LINEAR HOMOGENEOUS WITH COMPLEX ROOTS : §4.3

§4.3 #2 $y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow \boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

§4.3 #4 $y'' - 10y' + 26y = 0$

$\lambda^2 - 10\lambda + 26 = 0 \Rightarrow \lambda = \frac{10 \pm \sqrt{100 - 4(26)}}{2} = 5 \pm \frac{\sqrt{-4}}{2} = 5 \pm i$

Identify $\alpha = 5$ and $\beta = 1 \Rightarrow \boxed{y = e^{5x}(c_1 \cos(x) + c_2 \sin(x))}$

§4.3 #8 $4y'' + 4y' + 6y = 0$

$4\lambda^2 + 4\lambda + 6 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4(4)6}}{8} = \frac{-4 \pm \sqrt{-80}}{8} = \underbrace{-\frac{1}{2}}_{\alpha} \pm i \underbrace{\sqrt{\frac{80}{64}}}_{\beta}$

$\sqrt{\frac{80}{64}} = \sqrt{\frac{10}{8}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} = \beta \Rightarrow \boxed{y = e^{-x/2}\left(c_1 \cos\left(\frac{\sqrt{5}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{5}}{2}x\right)\right)}$

Remark: could also write:

$$y = Ae^{-x/2} \sin\left(\frac{\sqrt{5}}{2}x + \varphi\right)$$

exercise, relate c_1 & c_2 to A & φ.

§4.3 #10 $y'' + 4y' + 8y = 0$

$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i = \alpha \pm i\beta$.

$\boxed{y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))}$

§4.3 #12 $u'' + 7u = 0$

$\lambda^2 + 7 = 0 \Rightarrow \lambda^2 = -7 \Rightarrow \lambda = \pm\sqrt{-7} = \pm\sqrt{7}\, i \quad (\alpha = 0, \ \beta = \sqrt{7})$

$\boxed{u = c_1 \cos\sqrt{7}x + c_2 \sin\sqrt{7}x}$

§4.3 #22 $y'' + 2y' + 17y = 0$

$\lambda^2 + 2\lambda + 17 = 0 \rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(17)}}{2} = -1 \pm 4i = \alpha \pm i\beta$ *

$* \Rightarrow y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$

$y' = -e^{-x}(c_1 \cos 4x + c_2 \sin 4x) + e^{-x}(-4c_1 \sin 4x + 4c_2 \cos 4x)$ (product rule)

using the given conditions $\begin{cases} y(0) = c_1 = 1 \\ y'(0) = -c_1 + 4c_2 = -1 \Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0 \end{cases}$ $\therefore \boxed{y = e^{-x} \cos 4x}$

§4.3 #32 omit units in the math-models below,

a.) If $m = 10$, $k = 250$, $y(0) = 0.3$, $y'(0) = -0.1$ then find motion of undamped spring.

b.) find frequency of the oscillation described in part a.

Newton's 2nd law reads $\underbrace{my'' + ky = 0}$

$$10y'' + 250y = 0$$
$$y'' + 25y = 0$$
$$\lambda^2 + 25 = 0 \quad \therefore \quad \underline{\lambda = \pm 5i}$$

$$y = c_1 \cos(5t) + c_2 \sin(5t)$$
$$y' = -5c_1 \sin(5t) + 5c_2 \cos(5t)$$

$$\left.\begin{aligned} y(0) &= 0.3 = c_1 \\ y'(0) &= -0.1 = 5c_2 \end{aligned}\right\} \quad \boxed{y = 0.3\cos(5t) - 0.02\sin(5t)}$$

(answer to part a)

An equation of the form $y = A\cos(\omega t + \phi)$ has angular frequency $\omega = 2\pi f \Rightarrow \omega = 5 = 2\pi f \therefore \boxed{f = \frac{5}{2\pi}}$

Remark: We can also claim the general sol^{n} $y = A\cos(5t + \phi)$ where A is the amplitude and ϕ is the phase. For this problem,

$$\left.\begin{aligned} y(0) &= A\cos\phi = 0.3 \\ y'(0) &= -5A\sin(\phi) = -0.1 \end{aligned}\right\} \quad \frac{\sin\phi}{\cos\phi} = \frac{0.1/5}{0.3}$$

$$\tan\phi = \frac{1}{15} \Rightarrow \underline{\phi \cong 0.0667}$$

$$\sin^2\phi + \cos^2\phi = \left(\frac{0.3}{A}\right)^2 + \left(\frac{0.1}{5A}\right)^2 = 1$$
$$0.09 + 0.0004 = A^2 \quad \Rightarrow \quad A \cong 0.3007$$

Thus $\underline{y = 0.3007\cos(5t + 0.0667)}$

this form of the eq^{n} of motion does have certain advantages.

§4.3 #33 Spring with damping. Let $m = 10$, $b = 60$, $k = 250$, $y(0) = 0.3$, $y'(0) = -0.1$ find motion (a.) frequency (b.) and comment (c.)

$$10y'' + 60y' + 250y = 0$$
$$y'' + 6y' + 25y = 0$$
$$\lambda^2 + 6\lambda + 25 = 0$$
$$(\lambda+3)^2 + 16 = 0 \Rightarrow \underline{\lambda = -3 \pm 4i}.$$
$$\Rightarrow \underbrace{y = Ae^{-3t}\cos(4t + \phi)}$$

same as
$y = c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t$
but A, ϕ convenient for physical analysis here.

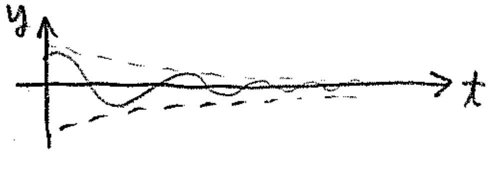
$$y(0) = A\cos(\phi) = 0.3$$
$$y'(t) = -3Ae^{-3t}\cos(4t+\phi) - 4Ae^{-3t}\sin(4t+\phi)$$
$$y'(0) = -3A\cos\phi - 4A\sin\phi = -0.1$$

Note $A = 0.3/\cos\phi$ thus,

$$-3\left(\frac{0.3}{\cos\phi}\right)\cos\phi - 4\left(\frac{0.3}{\cos\phi}\right)\sin\phi = -0.1$$
$$-0.9 - 1.2\tan\phi = -0.1$$
$$\tan\phi = -0.8/1.2 \Rightarrow \underline{\phi = -0.588}$$
$$\Rightarrow \underline{A = 0.361}.$$

$$\boxed{y(t) = 0.361\, e^{-3t}\cos(4t - 0.588)}$$

The "frequency"(*) is $\omega = 4 = 2\pi f \therefore f = \frac{4}{2\pi}$ which is slower than the same case with no damping.



(*) Remark: this function is not strictly speaking periodic so what does "frequency" mean?