LECTURE 11 : ON FRAMES AND CONGRUENCE IN $\mathbb{R}^3$

$p \bullet$ $\qquad E_1, E_2, E_3$ at p $\qquad \xrightarrow{\Phi}$ $\qquad q \bullet$ $\qquad \underbrace{F_1, F_2, F_3}_{\text{have attitude } B}$

$\Phi_* = R$

$\Phi(p) = q$ & $\Phi_*(E_i) = F_i$

$$R[E_1] = [F_1]$$
$$R[E_2] = [F_2]$$
$$R[E_3] = [F_3]$$

Recall $E_i = A_{i1} U_1 + A_{i2} U_2 + A_{i3} U_3$

$[E_i] = [A_{i1}, A_{i2}, A_{i3}]^T$

$$R\, row_1(A)^T = row_1(B)^T$$
$$R\, row_2(A)^T = row_2(B)^T$$
$$R\, row_3(A)^T = row_3(B)^T$$

$$R\left[col_1(A^T) \,|\, col_2(A^T) \,|\, col_3(A^T)\right] = B^T$$

$$RA^T = B^T \qquad A^T A = I$$

$$\underline{R = B^T A}$$

Th^m/ Let $E_1, E_2, E_3 \in T_p\mathbb{R}^3$ with attitude A and $F_1, F_2, F_3 \in T_q\mathbb{R}^3$ with attitude B then $\exists\, \Phi : \mathbb{R}^3 \to \mathbb{R}^3$ for which $\Phi(p) = q$ and $\Phi_*(E_j) = F_j$ for $j = 1, 2, 3$.

<u>Proof</u>: Let $R = B^TA$ and define $\Phi(x) = Rx + q - Rp$

Note $\Phi(p) = Rp + q - Rp = q$. Also, $[\Phi_*(E_j)] = R[E_j] = [F_j]$.

- see details on pg. ≈ 68 (§3.3 of my notes)
- also Example 3.3.2 is nice

Th^m/(3.3.3) $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $V, W \in \mathfrak{X}(\mathbb{R}^3)$

$$F_*(V) \cdot F_*(W) = V \cdot W$$

$$F_*(V) \times F_*(W) = \det(F_*)\, F_*(V \times W)$$

<u>Proof</u>

$$\begin{aligned} F_*(V) \cdot F_*(W) &= (RV) \cdot (RW) \\ &= (RV)^T RW \\ &= V^T \underbrace{\cancel{R^T R}}_{I} W \\ &= V^T W = V \cdot W. \end{aligned}$$

Proof continued

$$F_*(V) \times F_*(W) = (RV) \times (RW)$$

ignore $\left(R^T R = I \text{ and } F_*(v_1),\ F_*(v_2),\ F_*(v_3) \text{ forms a frame for } \mathbb{R}^3 \right.$

$\left. \text{note } F_*(v_j) = R v_j \right)$

$$\begin{aligned}
(F_*(V) \times F_*(W)) \cdot v_j &= (RV \times RW) \cdot v_j \\
&= \det [RV \mid RW \mid v_j] \qquad R^T R = I \\
&= \det [RV \mid RW \mid R R^T v_j] \\
&= \det R [V \mid W \mid R^T v_j] \\
&= \det R \det [V \mid W \mid R^T v_j] \\
&= (\det F_*)\, (V \times W) \cdot (R^T v_j) \qquad (Mx) \cdot y = x \cdot (M^T y) \\
&= (\det F_*)\, R(V \times W) \cdot v_j \\
&= \det F_* \,(F_*(V \times W)) \cdot v_j \quad \text{for } j = 1, 2, 3.
\end{aligned}$$

Thus $F_*(V) \times F_*(W) = (\det F_*)\, F_*(V \times W)$. //

Th^m 3.3.4 / FRENET APPARATUS OF ISOMETRIC IMAGE

Let α be a nonlinear, arclength parametrized curve with Frenet frame T, N, B and curvature κ, torsion τ. Suppose $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an <u>isometry</u> and <u>define</u> $\bar{\alpha} = F \circ \alpha$. If $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}$ form the Frenet frame and curvature & torsion of $\bar{\alpha}$ then

$$\bar{T} = F_*(T), \quad \bar{N} = F_*(N), \quad \bar{B} = \det(F_*)F_*(B), \quad \bar{\kappa} = \kappa, \quad \bar{\tau} = \tau(\det F_*)$$

