Power Series Solutions to Linear ODEs: §8.3

§8.3 #1 Determine all the singular points of the given differential eq$^{\underline{n}}$: $(x+1)y'' - x^2y' + 3y = 0$

To answer this question we divide by $x+1$ to place the given DEq$^{\underline{n}}$ into standard form,

$$y'' - \frac{x^2}{x+1}y' + \frac{3}{x+1}y = 0$$

Clearly $-x^2/x+1$ is not defined for $x = -1$ due to division by zero, hence it is not analytic. We deduce $x = -1$ is a singular point.

§8.3 #9 Find singular pts of $(\sin\theta)y'' - (\ln\theta)y = 0$

Put in standard form,

$$y'' - \frac{\ln\theta}{\sin\theta}y = 0$$

The coefficient function $-\ln\theta/\sin\theta$ is discontinuous at $\theta = n\pi$ for $n \in \mathbb{N}$ and clearly $\theta \leq 0$ will fall outside domain$(\ln\theta)$. Hence we have Singular points $\theta = n\pi$ for $n \in \mathbb{N}$ and $\theta \leq 0$.

§8.3 #13 Find 1$^{\underline{st}}$ four nonzero terms in Maclaurin series solution for $z'' - x^2z = 0$

Let $z = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots$

$z'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \cdots$

Thus we find,

$$\begin{aligned} z'' - x^2z &= 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + 42c_7x^5 + \cdots \\ &\quad - c_0x^2 - c_1x^3 - c_2x^4 - c_3x^5 - c_4x^6 - c_5x^8 - \cdots \\ &= 2c_2 + 6c_3x + (12c_4 - c_0)x^2 + (20c_5 - c_1)x^3 + (30c_6 - c_2)x^4 + \cdots \end{aligned}$$

Now the coefficients of the expression above must all be zero, hence,

$c_2 = 0$, $c_3 = 0$, $12c_4 - c_0 = 0$, $20c_5 - c_1 = 0$, $30c_6 - c_2 = 0, \ldots$

I'll solve these

§8.3 #13 Continued: solve the eq^{n}'s for $c_0, c_1, c_2, \ldots$

$$c_2 = 0$$
$$c_3 = 0$$
$$12c_4 - c_0 = 0 \Rightarrow c_0 = 12c_4 \Rightarrow c_4 = \frac{1}{12}c_0$$
$$20c_5 - c_1 = 0 \Rightarrow c_1 = 20c_5 \Rightarrow c_5 = \frac{1}{20}c_1$$
$$30c_6 - c_2 = 0 \Rightarrow c_2 = 30c_6 \Rightarrow c_6 = 0$$
$$42c_7 - c_3 = 0 \Rightarrow c_3 = 42c_7 \Rightarrow c_7 = 0$$

Thus, in terms of arbitrary constants c_0, c_1 the general solution is

$$z = c_0 + c_1 x + \frac{1}{12}c_0 x^4 + \frac{1}{20}c_1 x^5 + \cdots$$

$$\therefore \boxed{z = c_0\left(1 + \frac{1}{12}x^4\right) + c_1\left(x + \frac{1}{20}x^5 + \cdots\right)}$$

Remark: The procedure here is as follows:

1. Assume sol^{n} has power series representation with unknown coefficients $c_0, c_1, c_2, \ldots$ [can use other letters obviously like $a_0, a_1, a_2, \ldots$]

2. Substitute sol^{n} $c_0 + c_1 x + c_2 x^2 + \cdots$ into given DEq^{n}

3. Collect terms of matching powers and set those coefficients to zero

4. Solve eq^{n}'s stemming from 3. You should find a way to keep just two coefficients unknown for a 2nd order DEq^{n} (for n^{th} order will get n-arbitrary constants in general sol^{n})

Sometimes it's possible to solve 4 for all coefficients, but the pattern might be hard to spot in general. Often we only need the first few terms.

$\boxed{\S 8.3 \# 17}$ Find 1st four terms in power series sol^n centered at zero for: $w'' - x^2 w' + w = 0$

Let
$$w = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$
$$w' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$$
$$w'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$$

Substitute into $w'' - x^2 w' + w = 0$ to find,

$$2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + $$
$$- x^2 c_1 - 2c_2 x^3 - 3c_3 x^4 - 4c_4 x^5 - 5c_5 x^6 + $$
$$+ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots = 0$$

Now collect terms with like-degree,

$$c_0 + 2c_2 + x(6c_3 + c_1) + x^2(12c_4 - c_1 + c_2) + x^3(20c_5 - 2c_2 + c_3) + $$
$$+ x^4(120c_6 - 3c_3 + c_4) + x^5(720c_7 - 4c_4 + c_5) + \cdots = 0$$

Each coefficient must vanish, this gives conditions on c_n:

x^0: $2c_2 = -c_0 \Rightarrow \underline{c_2 = -c_0/2}$.

x: $6c_3 = -c_1 \Rightarrow \underline{c_3 = -c_1/6}$

x^2: $12c_4 = c_1 - c_2 \Rightarrow c_4 = \frac{1}{12}\left[c_1 + \frac{c_0}{2}\right] = \underline{\frac{c_1}{12} + \frac{c_0}{24}}$.

