§7.8 #1

$$\int_{-\infty}^{\infty} (t^2-1)\,\delta(t)\,dt = (t^2-1)\Big|_{t=0} = \boxed{-1}$$

§7.8 #2

$$\int_{-\infty}^{\infty} e^{3t}\,\delta(t)\,dt = e^{3t}\Big|_{t=0} = e^{3(0)} = \boxed{1}$$

§7.8 #3

$$\int_{-\infty}^{\infty} \sin(3t)\,\delta(t-\pi/2)\,dt = \sin(3t)\Big|_{t=\pi/2} = \sin\left(\tfrac{3\pi}{2}\right) = \boxed{-1}$$

§7.8 #4

$$\int_{-\infty}^{\infty} e^{-2t}\,\delta(t+1)\,dt = e^{-2t}\Big|_{t=-1} = e^{-2(-1)} = \boxed{e^2}$$

§7.8 #7

$$\mathcal{L}\{\delta(t-1)\}(s) = e^{-s}$$ using eq ^{n}(6) of p. 409.

$$\mathcal{L}\{\delta(t-3)\}(s) = e^{-3s}$$

Thus $\mathcal{L}\{\delta(t-1)-\delta(t-3)\}(s) = \boxed{e^{-s}-e^{-3s}}$

§7.8 #11

$$\mathcal{L}\{\delta(t-\pi)\sin t\} = \int_0^{\infty} \delta(t-\pi)\sin t\, e^{-st}\,dt$$ ← def^{n} of Laplace Transform

$$= (\sin t)e^{-st}\Big|_{t=\pi}$$

$$= (\sin\pi)e^{-s\pi}$$

$$= \boxed{0}$$

Remark: δ-function makes integrals become evaluations instead, much easier!

§7.8 #13 $\quad w'' + w = \delta(t-\pi) \qquad w(0)=0 \ \& \ w'(0)=0$

$$s^2 W(s) + W(s) = \mathcal{L}\{\delta(t-\pi)\}(s) = e^{-\pi s} \quad \leftarrow \text{(used intial conditions)}$$

$$W(s)(s^2+1) = e^{-\pi s} \Rightarrow W(s) = \frac{1}{s^2+1} e^{-\pi s}$$

Now to solve for $w(t)$ we take inverse Laplace transform,

$$w(t) = \mathcal{L}^{-1}\{W\}(t) = \mathcal{L}^{-1}\{e^{-\pi s} \underbrace{\frac{1}{s^2+1}}_{F(s)}\} \quad : \quad f(t) = \sin(t)$$

$$= f(t-a)\,u(t-a) \quad : \text{see Th}^m (8) \text{ p. 387.}$$

$$= \sin(t-\pi)\,u(t-\pi)$$

$$= (\sin t \cos\pi - \sin\pi \cos t)\,u(t-\pi)$$

$$= \boxed{-\sin(t)\,u(t-\pi)}$$

Similar to #17

$$y'' + y = 4\delta(t-2) + t^2 \qquad y(0)=0, \quad y'(0)=2$$

$$s^2 Y - 2 + Y = 4e^{-2s} + \frac{2}{s^3}$$

$$Y(s^2+1) = 2 + 4e^{-2s} + 2/s^3$$

$$Y = \frac{2}{s^2+1} + \frac{4}{s^2+1}e^{-2s} + \underbrace{\frac{2}{(s^2+1)s^3}}_{\boxed{\frac{s}{s^2+1} - \frac{1}{s} + \frac{1}{s^3}}}$$

partial fractions

$$\boxed{y = 2\sin(t) + 4\sin(t-2)\,u(t-2) + \cos t - 1 + \tfrac{1}{2}t^2}$$

§7.8 #29 The following eqⁿ's model a mass on a spring that is struck by a hammer at $t = \pi/2$.

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t-\pi/2) \quad \text{with } x(0)=1 \ \& \ x'(0)=0$$

Taking the Laplace transform yields,

$$s^2 \mathcal{X} - s + 9\mathcal{X} = -3e^{-\frac{\pi s}{2}}$$

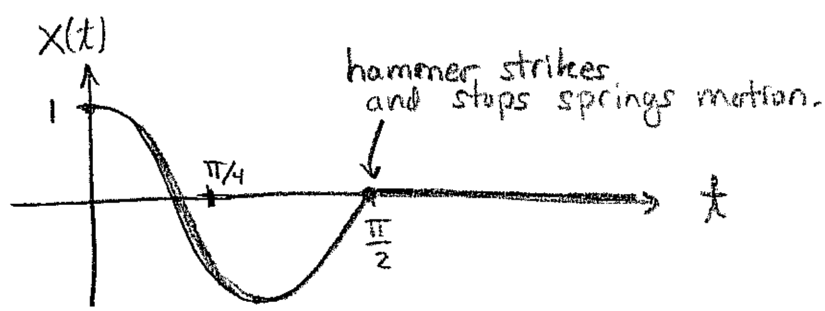
$$\mathcal{X}(s^2+9) = s - 3e^{-\pi s/2}$$

$$\mathcal{X}(s) = \frac{s}{s^2+9} - \frac{3}{s^2+9}e^{-\pi s/2}$$

$$\begin{aligned}
x(t) = \mathcal{L}^{-1}\{\mathcal{X}\}(t) &= \cos(3t) - \sin(3(t-\pi/2))\, u(t-\pi/2) \\
&= \cos(3t) - \sin(3t - 3\pi/2)\, u(t-\pi/2) \\
&= \cos(3t) - \left[\sin(3t)\cos(3\pi/2) - \sin(\tfrac{3\pi}{2})\cos(3t)\right] u(t-\pi/2) \\
&= \cos(3t) - \cos(3t)\, u(t-\pi/2) \\
&= \boxed{\cos(3t)\left[1 - u(t-\pi/2)\right] = x(t)}
\end{aligned}$$

$$x(t) = \begin{cases} \cos(3t) & \text{for } t < \pi/2 \\ 0 & \text{for } t > \pi/2 \end{cases}$$

the mass ceases moving after being hit by the hammer.



Remark: we could also understand earlier problems to describe a similar physical situation, for example #13 had $w(t) = -\sin(t)\,u(t-\pi)$ with $w(0) = w'(0) = 0$ this means the hammer struck the spring and set it vibrating for $t > \pi$. The possibilities are endless, but the fact the δ-function models such impulses is neat.