<u>Proof</u> $T(s) = \alpha'(s)$ & $\bar{T}(s) = \bar{\alpha}'(s) = (F \circ \alpha)'(s) = F_*(\alpha'(s)) = F_*(T(s))$

Hence $F_*(T) = \bar{T}$. Next consider the Frenet normals

$$\kappa = \|\alpha''\| \quad \text{and} \quad \bar{\kappa} = \|\bar{\alpha}''\| = \|(F \circ \alpha)''\| = \|F_*(\alpha'')\| = \|\alpha''\| = \kappa.$$

$$N = \frac{1}{\kappa}\alpha'' \quad \text{and} \quad \bar{N} = \frac{1}{\bar{\kappa}}\bar{\alpha}'' = \frac{1}{\kappa}(F \circ \alpha)'' = \frac{1}{\kappa}F_*(\alpha'') = F_*\left(\frac{\alpha''}{\kappa}\right) = F_*(N)$$

Thus $\bar{N} = F_*(N)$. Next consider $B = T \times N$ & $\bar{B} = \bar{T} \times \bar{N}$

$$\begin{aligned} \bar{B} = \bar{T} \times \bar{N} &= F_*(T) \times F_*(N) \\ &= \det(F_*)\, F_*(T \times N) \\ &= \det(F_*)\, F_*(B). \end{aligned}$$

Finally $\tau = -B' \cdot N$ and $\bar{\tau} = -\bar{B}' \cdot \bar{N}$

$$-\tau = B' \cdot N = F_*(B') \cdot F_*(N) = (F_*(B))' \cdot F_*(N) = \det(F_*)\overset{-\bar{\tau}}{(\bar{B}' \cdot \bar{N})} = \det F_*\, \bar{\tau}(-1)$$

Ex) $\alpha(s) = (R\cos \gamma s,\ R\sin \gamma s,\ m\gamma s)$ $\quad \gamma = \dfrac{1}{\sqrt{R^2+m^2}}$

$F(x,y,z) = (y, x, z)$ $\quad \theta = \gamma s$

$\bar{\alpha}(s) = (F\circ\alpha)(s) = (R\sin\theta,\ R\cos\theta,\ m\theta)$ $\quad \dfrac{d\theta}{ds} = \gamma.$

$$\bar{T} = \bar{\alpha}'(s) = R\gamma\cos\theta\, U_1 - R\gamma\sin\theta\, U_2 + m\gamma\, U_3$$

$$\|\bar{T}\| = R^2\gamma^2 + m^2\gamma^2 = \frac{R^2+m^2}{R^2+m^2} = 1.$$

$$\bar{\alpha}''(s) = -R\gamma^2\sin\theta\, U_1 - R\gamma^2\cos\theta\, U_2$$

$$\|\bar{\alpha}''(s)\| = R\gamma^2 = \frac{R}{R^2+m^2} = \boxed{\bar{\kappa}}$$

$\bar{N} = -\sin\theta\, U_1 - \cos\theta\, U_2$

Likewise, $\rightarrow$ $\bar{B} = \dfrac{1}{\sqrt{R^2+m^2}}\left[m\cos\theta\, U_1 - m\sin\theta\, U_2 - R\, U_3\right]$

$$\bar{B}' = -m\gamma^2\sin\theta\, U_1 - m\gamma^2\cos\theta\, U_2$$

$$\bar{\tau} = -\bar{B}'\cdot\bar{N} = -m\gamma^2 = \frac{-m}{R^2+m^2} = -\tau_\alpha$$

$$\det(F_*) = \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

Defn/ congruence of parametrized curves

We say $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ are congruent if there exits an isometry F for which $\beta = F \circ \alpha$.

