§4.7 #45

Given that $t^2y'' - 2ty' - 4y = 0$, $t > 0$ has sol^{n} $y = t^{-1}$ find a 2^{nd} linearly independent sol^{n} via reduction of order

$$y_2(t) = y_1(t)\int \frac{\exp\left(-\int P(t)\,dt\right)}{y_1(t)^2}\,dt$$

: reduction of order formula, see Th^{m}(8)

$$P(t) = \frac{-2t}{t^2} = \frac{-2}{t}.$$

$$= \frac{1}{t}\int \frac{\exp\left(\int \frac{2}{t}\,dt\right)}{(1/t)^2}\,dt$$

$$= \frac{1}{t}\int \frac{e^{\ln|t|^2}}{(1/t)^2}\,dt$$

$$= \frac{1}{t}\int t^4\,dt$$

$$= \frac{1}{t}\,\frac{t^5}{5} \quad \therefore \quad \boxed{y_2(t) = \frac{1}{5}t^4}$$

<u>Remark</u>: the $1/5$ can be left off. The general sol^{n} $y = c_1y_1 + c_2y_2$ allows all the same functions since c_2 can be adjusted accordingly. There is always some ambiguity in the choice of y_1, y_2. We can however say y_1, y_2 are unique up to <u>linear independence</u>. If z_1, z_2 are two other linear independent sol^{n}'s then $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2,\mathbb{R})$ such that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \left(\text{where } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \text{ since } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{Gl(2,\mathbb{R})}\right)$$

this denotes the set of all 2×2 invertible matrices with real entries.

§4.7#47 find 2^{nd} linearly independent sol^{n} for $tx'' - (t+1)x' + x = 0$, $t > 0$ given that $y_1(t) = e^t$ is a sol^{n}.

To begin we need to put the differential eq^{n} in form

$$x'' - \left(1 + \frac{1}{t}\right)x' + \frac{1}{t}x = 0$$

We find $P(t) = -1 - 1/t$ hence

$$\begin{aligned} \exp\left(-\int P(t)dt\right) &= \exp\left[-\int\left(-1 - \frac{1}{t}\right)dt\right] \\ &= \exp\left[t + \ln|t|\right] \\ &= \exp[t]\exp[\ln(t)] \qquad \begin{pmatrix} |t| = t \\ \text{since } t > 0 \end{pmatrix} \\ &= te^t \end{aligned}$$

Now use the Reduction of Order Th^{m};

$$\begin{aligned} y_2(t) &= y_1(t)\int \frac{\exp\left[-\int P(t)\,dt\right]}{y_1(t)^2}\,dt \\ &= e^t \int \frac{te^t}{(e^t)^2}\,dt \\ &= e^t \int te^{-t}\,dt \quad : \quad \int V\,du = uv - \int u\,dV \\ &= e^t\left[-te^{-t} + \int e^{-t}\,dt\right] \\ &= -te^te^{-t} - e^te^{-t} \\ &= -t - 1 \end{aligned}$$

Thus the second linearly independent sol^{n} is $\boxed{y_2(t) = -t - 1}$

Or if you prefer, $\boxed{y_2(t) = t + 1}$ etc...

<u>Remark</u>: the theory of §4.7 is quite satisfying ignoring one annoying detail, <u>how does one find the first sol^{n}</u> to a general variable coefficient DEq^{n}? We have methods for Cauchy-Euler but that's about it. Chapter 8 will help us find yuckier sol^{n}s.