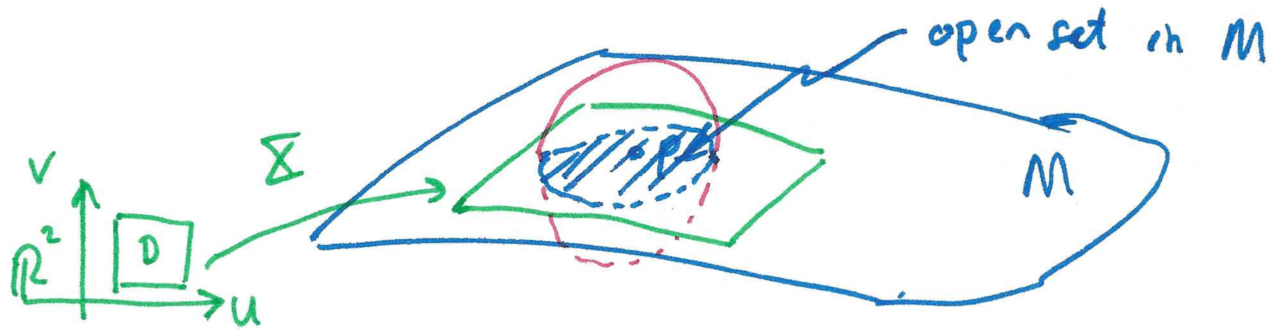
LECTURE 12 : SURFACES IN $\mathbb{R}^3$ (§4.1–4.3 of O'neill)

Def^n/ $\underbrace{\mathbf{X} : \underset{\substack{\uparrow \\ \text{open}}}{D} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3}_{\text{proper coordinate patch}}$

- $\mathbf{X}$ 1-1 (injective)
- $\mathbf{X}$ smooth
- $\mathbf{X}$ regular $\quad J_{\mathbf{X}}(p)$ has rank 2 at each $p \in D$
- $\mathbf{X}^{-1} : \mathbf{X}(D) \longrightarrow D$ is continuous

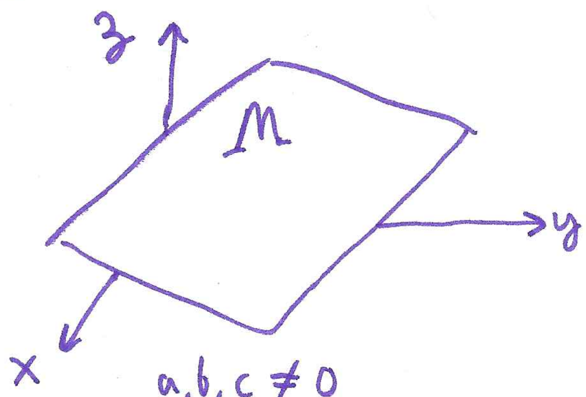
Consider $M \subseteq \mathbb{R}^3$ we say $\mathcal{U} \subseteq M$ is an open set if $\exists$ an open set $\tilde{\mathcal{U}} \subseteq \mathbb{R}^3$ such that $\mathcal{U} = \tilde{\mathcal{U}} \cap M$.
It suffices to use $\tilde{\mathcal{U}} = B_\epsilon(p) = \{ x \in \mathbb{R}^3 \mid d(x,p) < \epsilon \}$



Def^n/ M is surface if $M \subseteq \mathbb{R}^3$ and for each $p \in M$ $\exists$ an open set $\mathcal{U} \subseteq M$, containing p, and a proper patch $\mathbf{X} : D \to \mathbf{X}(D)$ for which $\mathcal{U} \subseteq \mathbf{X}(D)$. We say M is simple surface if $M = \mathbf{X}(D)$.

EXAMPLES OF SURFACES

$\boxed{E1}$ $ax+by+cz=d$



$a,b,c \neq 0$

$$X(x,y) = \left(x,\ y,\ \frac{d-ax-by}{c}\right)$$

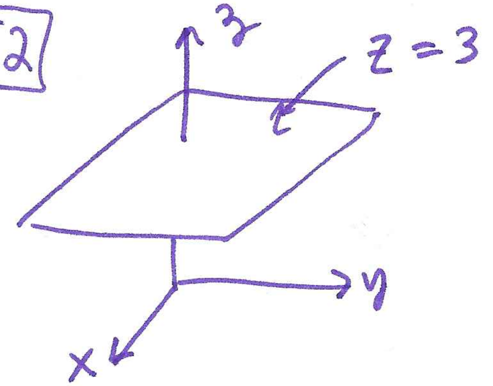
$$X(u,v) = \left(u,\ v,\ \tfrac{1}{c}(d-au-bv)\right)$$