Example/
$$\beta(t) = \Big(c_1 R\cos t + c_2 R\sin t + c_3 mt - 1 + 3/\sqrt{6},$$
$$c_3 R\cos t + c_3 R\sin t - 2c_3 mt - 2,$$
$$c_2 R\cos t + c_1 R\sin t + c_3 mt - 3 - 3/\sqrt{6}\Big)$$

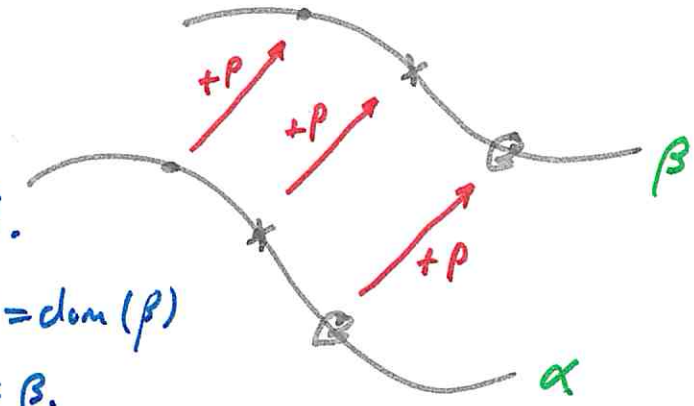
where $c_1 = 1/\sqrt{6} + 1/2$ and $c_2 = 1/\sqrt{6} - 1/2$ and $c_3 = 1/\sqrt{6}$

This is the isometric image of standard helix with radius R and slope m.

Helix: $\alpha(t) = (R\cos t,\ R\sin t,\ mt)$

Def^n/ parallel curves. We say α & $\beta : I \to \mathbb{R}^3$ are $\parallel$ if $\exists\, p \in \mathbb{R}^3$ for which $\beta(t) = \alpha(t) + p \;\; \forall\, t \in I$

Prop 3.3.9 Parametrized curves α & β are $\parallel$ iff $\alpha'(t) = \beta'(t)$ for all $t \in I$. Moreover, if $\alpha \parallel \beta$ and $\exists\, t_o \in \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ for which $\alpha(t_o) = \beta(t_o)$ then $\alpha = \beta$.



Proof $\beta(t) = \alpha(t) + p$ (assume $\alpha \parallel \beta$) $(I = \mathrm{dom}\,\alpha = \mathrm{dom}\,\beta)$

then $\beta'(t) = \alpha'(t)$ for all $t \in I$.

Conversely, if $\alpha'(t) = \beta'(t)$ for all $t \in I$

$$\sum_{j=1}^{3} \frac{d\alpha^j}{dt} U_j = \sum_{j=1}^{3} \frac{d\beta^j}{dt} U_j \qquad \frac{d\alpha^j}{dt} = \frac{d\beta^j}{dt}$$

