exhibited via the pplane tool. We also may find time to study the method of characteristics as presented in Zachmanoglou and Thoe and some of the deeper symmetry methods which are describe in Peter Hydon's text or Brian Cantwell's text on symmetries in differerential equations.

## 7.1 phase plane and stability

This section concerns largely qualitative analysis for systems of ODEs. We know from the existence theorems the solutions to a system of ODEs can be unique and will exist given continuity of the coefficient matrix which defines the system. However, certain points where the derivatives are all zero are places where interesting things tend to happen to the solution set. Often many solutions merge at such a **critical point**.

**Definition 7.1.1.** *critical point for a system of ODEs in normal form*

If the system of ODEs $\frac{d\vec{x}}{dt} = F(\vec{x}, t)$ has a solution $\vec{x}$ for which $t_o$ has $\frac{d\vec{x}}{dt}(t_o) = 0$ then $\vec{x}(t_o)$ is called a **critical point** of the system.

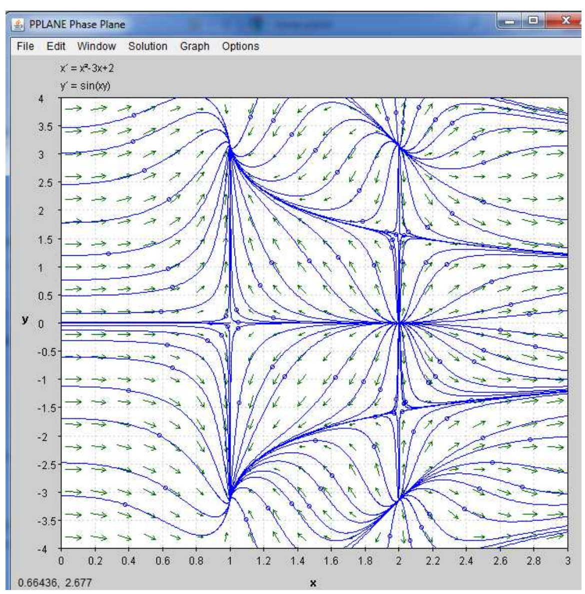
There are two major questions that concern us: (1.) where are the critical point(s) for a given system of ODEs ? (2.) do solutions near a given critical point tend to stay near the point or flow far away ? Let us begin by studying a system of two **autonomous ODEs**

$$\frac{dx}{dt} = g(x, y) \qquad \& \qquad \frac{dy}{dt} = f(x, y)$$

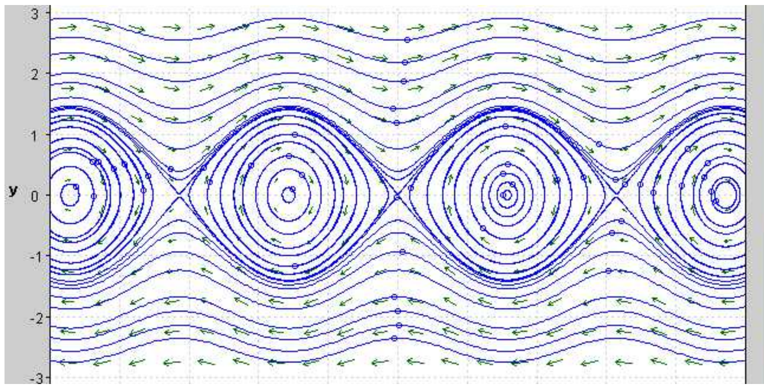
The location of critical points becomes an algebra problem: the system above has a critical point wherever both $f(x_o, y_o) = 0$ and $g(x_o, y_o) = 0$.

**Example 7.1.2. Problem:** *find critical points of the system* $\frac{dx}{dt} = x^2 - 3x + 2$ , $\frac{dy}{dt} = \sin(xy)$.