$$X^{-1}(p^1,p^2,p^3) = (p^1,p^2)$$

$$Y(u,v) = \left(u,\ \tfrac{1}{b}(d-au-cv),\ v\right)$$

$$Z(u,v) = \left(\tfrac{1}{a}(d-bu-cv),\ u,\ v\right)$$

$\boxed{E2}$



$$X(u,v) = (u,v,3)$$

$$J_X = \left[\frac{\partial X}{\partial u}\,\middle|\,\frac{\partial X}{\partial v}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

assume open

$\boxed{E3}$ <u>graph</u> $\text{graph}(f) = \{(x,y,f(x,y)) \mid (x,y) \in \text{dom}(f)\}$

is simple surface with Monge patch $\underline{X(u,v) = (u,v,f(u,v))}$.

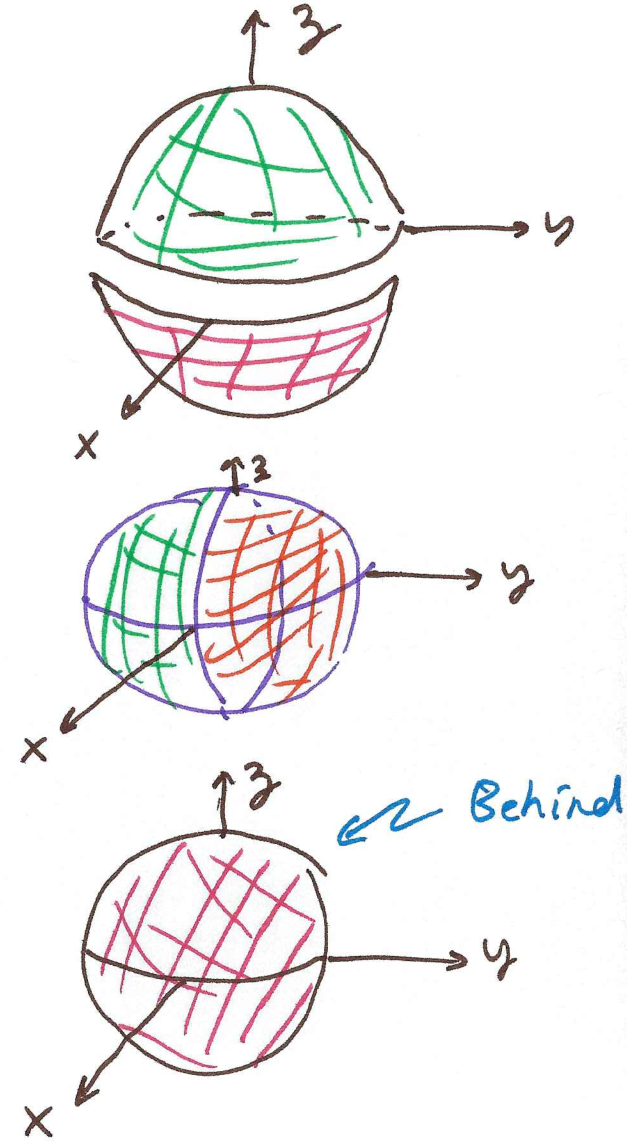
$$J_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{bmatrix}$$ rank 2 everywhere.

$\boxed{E4}$ $\Sigma : x^2+y^2+z^2=1$

$$\mathbb{X}(u,v) = \left(u, v, \pm\sqrt{1-u^2-v^2}\right)$$

$$\mathbb{Y}(u,v) = \left(u, \pm\sqrt{1-u^2-v^2}, v\right)$$

$$\mathbb{Z}(u,v) = \left(\pm\sqrt{1-u^2-v^2}, u, v\right)$$



Th$^{\underline{m}}$/ Let g be $g: \mathrm{dom}(g) \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ then $g^{-1}\{c\} = \{(x,y,z) \mid g(x,y,z) = c\} = M \neq \emptyset$ is a surface if $d_p g \neq 0 \ \forall p \in M$

<u>Proof</u>: $\S$ $M \neq \emptyset$ and $p \in M$ and $d_p g = \underline{\frac{\partial g}{\partial x}} dx + \underline{\frac{\partial g}{\partial y}} dy + \underline{\frac{\partial g}{\partial z}} dz \neq 0$

wlog, $\S$ $\frac{\partial g}{\partial z} \neq 0$ at p. Then the implicit fnct theorem tells us $\exists$ h for which $g(x,y,h(x,y)) = c$ for all (x,y) near enough to (p^1, p^2). Thus $\Phi(u,v) = (u,v,h(u,v))$. Then p was arbitrary $\Rightarrow \exists$ a proper patch at each $p \in M$.

