Power Series Basics: §8.2

§8.2#1 determine the interval of convergence,

$$\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n \equiv g(x)$$

Apply ratio test

$$L = \left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{2^{-(n+1)}}{n+2}(x-1)^{n+1}\right)\left(\frac{n+1}{2^{-n}(x-1)^n}\right)\right| = \frac{1}{2}\left(\frac{n+1}{n+2}\right)|x-1|$$

Notice $L \longrightarrow \frac{1}{2}|x-1|$ as $n \longrightarrow \infty$. The ratio test states $L < 1 \Rightarrow$ convergence, $L > 1 \Rightarrow$ divergence, $L = 1$ no info. Clearly $\frac{1}{2}|x-1| < 1 \iff |x-1| < 2 \iff -1 < x < 3$, thus the power series converges on at least $(-1,3)$. Check end-points,

$$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$, converges by alternating series test.

$$x = 3 \Rightarrow \sum_{n=0}^{\infty} \frac{2^{-n}2^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$, diverges by $p=1$ series test.

Hence, $\boxed{\text{dom}(g) = [-1, 3)}$.

§8.2#5

$$f(x) = \sum_{n=1}^{\infty} \frac{3}{n^3}(x-2)^n \Rightarrow L = \left|\frac{a_{n+1}}{a_n}\right| = \frac{n^3}{(n+1)^3}|x-2| \longrightarrow |x-2|$$
as $n \to \infty$

Hence dom(f) includes $(1,3)$. Moreover

1. $x = 1 \Rightarrow f(1) = \sum_{n=1}^{\infty} \frac{3(-1)^n}{n^3}$, converges by alternating series test
2. $x = 3 \Rightarrow \sum_{n=1}^{\infty} \frac{3}{n^3}$, converges by $p = 3$ series test.

Hence, $\boxed{\text{dom}(f) = [1, 3]}$

§8.2 #9 Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n$ and $g(x) = \sum_{n=1}^{\infty} 2^{-n} x^{n-1}$

$$f(x) + g(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n + \sum_{n=1}^{\infty} 2^{-n} x^{n-1}$$

$$= \underline{\underline{\sum_{k=0}^{\infty} \left(\frac{1}{k+1} + 2^{-k-1} \right) x^k}}.$$

let $k = n-1$
so $n = k+1$
to convert 2^{nd} summation to one which starts at zero.

We can change the label of summation back to n,

$$\boxed{f(x) + g(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + 2^{-n-1} \right) x^n}$$

§8.2 #11 Find first three nonzero terms in power series expansion of $f(x)g(x)$ for $f(x) = e^x$, $g(x) = \sin(x)$.

$$f(x)g(x) = \left(1 + x + \tfrac{1}{2}x^2 + \cdots\right)\left(x - \tfrac{1}{6}x^3 + \cdots\right)$$

$$= x - \tfrac{1}{6}x^3 + x^2 - \tfrac{1}{6}x^4 + \tfrac{1}{2}x^3 - \tfrac{1}{12}x^5 + \cdots$$

$$= \underline{\underline{x + x^2 + \tfrac{1}{3}x^3 + \cdots}}$$

Alternatively, could use Taylor's Thm, $h(x) = f(x)g(x)$

$$h(0) = 0$$

$$h'(x) = e^x(\sin(x) + \cos x) \dashrightarrow h'(0) = 1$$

$$h''(x) = e^x(\cancel{\sin x} + \cos x + \cos x - \cancel{\sin x}) \dashrightarrow h''(0) = 2$$

$$h'''(x) = e^x(2\cos x - 2\sin x) \dashrightarrow h'''(0) = 2$$

Thus,

$$h(x) = h(0) + h'(0)x + \tfrac{1}{2}h''(0)x^2 + \tfrac{1}{3!}h'''(0)x^3 + \cdots$$

$$= \underline{\underline{x + x^2 + \tfrac{1}{3}x^3 + \cdots}}.$$

(these are 1^{st} three terms in Maclaurin series for $h(x)$)

Remark: technically the question is ambiguous. If we centered the series at another point the first three terms would be different. For example, $h(x) = h(1) + h'(1)(x-1) + \frac{1}{2}h''(1)(x-1)^2 + \cdots = e^1\sin(1) + e^1(\sin(1) + \cos(1))(x-1) +$

§8.2 # 15 Find first few terms in power series of quotient

$$q(x) = \frac{\sum_{n=0}^{\infty} \frac{x^n}{2^n}}{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

by following the steps below

(a.) Let $q(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n are to be determined. Argue that $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ is Cauchy product of $q(x)$ and $\sum_{n=0}^{\infty} x^n/n!$

(b.) Use formula (6) of the Cauchy product to deduce eq^n's

$$\frac{1}{2^0} = a_0 \; , \quad \frac{1}{2} = a_0 + a_1$$

$$\frac{1}{2^2} = \frac{a_0}{2} + a_1 + a_2$$

$$\frac{1}{2^3} = \frac{a_0}{6} + \frac{a_1}{2} + a_2 + a_3, \ldots$$

(C.) Solve eq^n's in part (b.) to determine constants a_0, a_1, a_2, a_3.

