§4.6 #2 $y'' + y = \sec t$; note $y_h = c_1 \cos t + c_2 \sin t$. That is we have fundamental sol^{n}'s $y_1 = \cos t$ and $y_2 = \sin t$. Now then the method of variation of parameter begins with the ansatz,

$$y_p = V_1 y_1 + V_2 y_2$$

Here V_1 and V_2 play the role $A, B, C, \ldots$ played in the method of undetermined coefficients, big difference here is that V_1 & V_2 are unknown functions of t which we are to determine. Upon finding them we'll have a particular sol^{n} that we could not have found with previous methods. As derived on p. 194–195 of text we see that the conditions that V_1 & V_2 must meet for $Y_p = V_1 Y_2 + V_2 Y_2$ to solve $y'' + y = \sec t = g(t)$ are,

$$\left.\begin{aligned} y_1 V_1' + y_2 V_2' &= 0 \\ y_1' V_1' + y_2' V_2' &= g/a \end{aligned}\right\} \longrightarrow \boxed{\begin{aligned} \cos t\, V_1' + \sin t\, V_2' &= 0 \quad (1) \\ -\sin t\, V_1' + \cos t\, V_2' &= \sec(t) = \frac{1}{\cos t} \quad (2) \end{aligned}}$$

Solve (2) for $V_2' = \frac{\sin t}{\cos t} V_1' + \frac{1}{\cos^2 t}$ now substitute into Eq^{n} (1),

$$\cos t\, V_1' + \sin t \left(\frac{\sin t}{\cos t} V_1' + \frac{1}{\cos^2 t} \right) = 0$$

$$V_1' \left(\cos t + \frac{\sin^2 t}{\cos t} \right) = \frac{-\sin t}{\cos^2 t} \Rightarrow V_1' (\cos^2 t + \sin^2 t) = -\frac{\sin t}{\cos t}$$

$$\Rightarrow \frac{dV_1}{dt} = \frac{-\sin t}{\cos t} \rightarrow dV_1 = -\frac{\sin t\, dt}{\cos t}$$

$$dV_1 = \frac{du}{u}$$

$$\Rightarrow \boxed{V_1 = \ln|\cos t|}$$

Now since $V_1' = -\frac{\sin t}{\cos t}$ we find for V_2' that

$$V_2' = \frac{\sin t}{\cos t}\left(\frac{-\sin t}{\cos t}\right) + \frac{1}{\cos^2 t} = \frac{1 - \sin^2 t}{\cos^2 t} = \frac{\cos^2 t}{\cos^2 t} = 1 = \frac{dV_2}{dt}$$

$$\Rightarrow \boxed{V_2 = t}$$

Hence $y_p = \cos t \ln|\cos t| + t \sin t$

$$\Rightarrow y = y_h + y_p = \boxed{c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t}$$

general sol^{n}.

§4.6 #3 Consider $2x'' - 2x' - 4x = 2e^{3t}$

To begin ÷ by 2, $x'' - x' - 2x = e^{3t}$ now proceed,

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow x_1 = e^{2t} \ \& \ x_2 = e^{-t}$$

Solve then, in the goal of finding $x_p = v_1 x_1 + v_2 x_2$,

$$v_1' x_1 + v_2' x_2 = 0 \qquad v_1' e^{2t} + v_2' e^{-t} = 0 \quad (1)$$

$$v_1' x_1' + v_2' x_2' = e^{3t} \qquad 2v_1' e^{2t} - v_2' e^{-t} = e^{3t} \quad (2)$$

now we must solve these for v_1 & v_2.

Notice $(1) \Rightarrow v_2' = -v_1' e^{3t}$ thus (2) becomes

$$2v_1' e^{2t} - (-v_1' e^{3t}) e^{-t} = e^{3t}$$

$$\Rightarrow 3v_1' e^{2t} = e^{3t}$$

$$\Rightarrow v_1' = \frac{1}{3} e^{t} \Rightarrow \boxed{v_1 = \frac{1}{3} e^{t}}$$

Then $v_2' = -v_1' e^{3t} = -\frac{1}{3} e^{t} e^{3t} = \frac{-1}{3} e^{4t} = v_2'$

$$\Rightarrow \boxed{v_2 = \frac{-1}{12} e^{4t}}$$

Thus
$$x = x_h + x_p$$
$$= c_1 e^{2t} + c_2 e^{-t} + \frac{1}{3} e^{t} e^{2t} - \frac{1}{12} e^{4t} e^{-t}$$
$$= \boxed{c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4} e^{3t}}$$

Remark: this is a good example of what not to do. We can solve this problem more efficiently thru undet. coeff.

$$\left.\begin{aligned} x_p &= Ae^{3t} \\ x_p' &= 3Ae^{3t} \\ x_p'' &= 9Ae^{3t} \end{aligned}\right\} \quad \begin{aligned} x_p'' - x_p' - 2x_p &= e^{3t} \\ (9A - 3A - 2A)e^{3t} &= e^{3t} \\ 4A &= 1 \quad \therefore A = \tfrac{1}{4} \end{aligned}$$

$$\therefore x_p = \frac{1}{4} e^{3t}$$

Point: variation of parameters is a weapon of last resort.