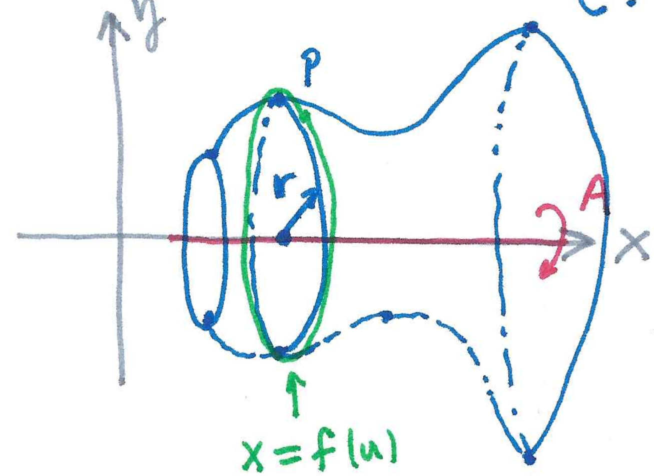
(e.g. $\Phi(u,v) = (u, h(u,v), v)$ good for p with $\frac{\partial g}{\partial y}(p) \neq 0$)

<u>$\widetilde{E4}$</u> $g(x,y,z) = x^2+y^2+z^2 = 1$

$dg = 2x\,dx + 2y\,dy + 2z\,dz \neq 0$

SURFACES OF REVOLUTION

E5



$C: (f(u), g(u), 0)$

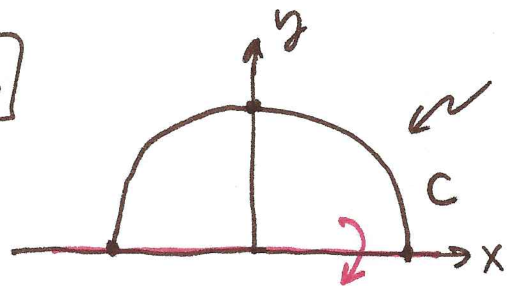
$r = g(u)$

$\gamma(v) = (f(u), g(u)\cos(v), g(u)\sin(v))$

$0 \leq v \leq 2\pi$

$$\Phi(u,v) = (f(u), g(u)\cos(v), g(u)\sin(v))$$

E6



$(\cos(u), \sin(u), 0) \quad 0 \leq u \leq \pi$

$$\Phi(u,v) = (\cos(u), \sin(u)\cos(v), \sin(u)\sin(v))$$

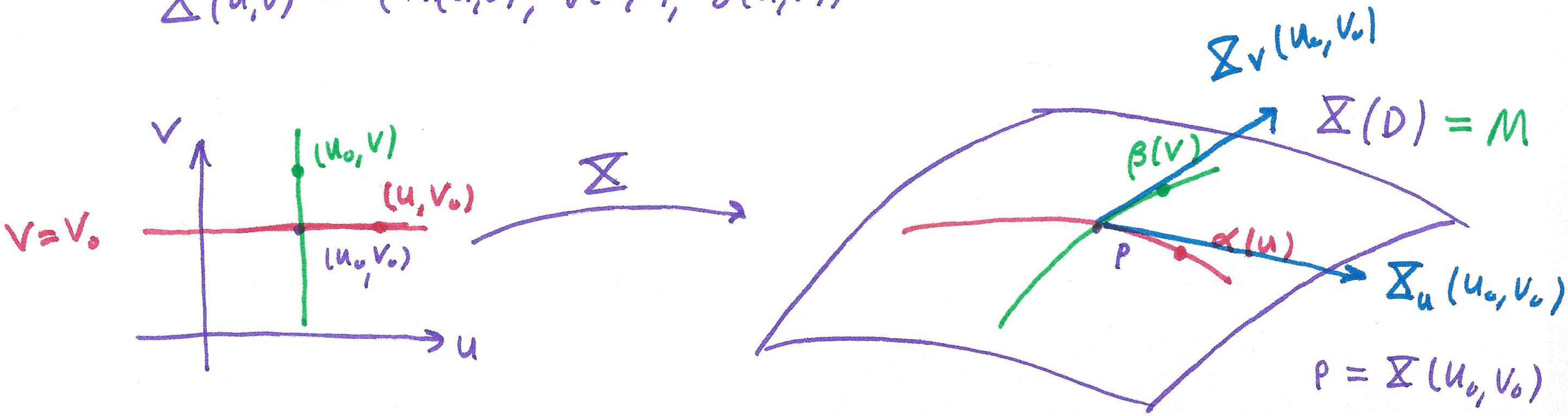
$0 \leq u < \pi$
$0 \leq v < 2\pi$

throw in = gives a "parametrization"

