INTRODUCTION TO DIFFERENTIAL EQUATIONS (DEq$^{\underline{n}}$s)

- A differential eq$^{\underline{n}}$ is an equation involving one or more dependent variables and one or more independent variables and their (dependent) derivatives.

<u>ODE</u> : Ordinary Differential Eq$^{\underline{n}}$: just <u>one</u> indep. variable

<u>PDE</u> : PARTIAL DIFFERENTIAL Eq$^{\underline{n}}$: two or more indep. variable

- A <u>sol$^{\underline{n}}$</u> or <u>solution</u> is a function or equation whose differential consequence includes the given DEq$^{\underline{n}}$. Or in simple terms, a sol$^{\underline{n}}$ is something that works when we plug it into the differential eq$^{\underline{n}}$

- <u>LINEAR DEq$^{\underline{n}}$</u> : the dependent variable and its derivatives appear linearly in the diff. eq$^{\underline{n}}$.

Examples:

① $\nabla^2 u = u_{xx} + u_{yy} = 0$ Laplace's Eq here

this is a linear PDE of $2^{\underline{nd}}$ order and the notation $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ (we discuss in CALC III this $\partial/\partial x$ notation, the partial der. w.r.t. x)

② $\nabla \cdot \vec{B} = 0$

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0$$

$1^{\underline{st}}$ order, linear PDE, one of Maxwell's Eq$^{\underline{n}}$s. It says no magnetic monopoles

③ $y'' + y = 0$

where $y' = \dfrac{dy}{dx}$ has sol$^{\underline{n}}$ $y = A\cos(t+\delta)$

for any A, δ since $y'' = (-A\sin(t+\delta))' = -A\cos(t+\delta)$

thus $\underbrace{y'' + y = 0}_{\text{this is linear, } 2^{\underline{nd}} \text{ order, constant coefficient, ODE}}$

(4) $\frac{dy}{dx} + y^2 = e^x$ 1^{st} order, non-linear, ODE

(5) $\frac{dy}{dx} + x^2 y = e^x$ 1^{st} order, linear ODE

(6) $\boxed{\frac{dy}{dx} = e^x y^2 \text{ and } y(0) = 2}$

1^{st} order, nonlinear, separable, initial value problem.

In this introductory treatment we learn just a few basic calculations

- How to solve $\frac{dy}{dx} = f(x)\,g(y)$ by separation of variables
- How to solve $\frac{dy}{dx} + Py = Q$ by integrating factor technique
- How to apply initial condition to either of the above
- Visualize sol$^{\underline{ns}}$ via Direction Field and approximate sol$^{\underline{n}}$ via Euler's Method
- How to check if given function is indeed a sol$^{\underline{n}}$ (we've already done this)

Remark: at this point, I'll use my Chapter 16 notes on "Introduction to DEq$^{\underline{ns}}$" which I wrote for Math 132 a few years ago. I also have sol$^{\underline{ns}}$ from $\boxed{H11}$, $\boxed{H12}$, $\boxed{H13}$, $\boxed{H10}$ to discuss. Also my handwritten (167) $\longrightarrow$ (170) discuss direction fields & Euler's Method.

16. INTRODUCTION TO DIFFERENTIAL EQUATIONS

What is a differential equation? It is an equation which involves derivatives. Differential equations are equations which relate the changes in various quantities. The natural world is filled with dynamic quantities, they depend on time. Often a differential equation will model how those quantities change with time.

The change need not be just with respect to time, we might consider something which varies as a result of time and space varying. For example, an electric or magnetic field components are functions of x,y,z and t. The differential equations which govern the electric and magnetic fields are known as Maxwell's Equations, these are *partial differential equations* (PDEs). In particular if $\vec{B} = < B_x, B_y, B_z >$ then

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

Another equation you might encounter in physics is the conservative force equation. A force $\vec{F}$ is called conservative if there exists a potential function U such that $\vec{F} = -\nabla U$. In particular, if the force $\vec{F} = < F_x, F_y, F_z >$ then the potential energy function U must satisfy three PDEs:

$$\vec{F} = -\nabla U \quad \Longleftrightarrow \quad \vec{F}_x = -\frac{\partial U}{\partial x}, \ \vec{F}_y = -\frac{\partial U}{\partial y}, \ \vec{F}_z = -\frac{\partial U}{\partial z}.$$

We will consider how to solve this sort of equation in calculus III. More general PDEs are treated in part in the differential equations course and beyond that there is an endless supply. Some mathematicians spend a whole career just studying one special PDE, the wealth of behavior contained in a simple equation is staggering.

In this course, we study the most basic type of differential equation, the *ordinary differential equation* (ODE). An ODE has one independent and one dependent variable. Sometimes we use "x" for the independent variable, in other situations we use "t" for time. The dependent variable is usually taken to be "y" but it is also taken to be "x" (but not at the same time that "x" is the independent variable).

1.) $\dfrac{dy}{dx} = x^2 + y + \sin(x)$ independent variable x, dependent variable y

2.) $\dfrac{dx}{dt} = x^2 + t + 3$ independent variable t, dependent variable x

3.) $\dfrac{ds}{d\theta} = se^{\theta}$ independent variable θ, dependent variable s

The above are *first order differential equations* because the highest derivative that appears is the first derivative. If both the dependent variable and its derivatives appear linearly then the ODEqn is said to be linear. Equations 1.) and 3.) are linear, but the appearance of x^2 ruins it in 2.). Homogeneous linear DEqns are especially nice since the sum of solutions is again a solution..

