§7.4#1 $G(s) = \frac{6}{(s-1)^4}$ looks like $\frac{3!}{s^2}$ but it's shifted by 1. So we should think about Th^m (3) of p. 386. $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$ Let $F(s) = 3!/s^4$ and $a = 1$ then recall from table 7.1 that $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$ put it all together,

$$\mathcal{L}\{e^t t^3\}(s) = \frac{3!}{(s-1)^4} = \frac{6}{(s-1)^4}$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}(t) = e^t t^3 = g(t) = \mathcal{L}^{-1}\{G\}(t)}$$

Remark: In terms of calculational difficulty taking the Laplace transform of f to get F is analogous to differentiation, in the sense that it is straight-forward, you just use the table & properties of the Laplace transform. On the other hand, taking the inverse Laplace transform is more like integration, you have to see where your going to get there, it's much harder. To take the inverse transform basically we just have to make an informed guess and then check our guess.

§7.4#2 $F(s) = \frac{2}{s^2+4}$ notice $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

Thus $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t) = \boxed{\sin(2t) = f(t)}$

§7.4#3 $F(s) = \frac{s+1}{s^2+2s+10}$. Since $s^2+2s+10$ cannot be factored over the real numbers $\Rightarrow$ we'll get sines & cosines. Complete the square: $s^2+2s+10 = (s+1)^2+9$. Thus

$$\frac{s+1}{s^2+2s+10} = \frac{s+1}{(s+1)^2+9} = \mathcal{L}\{e^{-t}\cos(3t)\}(s) \quad \text{(last entry of table.)}$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}(t) = e^{-t}\cos(3t)}$$

§7.4 #8 $F(s) = \frac{1}{s^5} = \frac{1}{4!}\frac{4!}{s^5} = \frac{1}{4!}\mathcal{L}\{t^4\}(s) = \mathcal{L}\left\{\frac{t^4}{24}\right\}(s)$

$\Rightarrow \boxed{\mathcal{L}^{-1}\{1/s^5\}(t) = t^4/24}$

§7.4 #9 $\frac{3s-15}{2s^2-4s+10} = F(s)$. We need to refine the expression algebraically so we can see what to do. Notice,

$$2s^2-4s+10 = 0 \quad \text{when} \quad s = \frac{4 \pm \sqrt{16-80}}{4} = 1 \pm \frac{8i}{4}$$

So it cannot be factored over $\mathbb{R}$, so complete the square,

$$2s^2-4s+10 = 2(s^2-2s+5) = 2((s-1)^2+4)$$

I like to give the s^2 term a coefficient of one before completing square. Just makes it easier.

Thus the denominator suggests we'll get $\sin(2t)$ or $\cos(2t)$ apparently shifted by e^t. Let's work out the details,

$$\frac{3s-15}{2s^2-4s+10} = \frac{1}{2}\left(\frac{3s-15}{(s-1)^2+4}\right)$$

$$= \frac{1}{2}\left(\frac{3(s-1)+3-15}{(s-1)^2+4}\right)$$

$$= \frac{3}{2}\left(\frac{s-1}{(s-1)^2+4}\right) - \frac{6}{(s-1)^2+4}$$

$$= \frac{3}{2}\left(\frac{s-1}{(s-1)^2+2^2}\right) - 3\left(\frac{2}{(s-1)^2+2^2}\right)$$

$$= \frac{3}{2}\mathcal{L}\{e^t \sin 2t\} - 3\mathcal{L}\{e^t \cos 2t\}$$

I want a function of $(s-1)$ so I add & subtract to make that happen explicitly.

trying to manipulate it till it looks like last entries in table 7.

Thus in view of the algebra above

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\}(t) = \frac{3}{2}e^t \sin 2t - 3e^t \cos(2t)}$$

<u>Remark</u>: to begin you must determine the nature of the quadratic in the denominator. If it's irreducible (can't factor) then complete the square and proceed as I did above. Or you can be boring and use logic like example 7 of p. 399.

§7.4 #21 $\dfrac{6s^2-13s+2}{s(s-1)(s-6)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{s-6}$ ← partial fractions. aka. undoing a common-denominator

$$6s^2-13s+2 = A(s-1)(s-6)+Bs(s-6)+Cs(s-1)$$

$s=0$ | $2 = A(-1)(-6) = 6A \Rightarrow A = 1/3$

$s=1$ | $6-13+2 = -5 = -5B \Rightarrow B=1$

$s=6$ | $216-78+2 = 140 = C\cdot 30 \Rightarrow C = 14/3$

$$\Rightarrow \quad \frac{6s^2-13s+2}{s(s-1)(s-6)} = \frac{1}{3}\left(\frac{1}{s}\right) + \frac{1}{s-1} + \frac{14}{3}\left(\frac{1}{s-6}\right)$$