$0 \leq u \leq \pi$
$0 \leq v \leq 2\pi$

CALCULUS ON M:

$$\mathbf{X}(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$$



$$\alpha(u) = \mathbf{X}(u, v_0)$$
$$\beta(v) = \mathbf{X}(u_0, v)$$

$$\mathbf{X}_u(u_0,v_0) = \alpha'(u_0) = \frac{d}{dt}\Big[\mathbf{X}(t,v_0)\Big]\Big|_{t=u_0}$$

$$\mathbf{X}_v(u_0,v_0) = \beta'(v_0) = \underbrace{\frac{d}{dt}\Big[\mathbf{X}(u_0,t)\Big]\Big|_{t=v_0}}_{\text{understood as } \underline{\text{vectors}} \text{ in } T_p\mathbb{R}^3}$$

$$\mathbf{X}_u(u_0,v_0) = \underbrace{\frac{d}{dt}\Big|_{t=u_0}\Big[x(t,v_0)\Big]}\, U_1(p) + \frac{d}{dt}\Big|_{t=u_0}\big(y(t,v_0)\big)\, U_2(p) + \frac{d}{dt}\Big|_{t=u_0}\big(z(t,v_0)\big)\, U_3(p)$$

$$U_i(p) = \frac{\partial}{\partial x^i}\Big|_p$$

$$\frac{d}{dt}\Big|_{t=u_0}\Big[x(t,v_0)\Big] = \frac{\partial x}{\partial u}\underbrace{\frac{du}{dt}}_{1} + \frac{\partial x}{\partial v}\underbrace{\frac{dv}{dt}}_{0} = \frac{\partial x}{\partial u}(u_0,v_0)$$

$u = t$
$v = v_0$

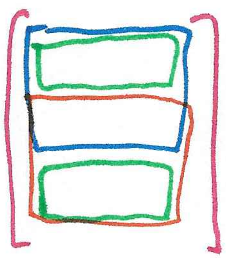
$$\Sigma_u = \frac{\partial x}{\partial u} U_1 + \frac{\partial y}{\partial u} U_2 + \frac{\partial z}{\partial u} U_3$$

$$\Sigma_v = \frac{\partial x}{\partial v} U_2 + \frac{\partial y}{\partial v} U_2 + \frac{\partial z}{\partial v} U_3$$

Def'n/ A regular mapping $\Sigma : D \longrightarrow \mathbb{R}^3$ with $\Sigma(D) \subseteq M$ is a parametrization of the region $\Sigma(D)$ in M.

REGULAR : $\operatorname{rank}(J_\Sigma) = 2 \iff J_\Sigma = \left[\begin{array}{c|c} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{array}\right] = \underbrace{\left[\begin{array}{c|c} \vec{\Sigma}_u & \vec{\Sigma}_v \end{array}\right]}_{\text{LI gives rank 2.}}$

3x2



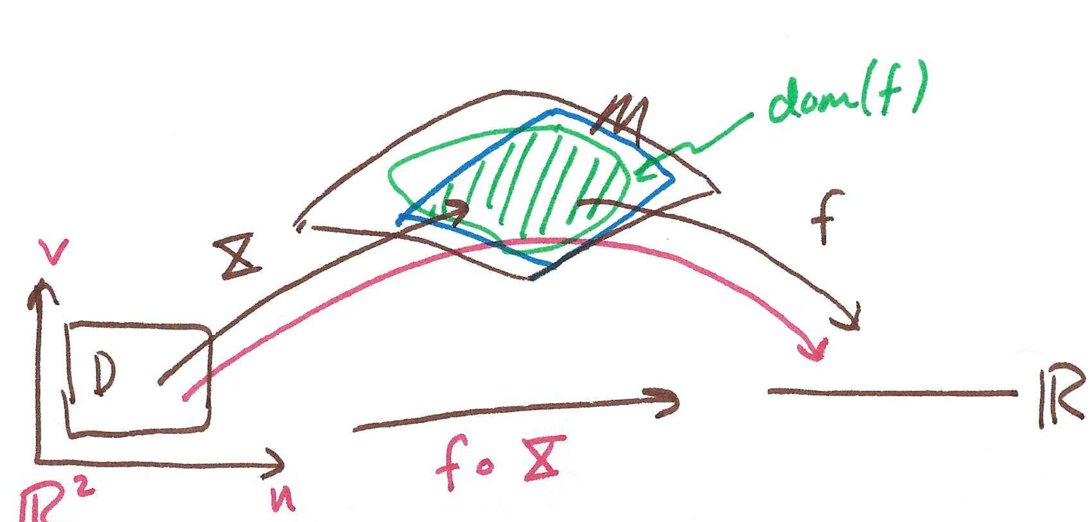
$\underbrace{\Sigma_u \times \Sigma_v \neq 0}_{} \Rightarrow \Sigma_u \nparallel \Sigma_v$