If the highest derivative that appears in the DEqn is the second derivative then we say the DEqn is a *second order differential equation*. If the highest order that appears in the DEqn is n-th order derivative then the DEqn is said to be an *n-th order differential equation*. If the differential equation can be written as a sum of the dependent variable and its derivatives set to zero without the independent variable appearing on its own then the DEqn is said to be homogeneous, otherwise the DEqn is said to be nonhomogeneous. If the differential equation can be written as a linear combination of the dependent variable and its derivatives such that the coefficients in the sum are just numbers then it is said to be a constant coefficient DEqn.

4.) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$ second order, linear, constant coeff., homogeneous

5.) $y''' + y' + t^2 y = t$ third order, linear, nonhomogeneous

6.) $y^{(n)}(t) = y$ n-th order, linear, constant coeff., homogeneous

Ok, our vocabulary lesson is over now. Some examples in this chapter are inspired by homework problems in the excellent DEqns text by Nagle Saff and Snider.

$\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$: 1^{st} order nonlinear ODE, dependent var. = y, indep. var. = x

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$: 2^{nd} order linear ODE, dep. var. = y, indep. var. = x.

$\dfrac{\partial N}{\partial t} = \dfrac{\partial^2 N}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial N}{\partial r} + kN$: 2^{nd} order PDE

16.1. WHAT IS A SOLUTION?

The question that titles this section begs another question; **"to what?"** We already know what the solution to an arithmetic problem is, it's a number. Or what is the solution to most algebra problems? Also a number, perhaps several. For example, $x^2 - 5x + 6 = (x-2)(x-3) = 0$ has solutions $x = 2$ and $x = 3$.

We mean to ask the question now, **"what is the solution to a differential equation?".**

- *A* ***explicit solution*** *to a differential equation is a function which satisfies the rule of the differential equation. In other words, it a function which works when substituted into the differential equation. Symbolically, if the differential equation (*) has the form* $F(x, y, y', y'', \dots, y^n(x)) = 0$ *then* f *is a solution of (*) if and only if* $F(x, f(x), f'(x), f''(x), \dots, f^{(n)}(x)) = 0$

- *An* ***implicit solution*** is some equation which satisfies the differential equation. Unless said otherwise when I say solution I mean implicit solution, but we like to find explicit solutions if possible.

- *The* ***general solution*** allows for all possible initial conditions. Technically speaking, it is not a function, rather it is a whole family of functions.

$$\boxed{\int x\,dx = \tfrac{1}{2}x^2 + C}$$

It probably helps to see a few examples at this point.

$$\frac{dy}{dx} = x \Rightarrow y = \tfrac{1}{2}x^2 + C$$

Example 16.1.1

Let $f(x) = x^2$ then f is a solution to $\frac{dy}{dx} = 2x$ since $\frac{df}{dx} = 2x$.
Let $g(x) = x$ then g is not a solution to $\frac{dy}{dx} = 2x$ since $\frac{dg}{dx} = 1 \neq 2x$.
We call f an explicit solution since it has the graph $y = x^2$ and you can see that y is an explicit function of x, they're not mixed together.

Example 16.1.2

Let $x^2 + y^2 = 4$ this implicitly defines a solution of $\frac{dy}{dx} = -\frac{x}{y}$ since implicit differentiation of the proposed solution yields $2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$. This is called an implicit solution because we cannot just solve for y as a single function of x. It is only possible to find an explicit solution locally, this is a circle, the upper and lower pieces are separately explicit solutions $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. The implicit solution contains both of these explicit solutions.

So how do we find solutions? I didn't mention that yet. You can see that in Example 10.1.1 the solution could be found by integration.

$$\frac{dy}{dx} = 2x \implies \int \frac{dy}{dx}\,dx = \int 2x\,dx = x^2 + C$$

Select the case $C = 0$ we get $f(x) = x^2$. If you think about it, every time we integrated we solved a differential equation. Think about it:

$$\boxed{\int f(x)\,dx = y \iff \frac{dy}{dx} = f(x)}$$

The general solution is thus analogous to the indefinite integral. In fact, when we solve an n-th order ODEn it amounts to integrating n-times. It is not surprising then that the general solution will have n-arbitrary constants of integration. When we solve a first order ODEn we get just one constant.

Example 16.1.3:
The general solution to $\frac{dy}{dx} = 2x$ is $y = x^2 + C$. Geometrically, we have a family of parabolas which open upward and differ just by a vertical shift.

Example 16.1.4:
The general solution to $\frac{dy}{dx} = -\frac{x}{y}$ is $x^2 + y^2 = R^2$. Geometrically, we have a family of circles. Example 10.1.2 was just one case. Given the differential equation we would need additional information in order to select a specific solution.

Bad News? Solving differential equations is not just integration in general. The differential equation in Example 10.1.1 was very special. More often than not a given DEqn will not allow us to find an equation of the form $\frac{dy}{dx}$ = stuff in x. Usually the dependent variable and its various derivatives are all jumbled together at once. We need other tricks to unravel the equation and find solutions. I suppose there are nearly as many techniques as there are types of DEqns. For this chapter there will be three main tricks to find general solutions:

- (16.2) Separate variables: $\frac{dy}{dx} = f(x)g(y) \implies \int g(y)dy = \int f(x)dx$.
- (16.4) Use Integrating Factor Technique: put in standard form $\frac{dy}{dx} + Py = Q$
  Then calculate $\mu = exp(\int p\,dx)$, multiply by μ, use reverse product rule, separate and integrate.
- (16.5) 2nd order constant coefficient ODE, find quadratic auxillary equation, solve the quadratic, write down general solution.