Forced Vibrations : §4.10

§4.10#9 An 8kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch 1.96m before reaching a resting equilibrium. At time $t = 0$ an external force $F(t) = [\cos(2t)]$ N is applied to the system. The damping constant is 3 Nsec/m. Determine steady state sol$^{\underline{n}}$ for this system

Gravity pulls the mass downward, at equilibrium the spring force and gravity must balance. This reveals the stiffness (which they cutely left out)

$$mg = k(\Delta x)$$

$$8(9.8) = k(1.96) \quad \therefore \quad \underline{k = 40} \quad \text{(omitting units)}$$

We must solve, for $t > 0$,

$$8y'' + 3y' + 40y = \cos(2t)$$

$$8\lambda^2 + 3\lambda + 40 = 0 \longrightarrow \lambda = \frac{-3 \pm \sqrt{9 - 4(8)(40)}}{16}$$

Thus $\underline{\lambda = -0.1875 \pm 2.228i}$,

$$y_h = Ae^{-0.1875t}\cos(2.228t + \phi)$$

This homogeneous sol$^{\underline{n}}$ is the system's response to the initial conditions. Since $y(0) = 0$ and $y'(0) = 0$ it will follow $y_h = 0$ but I wanted to mention it (technically y_h is <u>not</u> needed to find steady state sol since $y_h \longrightarrow 0$ as $t \longrightarrow \infty$ if $b > 0$ physically the friction force kills the initial motion of the system.)

§ 4.10 #9 Finally actually finding steady state solⁿ.

$$8y'' + 3y' + 40y = \cos(2t)$$

Use undetermined coeff.

$$y_p = A\cos(2t) + B\sin(2t)$$

$$y_p' = -2A\sin(2t) + 2B\cos(2t)$$

$$y_p'' = -4y_p$$

Substitute,

$$8y_p'' + 3y_p' + 40y_p = \cos(2t)$$

$$3\big(-2A\sin(2t) + 2B\cos(2t)\big) + 8\big(A\cos(2t) + B\sin(2t)\big) = \cos(2t)$$

$$\sin(2t)\big[-6A + 8B\big] + \cos(2t)\big[6B + 8A\big] = \cos(2t)$$

$\sin(2t)$: $-6A + 8B = 0 \longrightarrow B = \frac{3}{4}A$

$\cos(2t)$: $6B + 8A = 1$

$$6\left(\tfrac{3}{4}A\right) + 8A = 1$$

$$18A + 32A = 4$$

$$50A = 4 \quad \therefore \quad A = \frac{2}{25}$$

$$\Rightarrow B = \frac{3}{4}\left(\frac{2}{25}\right) = \frac{3}{50}$$

Hence $\boxed{y_p(t) = \frac{2}{25}\cos(2t) + \frac{3}{50}\sin(2t)}$

Remark: As $t \to \infty$ the general solⁿ will converge to the particular solⁿ. Physically $y_p(t)$ is the response of the system to the external force.

§4.10 #11) Omitting units and adjusting to compensate for idiotic British units,

$$m = \frac{8}{32}, \quad k = 10, \quad b = 1, \quad F_{ext}(t) = 2\cos 2t$$

where $y(0) = 0$ and $y'(0) = 0$. Find eq$^{\underline{n}}$ of motion and the resonant frequency

$$\frac{1}{4}y'' + y' + 10y = 2\cos 2t$$

$$y'' + 4y' + 40y = 8\cos(2t)$$

$$\lambda^2 + 4\lambda + 40 = 0$$

$$(\lambda+2)^2 + 36 = 0 \Rightarrow \underline{\lambda = -2 \pm 6i}.$$

Hence, $\underline{y_h = c_1 e^{-2t}\cos 6t + c_2 e^{-2t}\sin 6t}$. Save for later.

The particular sol$^{\underline{n}}$ can be found via undetermined coefficients,

$$y_p = A\sin(2t + \theta)$$

$$y_p' = 2A\cos(2t+\theta)$$

$$y_p'' = -4y_p$$

Remark: Could use $y_p = A\cos 2t + B\sin 2t$ but I want to check against text's answer, so I'll make due with $A\sin(2t+\theta)$.

Substitute into DEq$^{\underline{n}}$,

$$y_p'' + 4y_p' + 40y_p = 8\cos 2t$$

$$8A\cos(2t+\theta) + 36A\sin(2t+\theta) = 8\cos 2t$$

$$8A[\cos 2t\cos\theta - \sin 2t\sin\theta] + 36A[\sin 2t\cos\theta + \sin\theta\cos 2t] = 8\cos(2t)$$

$\underline{\cos 2t|}$ $\quad 8A\cos\theta + 36A\sin\theta = 8$

$\underline{\sin 2t|}$ $\quad -8A\sin\theta + 36A\cos\theta = 0 \Rightarrow A\big(\underbrace{-8\sin\theta + 36\cos\theta}_{\text{zero since } A \neq 0}\big) = 0$

$$\tan\theta = \frac{36}{8} = \frac{9}{2}$$

Thus $\theta = \tan^{-1}(9/2) = 1.352$

$\Rightarrow 36.878A = 8 \Rightarrow A = 0.2169.$

$$\underline{y_p = 0.2169\sin(2t + 1.352)}.$$

§4.10 #11 Continued

The general sol^{n} has the form

$$y(t) = e^{-2t}\left[c_1\cos(6t) + c_2\sin(6t)\right] + 0.2169\sin(2t + 1.352)$$

$$y'(t) = -2e^{-2t}\left[c_1\cos 6t + c_2\sin 6t\right] + e^{-2t}\left[-6c_1\sin 6t + 6c_2\cos 6t\right] + 2(0.2169)\cos(2t + 1.352)$$

