CAUCHY - EULER EQUATIONS : §8.5

PH-104

§8.5 #1 Use substitution $y = x^r$ to find general solution for $x > 0$ of $x^2 y'' + 6xy' + 6y = 0$

If $y = x^r$ then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$,

$$x^2(r(r-1)x^{r-2}) + 6xrx^{r-1} + 6x^r = 0$$

$$[r(r-1) + 6r + 6]x^r = 0 \quad (\text{for } \underline{x > 0})$$

$$\Rightarrow r^2 - 5r + 6 = 0$$

$$\Rightarrow (r-3)(r-2) = 0$$

$$\Rightarrow r_1 = 3,\ r_2 = 2 \quad \therefore \quad \boxed{y = c_1 x^3 + c_2 x^2}$$

§8.5 #5 Assume sol^n has form $y = x^r$ to find gen. sol^n for

$$\frac{d^2y}{dx^2} = \frac{5}{x}\frac{dy}{dx} - \frac{13}{x^2}y$$

Substitute, $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$r(r-1)x^{r-2} = \frac{5}{x}rx^{r-1} - \frac{13}{x^2}x^r$$

$$\Rightarrow x^{r-2}\left[r(r-1) - 5r + 13\right] = 0$$

Since $x > 0 \Rightarrow r^2 - 6r + 13 = (r-3)^2 + 4 = 0$. Thus $r = 3 \pm 2i$ which means $y = x^{3+2i}$ is a complex sol^n. Since we have a linear DEq^n we know we'll find two real sol^n's from $y = x^{3+2i}$. Let's see how,

$$x^{3+2i} = x^3 x^{2i} = x^3 e^{\ln(x^{2i})} = x^3 e^{2i\ln(x)}$$

$$= x^3\cos(2\ln(x)) + i x^3 \sin(2\ln(x))$$

Euler's Identity

As we discussed in general (for any linear ODE) the real-valued sol^n's in a complex sol^n are simply the real and imaginary parts. $y_1 = Re(x^{3+2i})$, $y_2 = Im(x^{3+2i})$

$$\therefore \quad \boxed{y = c_1 x^3 \cos(2\ln(x)) + c_2 x^3 \sin(2\ln(x))}$$

$\boxed{\S 8.5 \# 13}$ Solve $x^2 y'' - 2xy' + 2y = x^{-1/2}$ by variation of parameters

To use variation of parameters we must first find the fundamental sol$^{\underline{n}}$ set. As usual we try $y = x^r$ and see what conditions the DEq$^{\underline{n}}$ placer on r,

$$x^2(r(r-1)x^{r-2}) - 2xrx^{r-1} + 2x^r = 0$$

(no $x^{-1/2}$ we're finding the fundamental sol$^{\underline{n}}$s.)

$$\Rightarrow \quad r^2 - 3r + 2 = (r-2)(r-1) = 0$$

Thus $y_1 = x^2$ and $y_2 = x$. Fortunately variation of parameters applies to Cauchy – Euler problems,

$$\underbrace{y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{x^{-1/2}}{x^2}}_{\text{standard form}} \quad \text{with } g(x) = \frac{x^{-1/2}}{x^2} = x^{-5/2}, \; a = 1$$

Equation (10) of § 4.6 tells us how to find the parameters $V_1(x)$, $V_2(x)$ in the sol$^{\underline{n}}$ $y = V_1 y_1 + V_2 y_2$,

$$V_1(x) = \int \frac{-g(x)\,y_2(x)}{a[y_1y_2' - y_2y_1']}\,dx = \int \frac{-x^{-5/2}x}{x^2(1) - x(2x)}\,dx = \int \frac{-x^{-3/2}}{-x^2}\,dx = \int x^{-7/2}\,dx = -\frac{2}{5}x^{-5/2}$$

$$V_2(x) = \int \frac{g(x)\,y_1(x)}{a[y_1y_2' - y_2y_1']}\,dx = \int \frac{x^{-5/2}x^2}{x^2(1) - x(2x)}\,dx = \int \frac{x^{-1/2}}{-x^2}\,dx \doteq \int -x^{5/2}\,dx = \frac{2}{3}x^{-3/2}$$

Thus, $y = c_1x^2 + c_2x - \frac{2}{5}x^{-5/2}x^2 + \frac{2}{3}x^{-3/2}x$

$$\Rightarrow \quad \boxed{y = c_1x^2 + c_2x - \frac{4}{15}x^{-1/2}}$$

Remark: if you examine the derivation of eq$^{\underline{n}}$ (9) (from which eq$^{\underline{n}}$ 10 follows) on pg. 89 of these notes you'll find the derivation did not assume that a, b, c were constants, in fact a, b, c could be functions and the calculation stands. The tricky part is how to find y_1, y_2 when a, b, c nonconstant.