$\hookrightarrow \Sigma$ regular

DERIVATIVES on M

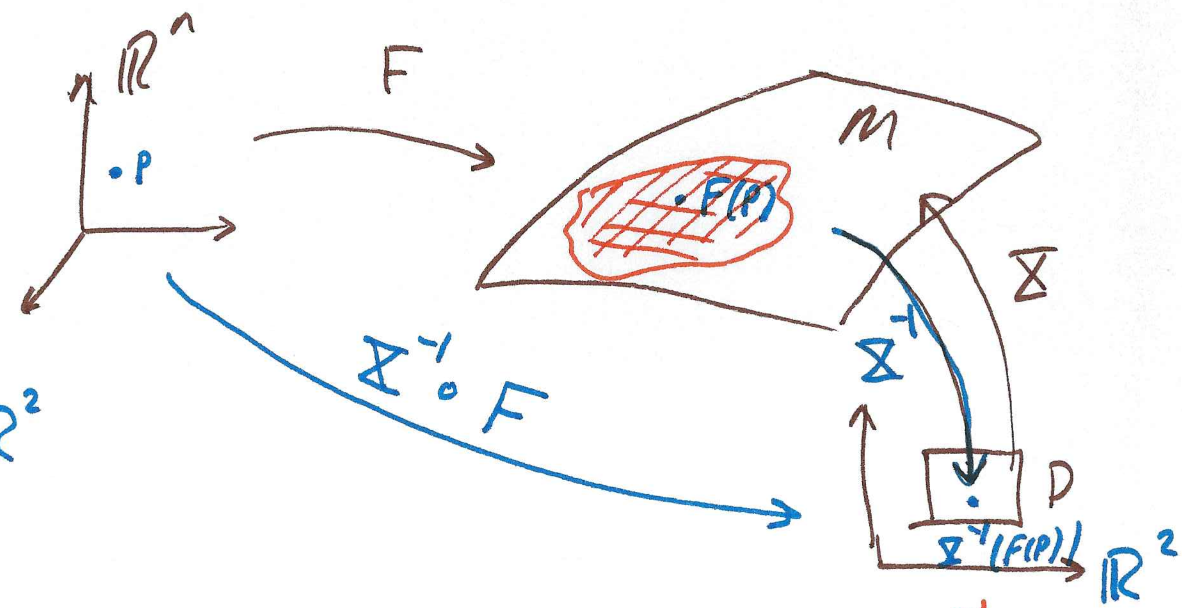
$f : M \longrightarrow \mathbb{R}$

is diff. on M provided the local coord. rep $f \circ \mathbf{X}$ are diff in the usual sense for each patch on M.



$F : \mathbb{R}^n \longrightarrow M$

is diff. provided $\mathbf{X}^{-1} \circ F$ is diff. fnct. from appropriate open sets in $\mathbb{R}^n \longrightarrow \mathbb{R}^2$



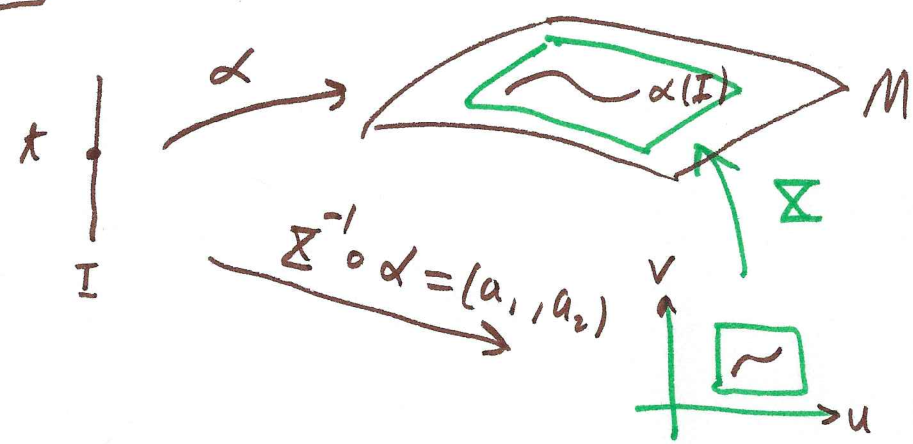
$\underbrace{\alpha : \mathbb{R} \longrightarrow M}_{\text{smooth curve}}$ if $\mathbf{X}^{-1} \circ \alpha$ is smooth.