Apply the given initial conditions,

$$y(0) = 0 = c_1 + 0.2169\sin(1.352) \quad \therefore \quad \underline{c_1 = -0.212}$$

$$y'(0) = 0 = -2c_1 + 6c_2 + 2(0.2169)\cos(1.352)$$

$$\Rightarrow c_2 = \frac{2(-0.212) - 2(0.2169)\cos(1.352)}{6} = \underline{-0.0864}$$

$$\therefore \boxed{y(t) \cong -0.212e^{-2t}\cos 6t - 0.0864e^{-2t}\sin 6t + 0.2169\sin(2t + 1.352)}$$

(equation of motion)

The resonant frequency is the frequence for $F_{ext}(t)$ which would result in the steady state sol^{n} with maximal amplitude. The text shows that the frequence where this occurs depends on m, b and k as follows:

$$f_{resonance} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{10}{1/4} - \frac{1}{2/16}}$$

$$= \frac{1}{2\pi}\sqrt{40 - 8}$$

$$= \frac{\sqrt{8}}{\pi}$$

$$\boxed{f_{resonance} = \frac{2\sqrt{2}}{\pi}\ \text{Hz}}$$

§4.10 (extra) Omitting units,
$m = 8$, $k = 10$, $b = 1$, $F_{ext.}(t) = 2\cos(2t)$ for $t > 0$
and $y(0) = 0$, $y'(0) = 0$. Find eq^{n} of motion for
the mass.

$$8y'' + y' + 10y = 2\cos(2t)$$

$$8\lambda^2 + \lambda + 10 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4(8)(10)}}{16}$$

$$\Rightarrow \lambda \cong -0.0625 \pm 1.1163\,i$$

$$\Rightarrow \underline{y_h = A_0 e^{-0.0625t}\cos(1.1163t + \phi_0)}.$$

Use undetermined coefficients, $y_p = A\cos 2t + B\sin 2t$
thus $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4y_p$. Substitute,

$$8y_p'' + y_p' + 10y_p = y_p' - 22y_p = 2\cos(2t)$$

$$-2A\sin 2t + 2B\cos 2t - 22A\cos 2t - 22B\sin 2t = 2\cos 2t$$

$$\sin(2t)\left[-2A - 22B\right] + \cos(2t)\left[2B - 22A\right] = 2\cos 2t$$

Equating coefficients: $-2A + 22B = 0$ & $2B - 22A = 2$ which
yields $A = 11B$ hence $2B - 22(11B) = 2 \Rightarrow \boxed{B = \frac{1}{120}}$
and so $\boxed{A = 11/120}$ we find the general sol$^{\underline{n}}$:

$$\boxed{y(t) = A_0 e^{-0.0625t}\cos(1.1163t + \phi_0) + \frac{11}{120}\cos(2t) + \frac{1}{120}\sin(2t)}$$

Now, we must select A_0 and ϕ_0 that cause $y(0) = 0$ & $y'(0) = 0$.

$$y'(t) = A_0 e^{-0.0625t}\left[-0.0625\cos(1.1163t + \phi_0) - 1.1163\sin(1.1163t + \phi_0)\right] + 2$$
$$\quad - \frac{22}{120}\sin(2t) + \frac{2}{120}\cos(2t)$$

$$y(0) = A_0 \cos\phi_0 + \frac{11}{120} = 0$$

$$y'(0) = A_0\left[-0.0625\cos(\phi_0) - 1.1163\sin(\phi_0)\right] + \frac{2}{120} = 0$$

Need to solve for A_0, ϕ_0

$$\frac{2}{120} = A_0\left[0.0625\cos\phi_0 + 1.1163\sin\phi_0\right]$$

$$\frac{-11}{120} = A_0\cos\phi_0$$

Dividing one eq$^{\text{n}}$ by the other

$$\frac{\frac{2}{120}}{\frac{-11}{120}} = \frac{-2}{11} = \frac{0.0625\cos\phi_0 + 1.1163\sin\phi_0}{\cos\phi_0} = 0.0625 + 1.1163\tan\phi_0$$

$$\tan\phi_0 = \frac{-0.244318}{1.1163} \quad \therefore \quad \underline{\phi_0 = 2.92613}$$

$$A_0 = \frac{-11/120}{\cos\phi_0} = 0.093865$$

Hence,

$$\boxed{y(t) = 0.093865\, e^{-0.0625t}\left[-0.0625\cos(1.1163t + \phi_0) - 11.1163\sin(1.1163t + \phi_0)\right] + -\frac{22}{120}\sin(2t) + \frac{2}{120}\cos(2t)}$$