Thus $\boxed{\mathcal{L}^{-1}\left\{\frac{6s^2-13s+2}{s(s-1)(s-6)}\right\}(t) = \frac{1}{3} + e^t + \frac{14}{3}e^{6t}}$

§7.4 #22 $\dfrac{s+11}{(s-1)(s+3)} = \dfrac{A}{s-1} + \dfrac{B}{s+3}$

$$s+11 = A(s+3) + B(s-1) \xrightarrow{s=-3} 8 = -4B \therefore B=-2$$

$$\xrightarrow[s=1]{} 12 = 4A \therefore A = 3$$

Thus $\dfrac{s+11}{(s-1)(s+3)} = \dfrac{3}{s-1} - \dfrac{2}{s+3} = F(s)$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\{F\}(t) = 3e^t - 2e^{-3t}}$$

§7.4 #26 $\dfrac{7s^3-2s^2-3s+6}{s^3(s-2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{D}{s-2}$

$$7s^3-2s^2-3s+6 = As^2(s-2)+Bs(s-2)+C(s-2)+Ds^3$$
$$= s^3[A+D] + s^2[-2A+B]+s[-2B+C]-2C$$

Equating Coefficients gives us,

logic here works ↑ the equations starting with the last one,

s^3: $A+D = 7$
s^2: $-2A+B = -2$
s: $-2B+C = -3$
1: $-2C = 6 \rightarrow C=-3 \rightarrow -2B-3=-3 \Rightarrow B=0 \Rightarrow A=1$
$\Rightarrow D=6$

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^3} + \frac{6}{s-2}\right\}(t) = \boxed{1-\frac{3}{2}t^2+6e^{2t}}$$

§7.4 #28 We'll solve for $F(s)$,

$$s^2 F(s) + sF(s) - 6F(s) = \frac{s^2+4}{s^2+s}$$

$$F(s)\,(s^2+s-6) = \frac{s^2+4}{s^2+s} \Rightarrow F(s) = \frac{s^2+4}{(s^2+s-6)(s^2+s)}$$

We can factor $(s^2+s-6)(s^2+s) = (s+3)(s-2)s(s+1)$. Then do partial fractions to find (after some algebra)

$$F(s) = \frac{s^2+4}{(s^2+s-6)(s^2+s)} = \frac{-13}{30(s+3)} + \frac{5}{6(s+1)} + \frac{4}{15(s-2)} - \frac{2}{3s}$$

Then it's clear how to find the inverse transform,

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \boxed{\frac{-13}{30}e^{-3t} + \frac{5}{6}e^{-t} + \frac{14}{5}e^{2t} - \frac{2}{3}}$$

§7.4 #31

a.) $f_1(t) = \begin{cases} 0 & t=2 \\ t & t\neq 2 \end{cases}$

$$\mathcal{L}\{f_1\}(s) = \int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty te^{-st}dt = \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

b.) $f_2(t) = \begin{cases} 5 & t=1 \\ 2 & t=6 \\ t & t\neq 1,6 \end{cases}$ again $\mathcal{L}\{f_2\}(s) = \mathcal{L}(t)(s)$

the integral ignores a finite number of jump-discontinuities.

c.) $\mathcal{L}\{f_3\}(s) = \mathcal{L}\{t\}(s) = 1/s^2$

<u>ALL</u> of the above are "the" inverse Laplace transform of $1/s^2$.

§7.4 # 33 $\mathcal{L}^{-1}\left\{ \frac{d^nF}{ds^n} \right\}(t) = (-t)^n f(t)$ use this to find the $\mathcal{L}^{-1}$ of $\ln((s+2)/(s-5))$. Notice

PH-71

$$\ln\left(\frac{s+2}{s-5}\right) = \ln(s+2) - \ln(s-5) = F(s)$$

$$\frac{dF}{ds} = \frac{1}{s+2} - \frac{1}{s-5}$$

$$\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t) = -t f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s-5}\right\}$$

$$= e^{-2t} - e^{5t}$$

$$\therefore \boxed{f(t) = \frac{-1}{t}\left(e^{-2t} - e^{5t}\right)}$$

§7.4 # 36

$F(s) = \tan^{-1}(1/s)$ find $\mathcal{L}^{-1}\{F\}(t) = f(t)$,

$$\frac{dF}{ds} = \frac{1}{1+(1/s)^2} \cdot \frac{-1}{s^2} = \frac{-1}{s^2+1} = -\mathcal{L}\{\sin t\}(s)$$

$$\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t) = -t f(t) = -\sin(t) \quad \therefore \boxed{f(t) = \frac{\sin t}{t}}$$