Btw $\mathbf{X}^{-1}$ is called a **CHART**

Lemma (3.1) If $\alpha : I \to M$ and $\alpha(I) \subset \mathbf{X}(D)$ then $\exists!$ diff. fncts a_1 & a_2 on I s.t. $\alpha(t) = \mathbf{X}(a_1(t), a_2(t))$

$$\alpha = \mathbf{X}(a_1, a_2)$$

Proof/ If $\alpha : I \to M$ smooth



a_1, a_2 coordinate fncts. of the curve α w.r.t. the patch $\mathbf{X}$

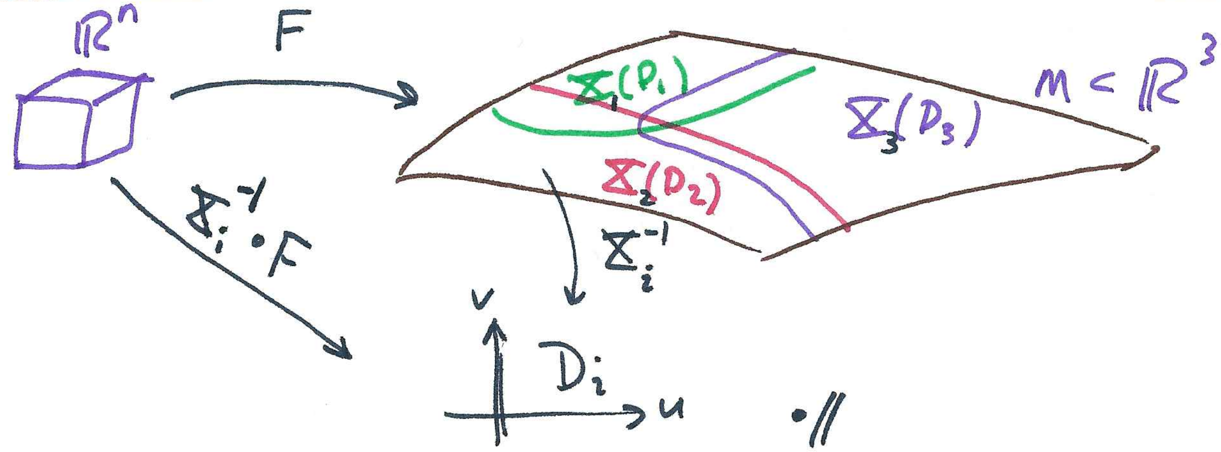
If $\mathbf{X}^{-1} \circ \alpha = (a_1, a_2)$

$\mathbf{X}(\mathbf{X}^{-1} \circ \alpha) = \mathbf{X}(a_1, a_2)$

$\alpha = \mathbf{X}(a_1, a_2)$ //

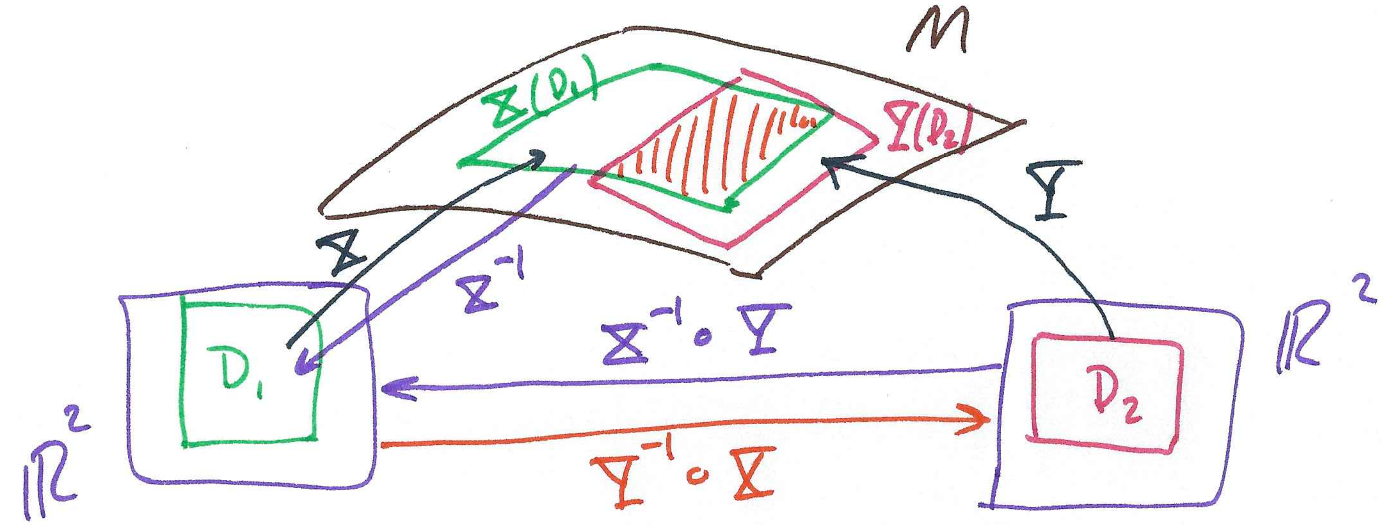
Thm/ $M \subset \mathbb{R}^3$, If $F : \mathbb{R}^n \to \mathbb{R}^3$ is a diff. map whose image $F(\mathbb{R}^n) \subseteq M$ then F is diff. (w.r.t. M)