**Solution:** *a critical point must simultaneously solve* $x^2 - 3x + 2 = 0$ *and* $\sin(xy) = 0$. *The polynomial equation factors to yield* $(x-1)(x-2) = 0$ *hence we require the point to have either* $x = 1$ *or* $x = 2$. *If* $x = 1$ *then* $\sin(y) = 0$ *hence* $y = n\pi$ *for* $n \in \mathbb{Z}$. *It follows that* $(1, n\pi)$ *is a critical point for each* $n \in \mathbb{Z}$. *Likewise, if* $x = 2$ *then* $\sin(2y) = 0$ *hence* $2y = k\pi$ *for* $k \in \mathbb{Z}$ *hence* $y = k\pi/2$. *It follows that* $(2, k\pi/2)$ *is a critical point for each* $k \in \mathbb{Z}$.



The plot above was prepared with the *pplane* tool which you can find online. You can study the plot and you'll spot the critical points with ease. If you look more closely then you'll see that some of the critical points have solutions which flow into the point whereas others have solutions which flow out of the point. If all the solutions flow into the point then we say the point is **stable** or **asymptotically stable**. Otherwise, if some solutions flow away from the point without bound then the point is said to be **unstable**. I will not attempt to give careful descriptions of these terms here. There is another type of stable point. Let me illustrate it by example. The plot below shows sample solutions for the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = \sin(2x)$. The points where $y = 0$ and $x = n\pi/2$ for some $n \in \mathbb{Z}$ are critical points.




These critical points are **stable centers**. Obviously I used pplane to create the plot above, but another method is known and has deep physical significance for problems such as the one illustrated above. The method I discuss next is known as the **energy method**, I focus on a class of problems which directly stem from a well-known physical problem.

Consider a mass $m$ under the influence of a conservative force $F = -dU/dx$. Note:

$$ma = F \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = -\frac{dU}{dx} \quad \Rightarrow \quad m\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{dx}{dt}\frac{dU}{dx} \quad \Rightarrow \quad m\frac{dx}{dt}\frac{dv}{dt} = -\frac{dx}{dt}\frac{dU}{dx}$$

However, $v\frac{dv}{dt} = \frac{d}{dt}[\frac{1}{2}v^2]$ and $\frac{dx}{dt}\frac{dU}{dx} = \frac{dU}{dt}$ hence,

$$m\frac{d}{dt}[\frac{1}{2}v^2] = -\frac{dU}{dt} \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = 0$$

In particular, we find that if $x, v$ are solutions of $ma = F$ then the associated **energy function:**

$$\boxed{E(x,v) = \frac{1}{2}mv^2 + U(x)}$$

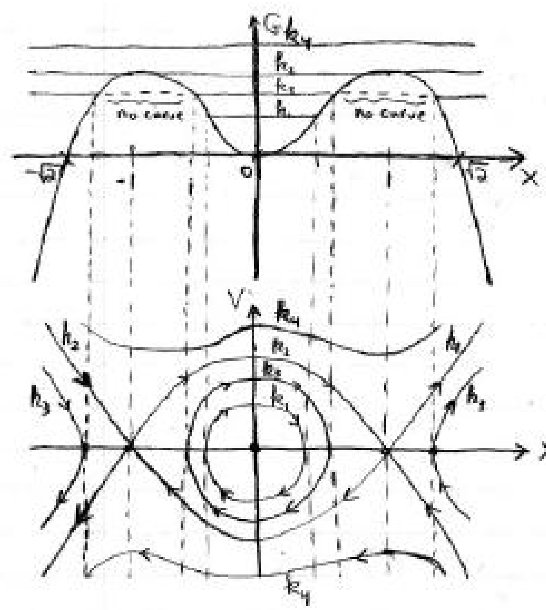
is constant along solutions of Newton's Second Law. Furthermore, consider $m\frac{d^2x}{dt^2} - F(x) = 0$ as a second order ODE. We can reduce it two a system of two ODEs in normal form by the standard substitution: $v = \frac{dx}{dt}$. Using velocity as an independent coordinate gives:

$$\boxed{\frac{dx}{dt} = v \qquad \& \qquad \frac{dv}{dt} = \frac{F}{m}}$$

Critical points of this system occur wherever both $v = 0$ and $F = 0$ since $m > 0$ by physical assumptions. Given our calculations concerning energy the solutions to this system must somehow parametrize the energy level curves as they appear in the $xv$-plane. This $xv$-plane is called the **phase plane** or the **Poincare plane** in honor of the mathematician who pioneered these concepts in the early 20-th century. Read Chapter 5 of Nagel Saff and Snider for a brief introduction to the concept of chaos and how the Poincare plane gave examples which inspired many mathematicians to work on the problem over the century that followed (chaos is still an active math research area).

