§2.6#9 Solve $(3x^2 - y^2)dx + (xy - x^3y^{-1})dy = 0$.

Since all terms have order 2 it follows $\frac{dy}{dx}$ will be able to be written as a function of $v = y/x$. If $v = y/x$ then $y = xv$. Consider then,

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{xy - x^3/y} = \frac{3x^2 - x^2v^2}{x^2v - x^3/xv} = \frac{3v - v^3}{v^2 - 1} = \underbrace{G(v)}_{\text{(notation in text)}}$$

Note that

$$\frac{dv}{dx} = \frac{d}{dx}\left[\frac{y}{x}\right] = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}\left[\frac{dy}{dx} - v\right]$$

$$\Rightarrow \quad x\frac{dv}{dx} = G(v) - v$$

$$\Rightarrow \quad \int \frac{dv}{G(v) - v} = \int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dv}{\frac{v^3 - 3v}{v^2 - 1} - v} = \int \frac{dv}{\frac{v^3 - 3v - v^3 + v}{v^2 - 1}}$$

$$= \int \frac{(v^2 - 1)\,dv}{-2v}$$

$$= \int \left(\frac{1}{2v} - \frac{v}{2}\right)dv$$

$$= \frac{1}{2}\ln|v| - \frac{1}{4}v^2 = \ln|x| + C$$

$$\Rightarrow \quad \boxed{\frac{1}{2}\ln\left|\frac{y}{x}\right| - \frac{1}{4}\frac{y^2}{x^2} = \ln|x| + C}$$

Remark: the symmetry
$x \mapsto \lambda x$
$y \mapsto \lambda y$
indicates we can use the invariant $v = y/x$ to make a change of variables which makes the DEq^{n} separable

Note, $2\ln\left|\frac{y}{x}\right| + 4\ln|x| - \frac{y^2}{x^2} = 4C \Rightarrow \ln(y^2/x^6) - y^2/x^2 = C$

§2.6 #17

$$\frac{dy}{dx} = \sqrt{x+y} - 1$$

Let $v = x + y$ then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ and we find

$$\frac{dy}{dx} = \frac{dv}{dx} - 1 = \sqrt{v} - 1$$

$$\Rightarrow \frac{dv}{\sqrt{v}} = dx$$

$$\Rightarrow 2\sqrt{v} = x + c$$

$$\Rightarrow v = \frac{1}{4}(x+c)^2$$

$$\Rightarrow x + y = \frac{1}{4}(x+c)^2$$

$$\boxed{y = \frac{1}{4}(x+c)^2 - x}$$

Note we divided by $v = x + y$ in the derivation above. This suggest we <u>might</u> have missed a sol^n with $v = 0 \Rightarrow x + y = 0$ or $y = -x$. We can check and see if $y = -x$ is a sol^n,

$$\underbrace{\frac{dy}{dx} = \sqrt{x+y} - 1}$$

put $y = -x$ on the RHS we get $\sqrt{x-x} - 1 = -1$ likewise $\frac{dy}{dx} = \frac{d}{dx}(-x) = -1$. Thus $y = -x$ solves this differential equation.

§2.6 #41 Use the substitution $v = x - y + 2$ to Solve $\frac{dy}{dx} \overset{(*)}{=} y - x - 1 + (x-y+2)^{-1}$.

Notice $y - x = 2 - v$. Also,

$$\frac{dv}{dx} = \frac{d}{dx}\left[x - y + 2\right] = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dv}{dx} = \underbrace{2 - v - 1 + v^{-1}}_{\text{using } (*)} = 1 - v + \frac{1}{v}$$

$$\Rightarrow \frac{dv}{dx} = v - \frac{1}{v} = \frac{v^2-1}{v}$$

$$\Rightarrow \int \frac{v\,dv}{v^2-1} = \int dx$$

$$\Rightarrow \frac{1}{2}\ln|v^2-1| = x + C$$

$$\Rightarrow \frac{1}{2}\ln|(x-y+2)^2 - 1| = x + C$$

$$\Rightarrow \ln|(x-y+2)^2 - 1| = 2x + 2C$$

$$\boxed{(x-y+2)^2 = 1 + ke^{2x}}$$

This time there are no extraneous sol^{n}'s since the initial eq^{n} also has $\frac{1}{v}$ thus $v = 0$ causes (*) to be undefined. The transformation does not lose sol^{n}'s this time.