Proof Sketch:



//

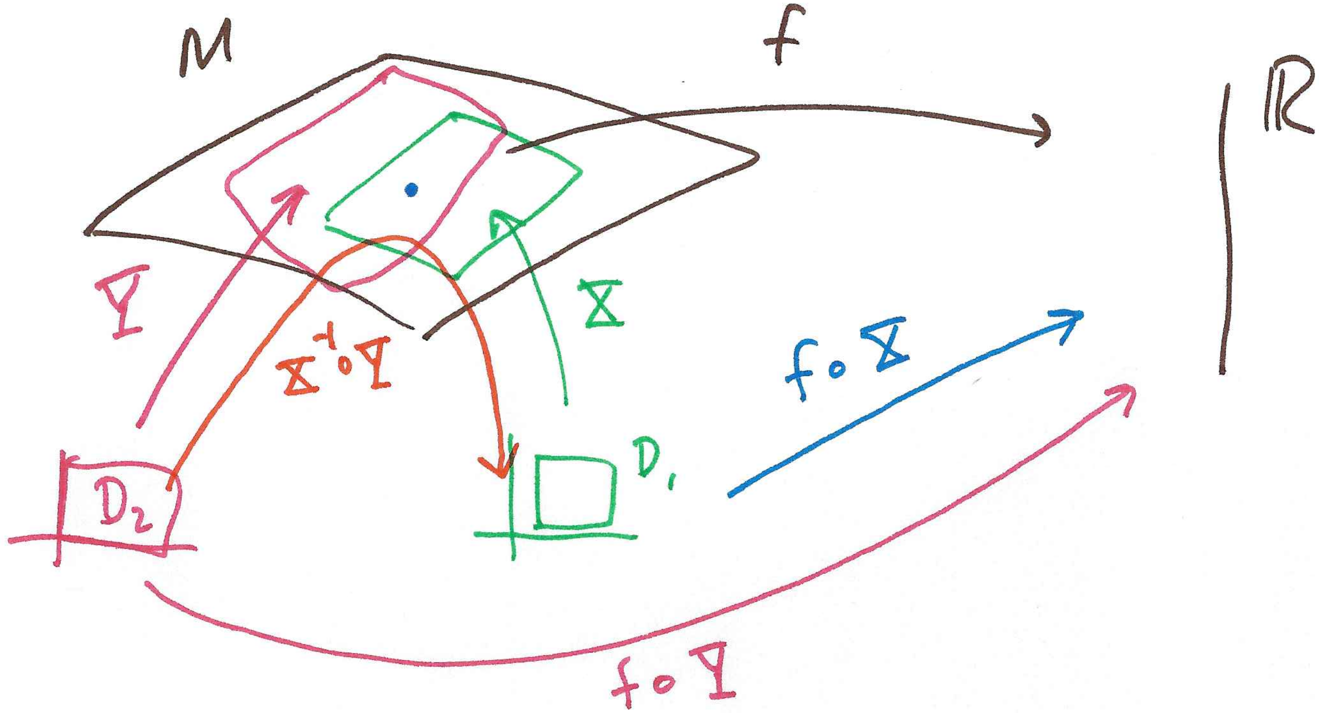
Cor (3.3) If $\mathbf{x}, \mathbf{y}$ are overlapping patches on M (meaning $\mathbf{x}(D_1) \cap \mathbf{y}(D_2) \neq \phi$) then $\mathbf{x}^{-1} \circ \mathbf{y}$ & $\mathbf{y}^{-1} \circ \mathbf{x}$ are smooth maps on $\mathbb{R}^2$.