§4.6 #7 $y'' + 16y = \sec 4\theta$

$y_1 = \cos 4\theta$ & $y_2 = \sin 4\theta$ not hard to see.

We suppose that $Y_p = V_1 y_1 + V_2 y_2$ thus solve,

$$V_1' y_1 + V_2' y_2 = 0$$
$$V_1' y_1' + V_2' y_2' = \sec 4\theta$$

These become,

$$V_1' \cos 4\theta + V_2' \sin 4\theta = 0 \quad : (1)$$
$$-4 V_1' \sin 4\theta + 4 V_2' \cos 4\theta = \sec 4\theta = \frac{1}{\cos 4\theta} \quad : (2)$$

Eq$^{\text{n}}$ (1) says $V_1' = -V_2' \tan 4\theta$ thus (2) becomes,

$$-4(-V_2' \tan 4\theta) \sin 4\theta + 4V_2' \cos 4\theta = \frac{1}{\cos 4\theta}$$

$$V_2' \left(\frac{\sin^2 4\theta}{\cos 4\theta} + \cos 4\theta \right) = \frac{1}{4 \cos 4\theta}$$ multiply by $\cos 4\theta$

$$V_2' (\sin^2 4\theta + \cos^2 4\theta) = 1/4 \Rightarrow \frac{dV_2}{d\theta} = 1/4 \therefore V_2 = \theta/4$$

Consider then (1) now that we know $V_2' = 1/4$ it simplifies,

$$V_1' \cos 4\theta + \tfrac{1}{4} \sin 4\theta = 0 \Rightarrow V_1' = \frac{-1}{4} \frac{\sin 4\theta}{\cos 4\theta}$$

$$V_1 = \int \frac{dV_1}{d\theta} d\theta = \int \frac{-1}{4} \frac{\sin 4\theta}{\cos 4\theta} d\theta = \frac{+1}{16} \int \frac{du}{u} = \frac{1}{16} \ln|\cos 4\theta| = V_1$$

Thus $Y = Y_h + Y_p = \boxed{c_1 \cos 4\theta + c_2 \sin 4\theta + \frac{1}{16} \cos 4\theta \ln|\cos 4\theta| + \frac{\theta}{4} \sin 4\theta}$

general sol$^{\text{n}}$.

using (10) instead

$$V_1 = \int \frac{-g(t)\, Y_2\, d\theta}{a[Y_1 Y_2' - Y_1' Y_2]} = \int \frac{-\sec 4\theta \sin 4\theta \, d\theta}{\cos 4\theta \cdot 4\cos 4\theta + 4 \sin 4\theta \sin 4\theta} = \int -\frac{\tan 4\theta}{4} d\theta = \frac{1}{16} \ln|\cos 4\theta|$$

$$V_2 = \int \frac{g(t)\, Y_1\, d\theta}{Y_1 Y_2' - Y_1' Y_2} = \int \frac{\sec 4\theta \cos 4\theta}{4} d\theta = \int \frac{d\theta}{4} = \frac{\theta}{4}$$

Clearly Eq$^{\text{n}}$ (10) tends to simplify these calculations, the sentence at the bottom of p. 195 is questionable. Here's the rub though, these are not as easy to memorize as Eq$^{\text{n}}$ (9). And besides to be really pure about working from bare principles you'd start with $Y_p = V_1 y_1 + V_2 y_2$ and derive eq$^{\text{n}}$ (9). I'll allow you to memorize the eq$^{\text{n}}$ (10) for more efficient calculation, however I may ask you to show how (9) $\Rightarrow$ (10). that'd be a reasonable ? .

