SPECIAL INTEGRATING FACTORS: § 2.5.

In § 2.3 we saw multiplication by $\mu = \exp(\int P\,dx)$ would cause the given ODEq$^{\underline{n}}$ to separate so we could integrate. We use a similar idea here; we multiply by some factor μ which causes the given DEq$^{\underline{n}}$ to morph into an exact eq$^{\underline{n}}$ (which we know how to solve). Unfortunately, we may lose some sol$^{\underline{n}}$'s since there is no garauntee $\mu \neq 0$ (in contrast to μ from § 2.3). (The method was already illustrated in § 2.4 # 29)

§ 2.5 # 7 (hopefully we can use the method boxed in § 2.5)

$$\underbrace{(3x^2 + y)}_{M}dx + \underbrace{(x^2y - x)}_{N}dy = 0 \longleftarrow (*)$$

$$\frac{\partial M}{\partial y} = 1 \qquad\qquad \frac{\partial N}{\partial x} = 2xy - 1$$

Notice, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 2xy - 2$ thus $\dfrac{N_x - M_y}{M} = \dfrac{2xy-2}{3x^2+y}$

Also $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2 - 2xy$ thus $\dfrac{M_y - N_x}{N} = \dfrac{2-2xy}{x^2y - x} = \dfrac{2(1-xy)}{x(xy-1)}$

thus $\dfrac{\partial M/\partial y - \partial N/\partial x}{N} = \dfrac{-2}{x}$ which depends only on x, use Th$^{\underline{m}}$ 3.

$$\mu = \exp\left[\int \frac{-2\,dx}{x}\right] = \exp\left[-2\ln|x|\right] = \exp\left[\ln\left(\frac{1}{|x|^2}\right)\right] = \frac{1}{|x|^2} = \frac{1}{x^2}.$$

Multiply $(*)$ by $\mu = 1/x^2$ to obtain,

$$(3 + y/x^2)\,dx + (y - 1/x)\,dy = 0 \qquad (**)$$

You can easily verify the eq$^{\underline{n}}$ above is $dF = 0$ for the function $F(x,y) = 3x - y/x + \frac{1}{2}y^2$. Hence we find sol$^{\underline{n}}$'s

$$\boxed{3x - y/x + y^2/2 = k}$$

and the sol$^{\underline{n}}$ $\boxed{x = 0}$ also solves $(*)$ but was lost when we multiplied by $\mu = 1/x^2$.

§2.5 #9

$$\underbrace{(2y^2 + 2y + 4x^2)}_{M} dx + \underbrace{(2xy + x)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = 4y + 2 \qquad \frac{\partial N}{\partial x} = 2y + 1$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y + 2 - 2y - 1 = 2y + 1$$

Notice $N = x(2y+1)$ thus $\dfrac{\partial M/\partial y - \partial N/\partial x}{N} = \dfrac{2y+1}{x(2y+1)} = \dfrac{1}{x}$.

With the insight of Th$^{\underline{m}}$ (3) we calculate $\mu = \exp\left(\int \frac{dx}{x}\right) = |x|$.
Multiply by μ, notice $|x| = \pm x$ so assume $x > 0$ or $x < 0$ in sol$^{\underline{n}}$ and we can drop $|\cdot|$,

$$\underbrace{(2xy^2 + 2xy + 4x^3)}_{\frac{\partial F}{\partial x}} dx + \underbrace{(2x^2y + x^2)}_{\frac{\partial F}{\partial y}} dy = \pm\mu \cdot 0 = 0.$$

Think. $F(x,y) = x^2y^2 + x^2y + x^4$. We find sol$^{\underline{n}}$'s $\boxed{x^2y^2 + x^2y + x^4 = k \ \text{ and } \ \underbrace{x = 0}_{}}$

substitute into given DEq$^{\underline{n}}$ directly noting $dx = 0$ and $N|_{x=0} = 0$.

§2.5 #11 $(y^2 + 2xy)dx - x^2 dy = 0$

Use Th$^{\underline{m}}$ (3) to find $\mu = 1/y^2$ hence

$$\underbrace{\left(1 + \frac{2x}{y}\right)}_{\frac{\partial F}{\partial x}} dx - \underbrace{\frac{x^2}{y^2}}_{\frac{\partial F}{\partial y}} dy = 0 \longrightarrow F(x,y) = x + \frac{x^2}{y}$$

we find sol$^{\underline{n}}$'s $\boxed{x + \dfrac{x^2}{y} = k \ \text{ and } \ y = 0}$

this one missed because μ undefined for $y = 0$.

§2.5 #13 Find an integrating factor of the form $x^n y^m$ and solve the eq$\underline{n}$

$$(2y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy \overset{(*)}{=} 0$$

An integrating factor $\mu = x^n y^m$ will force the given DEq$\underline{n}$ to become exact upon multiplication by μ. This means we have

$$x^n y^m (2y^2 - 6xy) = M$$
$$x^n y^m (3xy - 4x^2) = N$$

such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[2y^{m+2}x^n - 6x^{n+1}y^{m+1} \right]$$
$$= 2(m+2)y^{m+1}x^n - 6(m+1)x^{n+1}y^m$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[3x^{n+1}y^{m+1} - 4x^{n+2}y^m \right]$$
$$= 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

Equate $\partial M/\partial y$ with $\partial N/\partial x$ to insure $\mu(*)$ is exact,

$$2(m+2)y^{m+1}x^n - 6(m+1)x^{n+1}y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

Equate coefficients of matching powers,

$\underline{x^{n+1}y^m)}$ $\quad -6(m+1) = -4(n+2)$

$\underline{y^{m+1}x^n)}$ $\quad 2(m+2) = 3(n+1)$

Now solve these simultaneously. Note the 2^{nd} Eq$\underline{n}$ implies $6m + 12 = 9n + 9 \Rightarrow \underline{6m = 9n - 3}$ subst. into first eq$\underline{n}$

$$\underbrace{-6m - 6 = -4n - 8}$$
$$6m = 4n + 2 = 9n - 3 \Rightarrow \underline{5n = 5}$$

We found $5n = 5$ thus $n = 1$ and

$$m = \frac{1}{6}(9n-3) = \frac{1}{6}(9-3) = \frac{6}{6} = 1$$

Hence $\mu = xy$ should work. Multiply (*) by $\mu = xy$ to obtain,

$$\underbrace{(2xy^3 - 6x^2y^2)}_{\frac{\partial F}{\partial x}}dx + \underbrace{(3x^2y^2 - 4x^3y)}_{\frac{\partial F}{\partial y}}dy = 0$$

Think. $F(x,y) = x^2y^3 - 2x^3y^2$. Consequently we find $\boxed{\text{sol}^{\underline{n}}\text{s } x^2y^3 - 2x^3y^2 = k \text{ for } (*)}$

Also you can verify $\boxed{x = 0}$ is a sol$^{\underline{n}}$ and $\boxed{y = 0}$ is a sol$^{\underline{n}}$ for (*).

<u>Remark:</u> the hard part of §2.5 is finding μ. Clearly there are many possible patterns and choices depending on the structure of the given DEq$^{\underline{n}}$. You can look over the remaining problems in the text to see some of the other patterns we could look for. As I have mentioned previously, the methods and insights of §2.3 & §2.5 could be further eulicidated by a careful discussion of symmetries and differential eq$^{\underline{n}}$s.