Cor (3.4) $\mathbf{y}(u,v) = \mathbf{x}(\bar{u}(u,v), \bar{v}(u,v))$ for diff fncts $\bar{u}$ & $\bar{v}$ from $\mathbb{R}^2 \to \mathbb{R}$. (for all $(u,v) \in \mathrm{dom}(\mathbf{x}^{-1} \circ \mathbf{y})$)

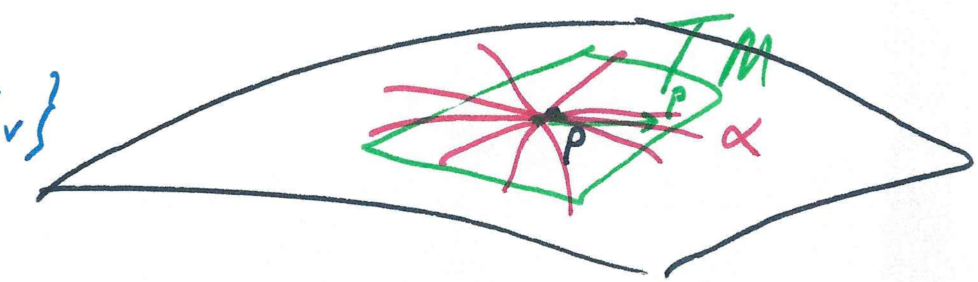
Prop $f(\mathbf{x})$ smooth iff $f(\mathbf{y})$ smooth for all overlapping patches $\mathbf{y}$ w.r.t. $\mathbf{x}$

Proof



$$f \circ Y = \underbrace{(f \circ X)}_{\text{smooth}} \circ \underbrace{(X^{-1} \circ Y)}_{\text{smooth}} \Longrightarrow \underline{f \circ Y \text{ smooth}}. \,//$$

Def^n/ $(P,V) \in T_pM$ iff $\exists \alpha : I \longrightarrow M$ and $\alpha'(t_0) = (P,V)$ for $P = \alpha(t_0)$. That is $T_pM \subset T_p\mathbb{R}^3$ formed by all possible tangents to curves on M



Lemma (3.6)/ $T_pM = \text{span}\{\mathbf{X}_u, \mathbf{X}_v\}$ where $\mathbf{X}_u(u_0,v_0)$, $\mathbf{X}_v(u_0,v_0)$ and $\mathbf{X}(u_0,v_0) = P$

Proof: $\alpha = \mathbf{X}(a_1, a_2)$ $\quad \alpha$ some curve on M, $\alpha(0) = P$.

$\alpha'(0) = ?$ $\quad \alpha(t) = (x(a_1(t), a_2(t)), y(a_1(t), a_2(t)), z(a_1(t), a_2(t)))$

$$\alpha'(t) = \sum_{i=1}^{3} \frac{dx^i(a_1(t), a_2(t))}{dt} U_i(\alpha(t))$$

$$= \sum_{i=1}^{3} \left(\frac{\partial x^i}{\partial u} \frac{da_1}{dt} + \frac{\partial x^i}{\partial v} \frac{da_2}{dt} \right) U_i(\alpha(t))$$

$$= \frac{da_1}{dt} \underbrace{\sum_{i=1}^{3} \frac{\partial x^i}{\partial u} U_i(\alpha(t))}_{\mathbf{X}_u(\alpha(t))} + \frac{da_2}{dt} \underbrace{\sum_{i=1}^{3} \frac{\partial x^i}{\partial v} U_i(\alpha(t))}_{\mathbf{X}_v(\alpha(t))} \Rightarrow \alpha'(0) \text{ is in span of } \mathbf{X}_u, \mathbf{X}_v$$

Proof continued:

~~$\alpha'(0) = (\alpha(0), v)$ notice~~

$$\alpha(t) = \mathbb{X}(u_0 + tc_1,\ v_0 + tc_2)$$

$$\alpha'(0) = c_1 \mathbb{X}_u(u_0, v_0) + c_2 \mathbb{X}_v(u_0, v_0)$$

Given anything in $\operatorname{span}\{\mathbb{X}_u(u_0, v_0), \mathbb{X}_v(u_0, v_0)\}$ we have

$(\alpha(0), \alpha'(0)) = (p, \alpha'(0)) \in T_pM \longrightarrow \operatorname{span}\{\mathbb{X}_u(u_0, v_0), \mathbb{X}_v(u_0, v_0)\} \subseteq T_pM.$

$$\therefore \quad T_pM = \langle \mathbb{X}_u(u_0, v_0), \mathbb{X}_v(u_0, v_0) \rangle$$

Defn/ Euclidean vector field on M is an assignment $p \longmapsto Z(p) \in T_p\mathbb{R}^3$ for each $p \in M$.

Normal vector field $Z(p) \in (T_pM)^{\perp}$ at each $p \in M$

Comment: $M: g = c$ then $\nabla g = \frac{\partial g}{\partial x} U_1 + \frac{\partial g}{\partial y} U_2 + \frac{\partial g}{\partial z} U_3$ gives NORMAL vector field to M.

E/ $g = x^2 + y^2 + z^2 = 1$ $\quad V(p) = -p_2 U_1 + p_1 U_2$

$$\nabla g = 2x U_1 + 2y U_2 + 2z U_3.$$