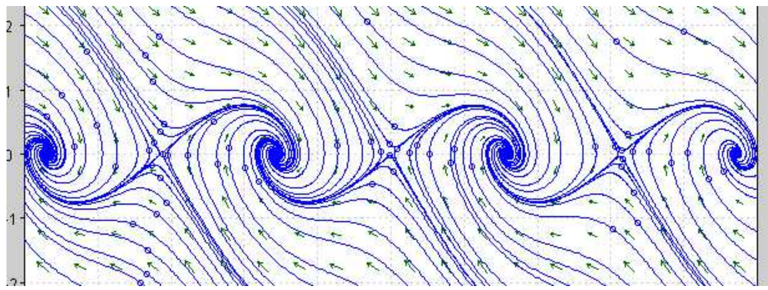
Think further about the critical points of $\frac{dx}{dt} = v$ & $\frac{dv}{dt} = \frac{F}{m}$. Recall we assumed $F$ was conservative hence there exists a **potential energy** function $U$ such that $F = -\frac{dU}{dx}$. This means the condition $F = 0$ gives $\frac{dU}{dx} = 0$. **Ah HA !** this means that the critical points of the phase plane solutions must be on the $x$-axis (where $v = 0$) at points where the potential energy $U$ has critical points in the $xU$-plane. Here I am contrasting the concept of critical point of a system with critical point ala calculus I. The $xU$-plane is called the **potential plane**.

The analysis of the last paragraph means that we can use the potential energy diagram to create the phase plane trajectories. This is closely tied to the specific mathematics of the energy function. Let us observe for a particular energy $E_o = \frac{1}{2}mv^2 + U(x)$ we cannot have motions where $U(x) < E_o$ since the kinetic energy $\frac{1}{2}mv^2 \geq 0$. Moreover, points where $E = U$ are points where $v = 0$ and these correspond to points where the motion either turns around or is resting.



In the plot above the top-graph is the **potential plane plot** whereas the lower plot is the corresponding **phase plane plot**. The point $(0, 0)$ is a stable center in the phase plane whereas $(\pm 1, 0)$ are unstable critical points. The trajectories in the phase plane are constructed such that the critical points match-up and the direction of all trajectories with $v > 0$ flow right whereas those with $v < 0$ flow left since $\frac{dx}{dt} = v$. Also, if $E > U$ at a critical point of $U$ then the corresponding trajectory will have a horizontal tangent since $\frac{dv}{dt} = 0$ at such points. These rules force you to draw essentially the same pattern plotted above.

All of the discussion above concerns the conservative case. In retrospect you should see the example $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = \sin(2x)$ is the phase plane DEqn corresponding to $m = 1$ with $F = \sin(2x)$. If we add a friction force $F_f = -v$ then $\frac{dv}{dt} = \sin(2x) - v$ is Newton's equation and we would study the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = \sin(2x) - y$. The energy $E = \frac{1}{2}v^2 - \frac{1}{2}\cos(2x)$ is not conserved in this case. I will not work out the explicit details of such analysis here, but perhaps you will find the constrast of the pplane plot below with that previously given of interest:




This material is discussed in §12.4 of Nagel Saff and Snider. The method of Lyaponov as discussed in §12.5 is a way of generalizing this energy method to autonomous ODEs which are not direct reductions of Newton's equation. That is a very interesting topic, but we don't go too deep here. Let us conclude our brief study of qualitative methods with a discussion of **homogeneous constant coefficient linear systems**. The problem $\frac{d\vec{x}}{dt} = A\vec{x}$ we solved explicitly by the generalized e-vector method and we can make some general comments here without further work:

1. if all the e-values were both negative then the solutions will tend towards $(0, 0)$ as $t \to \infty$ due to the exponentials in the solution.
2. if any of the e-values were positive then the solutions will be unbounded as $t \to \infty$ since exponentials in the solution.
3. if the e-value was pure imaginary then the motion is bounded since the formulas are just sines and cosines which are bounded
4. if the e-value was complex with negative real part then the associated motion is stable and tends to $(0, 0)$ as the exponentials damp the sines and cosines in the $t \to \infty$ limit.
5. if the e-value was complex with positive real part then the associated motion is unstable and becomes unbounded as the exponentials blow-up in the limit $t \to \infty$.

See the table in §12.2 on page 779 of Nagel Saff and Snider for a really nice summary. Note however, my comments apply just as well to the $n = 2$ case as the $n = 22$ case. In short, the **spectrum** of the matrix $A$ determines the stability of the solutions for $\frac{d\vec{x}}{dt} = A\vec{x}$. The spectrum is the list of the e-values for $A$. We could explicitly prove the claims I just made above, it ought not be too hard given all the previous calculations we've made to solve the homoegeneous constant coefficient case. What follows is far less trivial.

**Theorem 7.1.3.**

If the almost linear system $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t, \vec{x})$ has a matrix $A$ with e-values whose real parts are all negative then the zero solution of the almost linear system is asymptotically stable. However, if $A$ has even one e-value with a positive real part then the zero solution is unstable.

This theorem is due to Poincare and Perron as is stated in section §12.7 page 824 of Nagel Saff and Snider. Here is a sketch of the idea behind the theorem:

1. when $\vec{x}$ is sufficiently small the term $\vec{f}(t, \vec{x})$ tends to zero hence the ODE is well-approximated by $\frac{d\vec{x}}{dt} = A\vec{x}$
2. close to the origin the problem is essentially the same as that we have already solved thus the e-values reveal the stability or instability of the origin.

In the pure imaginary case the theorem is silient because it is not generally known whether that pure cyclicity of the localization of the ODE will be maintained globally or if it will be spoiled into spiral-type solutions. Spirals can either go out or in and that is the trouble for the pure imaginary case.

This idea is just another appearance of the linearization concept from calculus. We can sometimes replace a complicated, globally nonlinear ODE, with a simple almost linear system. The advantage is the usual one; linear systems are easier to analyze.

In any event, to go deeper into these matters it would be wiser to think about manifolds and general coordinate change since we are being driven to think about such issues like it or not. Have no fear, your course ends here.

# Chapter 8

# orthogonal functions

This chapter roughly corresponds to Chapter 13 of Ritger & Rose. It is somewhat unlikely for me to have time to type these notes for the Spring 2013 semester. I will probably post some pdf's and link to somewhich are already posted on my website when we approach this material in lecture. It is important you take complete notes in lecture. That said, most of what I'll do is just to disassemble fhe work of Ritger & Rose, so reading your text is a good starting point.

## 8.1 boundary value problems

## 8.2 eigenvalue problems

## 8.3 a generalization of the dot-product for functions

## 8.4 series of orthogonal functions

## 8.5 Sturm-Lioville theorem

## 8.6 ordinary Fourier series

# Chapter 9

# basic partial differential equations

This chapter roughly corresponds to Chapter 14 of Ritger & Rose. It is somewhat unlikely for me to have time to type these notes for the Spring 2013 semester. I will probably post some pdf's and link to somewhich are already posted on my website when we approach this material in lecture. It is important you take complete notes in lecture. That said, most of what I'll do is just to disassemble fhe work of Ritger & Rose, so reading your text is a good starting point. Moreover, Riger and Rose has some nice intutive explainations of how the PDEs which we study are derived. I have numerous problems solved from Nagle Saff and Snider and you'll want to look at those once we get into the material a lecture or two. The solution of a PDE with BV and initial conditions is fairly involved.

## 9.1 separation of variables

## 9.2 heat equations

## 9.3 wave equations

## 9.4 laplace's equation

## 9.5 noncartesian problems