x^3: $20c_5 = 2c_2 - c_3 \Rightarrow c_5 = \frac{1}{20}\left[2c_2 - c_3\right] = \frac{1}{20}\left[-c_0 + \frac{c_1}{6}\right] = \underline{\frac{-c_0}{20} + \frac{c_1}{120}}$.

x^4: $120c_6 = 3c_3 - c_4 \Rightarrow c_6 = \frac{1}{120}\left[3c_3 - c_4\right] = \frac{1}{120}\left[\frac{-c_1}{2} - \frac{c_1}{12} - \frac{c_0}{24}\right] = \frac{-7c_1}{1440} - \frac{c_0}{2880}$

Thus we find sol^n,

$$w = c_0 + c_1 x - (c_0/2)x^2 - (c_1/6)x^3 + (c_1/12 + c_0/24)x^4 + \left(\frac{-c_0}{20} + \frac{c_1}{120}\right)x^5 + \cdots$$

$$\boxed{\begin{aligned} w = {} & c_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{120}x^5 - \tfrac{1}{2880}x^6 + \cdots\right) \\ & + c_1\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \tfrac{1}{120}x^5 - \tfrac{7}{1440}x^6 + \cdots\right) \end{aligned}}$$

__Remark__: I computed a few more terms than I needed here, hopefully they're correct. I can only check the first two terms in each series.

§ 8.3 # 19 Find the complete power series sol$^{\underline{n}}$ for $y' - 2xy = 0$ <u>including</u> a general formula for the coefficients

Let $y = \sum_{n=0}^{\infty} c_n x^n$ then $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$

thus substitute into $y' - 2xy = 0$ to find,

$$\sum_{n=0}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

need to match series up so we have same type of index on the exponent of x.

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots - 2(c_0 x + c_1 x^2 + c_2 x^3 + \cdots) = 0$$

this suggests I want to single out first term in first sum since it falls outside pattern,

$$c_1 + \underbrace{\sum_{n=2}^{\infty} n c_n x^{n-1}} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

let $k+1 = n-1$
$k = n-2$ then
$n = 2 \longrightarrow k = 0$
and $n = k+2$

trying to match with $\sum_{n=0}^{\infty} c_n x^{n+1}$

$$c_1 + \sum_{k=0}^{\infty} (k+2) c_{k+2} x^{k+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\Rightarrow c_1 + \sum_{m=0}^{\infty} \left[(m+2)c_{m+2} - c_m\right] x^{m+1} = 0$$

We find $c_1 = 0$ and $c_m = (m+2)c_{m+2}$ for $m \geq 0$. Perhaps it's easier to use $c_j = \frac{1}{j} c_{j-2}$ for $j \geq 2$. Clearly all odd coefficients are zero; $c_3 = \frac{1}{3} c_1$, $c_5 = \frac{1}{5} c_3$, ... all even coefficients can be calculated iteratively from c_0,

$c_2 = \frac{1}{2} c_0$, $c_4 = \frac{1}{4} c_2 = \frac{1}{4 \cdot 2} c_0$, $c_6 = \frac{1}{6 \cdot 4 \cdot 2} c_0$, ... continued

We argued $c_{2k+1} = 0$ for $k \geq 0$. Examining the even coefficient eq$^{\underline{n}}$'s reveals the pattern below,

$$c_{2k} = \left(\frac{1}{(2k)(2(k-1))(2(k-2))\cdots(2(2))2(1)} \right) c_o$$
$$= \left(\frac{1}{2^k \; k(k-1)\cdots 2\cdot 1} \right) c_o$$
$$= \frac{c_o}{2^k k!} \quad \text{for } k \geq 0.$$

We find,

$$y = \sum_{k=0}^{\infty} c_{2k} x^{2k}$$
$$= \sum_{k=0}^{\infty} \frac{c_o}{2^k k!} x^{2k}$$
$$= c_o \left(\sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} \right)$$
$$\therefore \boxed{y = c_o \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}}$$

Sometimes $\underline{y = c_o \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots \right)}$ is more useful to a particular application, but technically to completely solve the problem we need to determine <u>all</u> coefficients (upto n arbitrary coefficients for $n^{\underline{th}}$ order ODE)

<u>Remark</u>: I don't usually explain quite as much steps as in this sol$^{\underline{n}}$. I'm trying to explain a few tricks that I later use w/o apology.