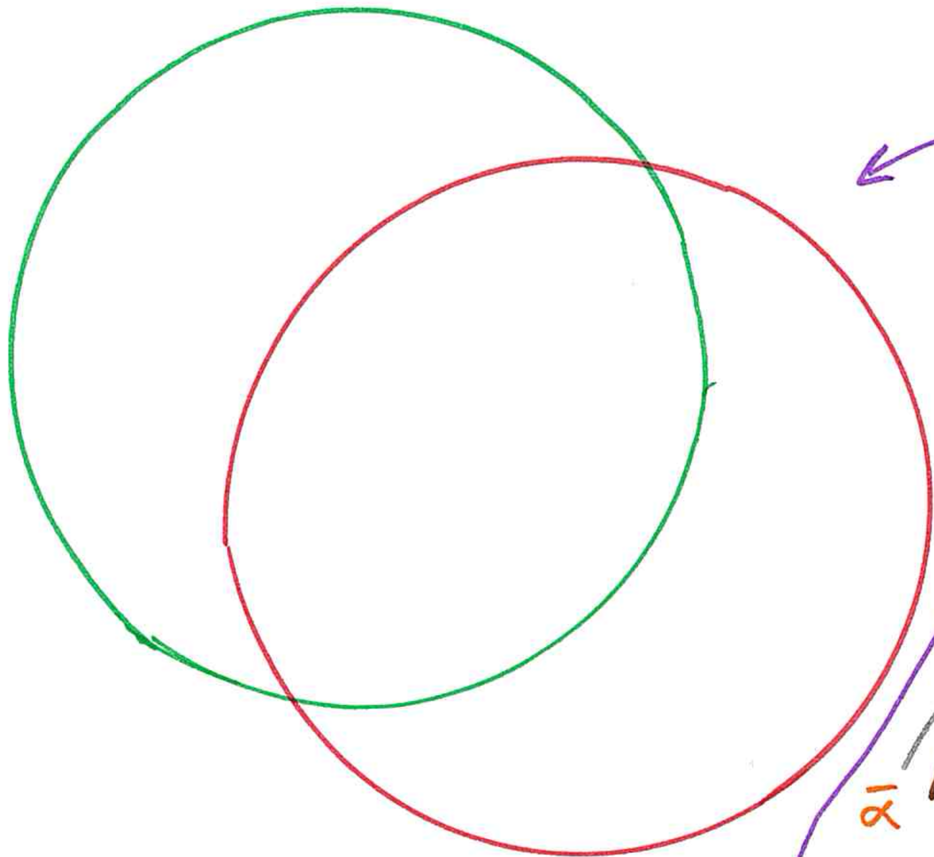
for $j = 1, 2, 3$

Hence $\alpha^j(t) = \beta^j(t) + p^j \longrightarrow \alpha(t) = \beta(t) + p$.

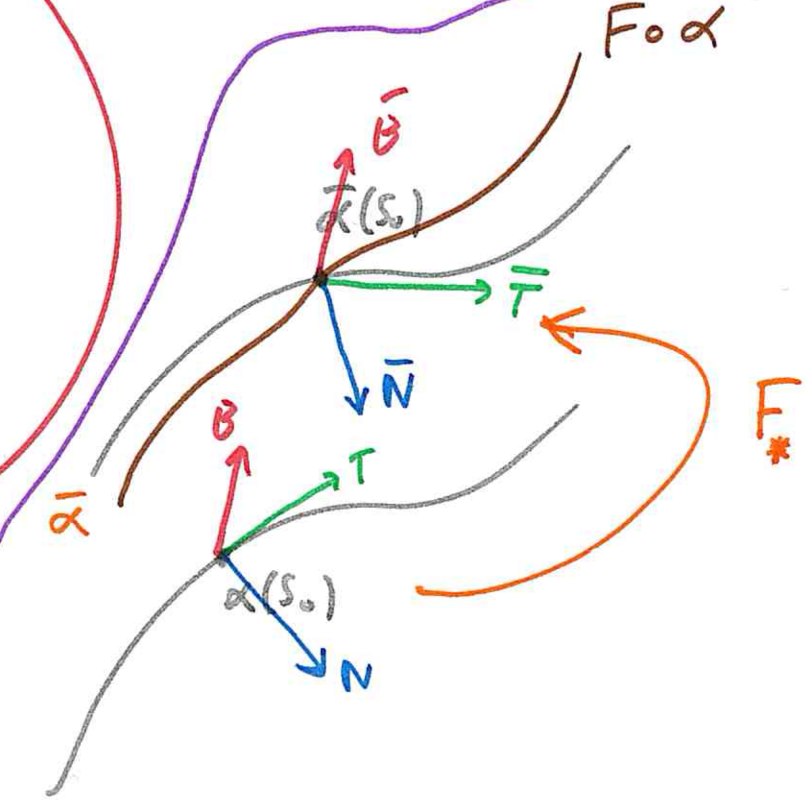
If $\alpha(t_o) = \beta(t_o) \Rightarrow \alpha(t_o) = \beta(t_o) + p = \beta(t_o) \;\therefore\; p = 0$

$\Rightarrow \alpha(t) = \beta(t) \;\forall t.$ //

Ex



these parallel circles intersect.



(proved Th$^{\underline{m}}$ 3.3.11 from pg. 72–73 & discussed Prop. 3.3.12 on 73 and Th$^{\underline{m}}$ 3.3.13 briefly)