(a.) multiply by denominator $\sum_{n=0}^{\infty} x^n/n!$ to find

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = q(x) \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} c_n x^n$$

(b.) notice a_n are coefficients of $q(x)$ (we don't know them yet) clearly $b_n = 1/n!$ [$q(x)$ plays role of $f(x)$ in (5) on p. 460 while $\sum_{n=0}^{\infty} x^n/n! = g(x)$ in (5) of p. 460]

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \implies \underbrace{\frac{1}{2^n} = \sum_{k=0}^{n} a_k \frac{1}{(n-k)!}}$$

Then evaluate for $n = 0, 1, 2, 3$ to obtain,

$$\frac{1}{2^0} = a_0, \quad \frac{1}{2} = a_0 + a_1, \quad \frac{1}{2^2} = \frac{a_0}{2} + a_1 + a_2, \quad \frac{1}{2^3} = \frac{a_0}{3!} + \frac{a_1}{2} + a_2 + a_3$$

(C.) $a_0 = 1$, $a_1 = \frac{1}{2} - 1 = \frac{-1}{2}$, $a_2 = \frac{1}{4} - \frac{1}{2} - \left(\frac{-1}{2}\right) = \frac{1}{4}$ and

$a_3 = \frac{1}{8} - \frac{1}{6} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} = \frac{-1}{24}$ $\therefore$ $\boxed{q(x) = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \cdots}$

§8.2#25 Express the given power series as a series with generic term x^k

Consider,

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k . = \sum_{j=1}^{\infty} a_{j-1} x^j$$

let $k = n+1$

note $n=0$ gives $k=1$ thus

Also solve for $n = k-1$

Of course we could just as well write the index of summation can be replaced with another letter in much the same way as $\int_0^1 f(x)dx = \int_0^1 f(u)du$.

We sometimes call n, k, x, u dummy - variables.

§8.2#29 Find Taylor Series about $x_o = \pi$ for $f(x) = \cos(x)$

$$\begin{aligned}
f(x) &= f(\pi) + f'(\pi)(x-\pi)^2 + \tfrac{1}{2} f''(\pi)(x-\pi)^2 + \tfrac{1}{3!} f'''(\pi)(x-\pi)^3 + \cdots \\
&= \cos(\pi) - \sin(\pi)(x-\pi)^2 - \tfrac{1}{2}\cos\pi\,(x-\pi)^2 + \tfrac{1}{3!}\sin\pi\,(x-\pi)^3 + \cdots \\
&= \cos\pi - \tfrac{1}{2}\cos(\pi)(x-\pi)^2 + \tfrac{1}{4!}(\cos\pi)(x-\pi)^4 - \tfrac{1}{6!}\cos\pi\,(x-\pi)^6 + \cdots \\
&= -1 + \tfrac{1}{2}(x-\pi)^2 - \tfrac{1}{4!}(x-\pi)^4 + \tfrac{1}{6!}(x-\pi)^6 + \cdots \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^n}
\end{aligned}$$

- If the Taylor series exists (and converges to the function) then it is unique. Thus if there is another way to find it we'll get the same series, here we can use a trig-identity trick to short-cut the work above,

$$\begin{aligned}
f(x) = \cos(x) = \cos(x-\pi+\pi) &= \cos(x-\pi)\cos\pi - \sin(x-\pi)\sin\pi \\
&= -\cos(x-\pi) \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi)^n
\end{aligned}$$

known Maclaurin Series for cosine

(alternate trick sol$^{\underline{n}}$)

§ 8.2 #31 Find Taylor Series for $f(x) = \frac{1+x}{1-x}$ centered at $x = 0$.

You could apply Taylor's Th$^{\underline{m}}$ directly, BUT it's much easier to use the geometric series result: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ which holds for all $r \in \mathbb{R}$ such that $|r| < 1$.

$$
\begin{aligned}
f(x) = \frac{1+x}{1-x} &= \frac{1}{1-x} + \frac{x}{1-x} \\
&= \sum_{n=0}^{\infty} 1(x)^n + \sum_{n=0}^{\infty} x(x)^n \\
&= 1 + \sum_{n=1}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} \\
&= 1 + \sum_{n=1}^{\infty} x^n + \sum_{k=1}^{\infty} x^k \\
&= \boxed{1 + 2\sum_{n=1}^{\infty} x^n}
\end{aligned}
$$

: let $k = n+1$ then $n = 0$ switches to $k = 1$ thus

§ 8.2 #33 Find Taylor Series for $f(x) = x^3 + 3x - 4$ centered at $x_o = 1$

Notice that $f'(x) = 3x^2 + 3$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(n)}(x) = 0$ for $n \geq 4$. Thus,

$$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3$$

$$\boxed{f(x) = 6(x-1) + 3(x-1)^2 + (x-1)^3}$$

<u>Remark:</u> if we need to recenter a polynomial it's pretty easy to do with Taylor's Th$^{\underline{m}}$.