§8.3 #21) Find general sol$^{\underline{n}}$ for $y'' - xy' + 4y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \quad , \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute into $y'' - xy' + 4y = 0$ and pull x into sum,

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\substack{\text{let } k = n-2 \\ n=2 \Rightarrow k=0 \\ \text{and } n = k+2}} - \underbrace{\sum_{n=0}^{\infty} n c_n x^n}_{\substack{\text{put zero} \\ \text{back in to} \\ \text{match other} \\ \text{sums.}}} + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1) c_{k+2} - k c_k + 4 c_k \Big] x^k = 0$$

The coefficient of x^k must vanish for each $k = 0, 1, 2, \dots$

$x^0)$ $2c_2 + 4c_0 = 0 \quad \therefore \quad c_2 = -2c_0$

$x^1)$ $6c_3 + 3c_1 = 0 \quad \therefore \quad c_3 = -\frac{1}{2} c_1$

$x^2)$ $4(3)c_4 + 2c_2 = 0 \quad \therefore \quad c_4 = \frac{-1}{6} c_2 = \frac{1}{3} c_0$

$x^3)$ $5(4)c_5 + c_3 = 0 \quad \therefore \quad c_5 = \frac{-1}{20} c_3 = \frac{1}{40} c_1$

$x^4)$ $6(5)c_6 = 0 \quad \therefore \quad c_6 = 0 \leftarrow$ neat.

$x^5)$ $7(6)c_7 - c_5 = 0 \quad \therefore \quad c_7 = \frac{1}{42} c_5 = \frac{1}{40(42)} c_1$

$x^6)$ $8(7)c_8 + 2c_6 = 0 \quad \therefore \quad c_8 = \frac{-2}{56} c_6 = 0.$

It's clear $c_{2k} = 0$ for $k = 3, 4, 5, 6, \dots$ Only $c_0, c_2, c_4 \neq 0$ for even powers. The odd coefficients are nontrivial for ∞ly many terms, let $k = 2j+1$ then

$$(2j+1+2)(2j+2) c_{2j+1+2} - (4 - 2j - 1) c_{2j+1} = 0$$

$$c_{2j+3} = \left[\frac{2j-3}{(2j+3)(2j+2)} \right] c_{2j+1} \quad \text{for } j = 0, 1, 2, \dots$$

$$C_{2j+3} = \left[\frac{2j-3}{(2j+3)\,2(j+1)}\right] C_{2j+1} \quad \text{for } j = 0, 1, 2, \ldots$$

$j=0$, $\quad C_3 = \frac{-3}{3(2)} C_1$

$j=1$, $\quad C_5 = \frac{-1}{5\cdot 4} C_3 = \frac{(-1)(-3)}{5\cdot 4\cdot 3\cdot 2} C_1$

$j=2$, $\quad C_7 = \frac{1}{7\cdot 6} C_5 = \frac{(1)(-1)(-3)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} C_1$

$j=3$, $\quad C_9 = \frac{3}{9\cdot 8} C_7 = \frac{3(1)(-1)(-3)}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} C_1$

$j=4$, $\quad C_{11} = \frac{5}{11\cdot 10} C_9 = \frac{5(3)(1)(-1)(-3)}{11\cdot 10\cdots 3\cdot 2\cdot 1} C_1$

I think the pattern is clear now, notice C_1 stands alone,

$$y = c_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^4\right) + ?$$
$$+ c_1\left(x + \sum_{j=0}^{\infty} \frac{(-3)(-1)(1)(3)\cdots(2j-3)}{(2j+3)!} x^{2j+3}\right)$$
$$= c_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^4 + \cdots\right) + c_1\left(x + \frac{(-3)}{3!}x^3 + \frac{(-3)(-1)}{5!}x^5 + \frac{(-3)(-1)(1)}{7!}x^7 + \cdots\right.$$

(this is equivalent to the text's answer-key)

§8.3 # 27 Find first four nontrivial terms in series solution centered at $x_0 = 0$ for $(x+1)y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$

Let $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$

$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + \cdots$

Substitute into $(x+1)y'' - y = 0$,

$$(x+1)(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots) - c_0 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4 = 0$$

$$2c_2 x + 6c_3 x^2 + 12c_4 x^3 + 20c_5 x^4 + 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$$
$$- c_0 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4 + \cdots = 0$$

$$2c_2 - c_0 + x(2c_2 + 6c_3 - c_1) + x^2(6c_3 + 12c_4 - c_2) + x^3(12c_4 + 20c_5 - c_3) = 0$$

Consequently,

$2c_2 - c_0 = 0 \Rightarrow c_2 = c_0/2$, since $y(0) = \underline{c_0 = 0} \Rightarrow \underline{c_2 = 0}$.

$2c_2 + 6c_3 - c_1 = 0 \Rightarrow c_3 = c_1/6$, since $y'(0) = \underline{c_1 = 1} \Rightarrow \underline{c_3 = 1/6}$.

$6c_3 + 12c_4 - c_2 = 0 \Rightarrow \underline{c_4 = -c_3/2 = -1/12}$.

$12c_4 + 20c_5 - c_3 = 0 \Rightarrow 20c_5 = -12\left(\frac{-1}{12}\right) + \frac{1}{6} = \frac{7}{6} \Rightarrow \underline{c_5 = \frac{7}{120}}$.

Fortunately we found 4 nonzero terms. It can happen that we need to keep terms of much higher order because more low order coefficients are trivial. Anyhow,

$$\boxed{y = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{120}x^5 + \cdots}$$

It's interesting we only used up through coefficients of x^3 and yet we found all the way up to the coefficient of x^5 in y itself.