§4.6 #11 $y'' + y = \tan^2 t$. Here $y_1 = \cos t$, $y_2 = \sin t$ and $g(t) = \tan^2 t$, $a = 1$.

$$V_1 = \int \frac{-\tan^2 t \sin t}{\cos t \cos t - (-\sin t)\sin t} = \int -\tan^2 t \sin t\, dt$$

$$= \int \frac{-\sin^3 t\, dt}{\cos^2 t}$$

$$= \int \frac{(1-\cos^2 t)}{\cos^2 t}(-\sin t\, dt)$$

$$= \int \frac{1-u^2}{u^2}\, du$$

$$= \frac{-1}{u} - u = \frac{-1}{\cos t} - \cos t = -\sec t - \cos t = V_1$$

$$V_2 = \int \frac{\tan^2 t \cos t}{1}\, dt = \int \frac{\sin^2 t \cos t\, dt}{\cos^2 t}$$

$$= \int \left(\frac{1-\cos^2 t}{\cos t}\right) dt$$

$$= \int (\sec t - \cos t)\, dt$$

$$= \ln|\sec t + \tan t| - \sin t = V_2$$

$$y = y_h + y_p = c_1 \cos t + c_2 \sin t + \cos t(-\sec t - \cos t) + \sin t(-\sin t + \ln|\sec t + \tan t|)$$

$$= c_1 \cos t + c_2 \sin t - 1 - \cos^2 t - \sin^2 t + \sin t \ln|\sec t + \tan t|$$

$$= \boxed{c_1 \cos t + c_2 \sin t - 2 + \sin t \ln|\sec t + \tan t| = y}$$

__Remark:__ Although in previous problems I have not utilized eq^n (10) I think it's clear that it's good to use eq^n (10) and we will usually begin with it as we did here. You should understand where the intial eq^n's for V_1 and V_2 come from, but I will not expect you to justify them every time.

§4.6 # 17 $\frac{1}{2}y'' + 2y = \tan 2t - \frac{1}{2}e^t$, I multiply by 2 to begin
$y'' + 4y = 2\tan 2t - e^t$. Now we'll break up the problem into sensible pieces

1.) find y_1 & y_2 the fundamental sol^{n}'s ($y_1 = \cos 2t$ & $y_2 = \sin 2t$)
2.) find y_{p_1} the particular sol^{n} to $y'' + 4y = -e^t$
3.) find y_{p_2} the particular sol^{n} to $y'' + 4y = 2\tan 2t$
4.) assemble total sol^{n} via superposition principle.

To begin then

2.) Let $y_{p_1} = Ae^t$ then $y_{p_1}' = y_{p_1}'' = y_{p_1} = Ae^t$ substitute to get

$$y_{p_1}'' + 4y_{p_1} = 5y_{p_1} = 5Ae^t = -e^t \Rightarrow 5A = -1 \Rightarrow A = \frac{-1}{5}$$

Thus by undetermined coefficients we find $y_{p_1} = \frac{-1}{5}e^t$

3.) Use variation of parameters to find $Y_{p_2} = V_1 y_1 + V_2 y_2$ for the D.eq^{n} $y'' + 4y = 2\tan 2t = g(t)$. Go straight to eq^{n} (10)

$$V_1 = \int \frac{-2\tan(2t)\sin(2t)\,dt}{\sin 2t(-2\cos 2t) - 2\cos 2t(\cos 2t)}$$

$$= \int \frac{\sin^2(2t)\,dt}{\cos(2t)}$$

$$= \int \frac{1 - \cos^2(2t)}{\cos(2t)}\,dt$$

$$= \int [\sec(2t) - \cos(2t)]\,dt = \frac{1}{2}\ln|\sec(2t) + \tan(2t)| - \frac{1}{2}\sin(2t) = V_1$$

$$V_2 = \int \frac{2\tan(2t)\cos(2t)\,dt}{-2}$$

using what we saw for V_1, same here. btw. note the denominator is the Wronskian! that is comforting as $W[Y_1, Y_2](t) \neq 0$ for L.I. functions Y_1 & Y_2.

$$= \int -\sin(2t)\,dt$$

$$= \frac{1}{2}\cos(2t) = V_2$$

Thus $Y_{p_2} = \frac{\cos(2t)}{2}\ln|\sec 2t + \tan 2t| - \frac{1}{2}\sin\cancel{2t}\cancel{\cos 2t} + \frac{1}{2}\sin\cancel{2t}\cancel{\cos 2t}$

4.) By the superposition principle (for linear differential eq^{n}'s, Th^{m}(3) pg. 187.

$$\boxed{y = C_1\cos 2t + C_2\sin 2t - \frac{1}{5}e^t + \frac{1}{2}\cos(2t)\ln|\sec 2t + \tan 2t|}$$

Remark: If we had not split up the problem and instead had tried to do it in one shot with variation of parameters it would have also worked but would've been messy.