LECTURE 13: DIFFERENTIAL FORMS, MAPS, $\int$ ON M

$\dim(T_pM) = 2$, BASIS $\{X_u, X_v\}$

0-forms: $f: M \longrightarrow \mathbb{R}$

1-forms: $\alpha \in \Lambda^1(M) \hookrightarrow \alpha(p) \in (T_pM)^*$ for each $p \in M$

2-forms: $\beta \in \Lambda^2(M) \hookrightarrow \beta(p): T_pM \times T_pM \longrightarrow \mathbb{R}$

$\beta(p)$ — bilinear; skew-symmetric $\beta(p)(V,W) = -\beta(p)(W,V)$

k-form: $\Lambda^k(M) = \{0\}$ for $k \geq 3$

1-form α smooth: $\alpha(V)$ is smooth fnct for all $V \in \Gamma(M)$, $V(p) \in T_pM$

2-form β smooth: $\beta(V,W)$ is smooth fnct. for all $V, W \in \Gamma(M)$.

0-form f smooth $\Longleftrightarrow$ $f \circ X: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is smooth $\forall X$ on M.

$$\boxed{\begin{array}{l}\textbf{Lemma 4.2: } \text{For } \{v,w\} \text{ LI in } T_pM, \text{ a 2-form } \eta, \\ \eta(av+bw,\ cv+dw) = \underbrace{(ad-bc)}\,\eta(v,w)\end{array}}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Proof: β η a 2-form,

$$\begin{aligned}\eta(av+bw,\ cv+dw) &= a\,\eta(v,\ cv+dw) + b\,\eta(w,\ cv+dw) \\ &= ac\,\cancel{\eta(v,v)}^{\,0} + ad\,\eta(v,w) + bc\,\eta(w,v) + bd\,\cancel{\eta(w,w)}^{\,0} \\ &= \underline{(ad-bc)\,\eta(v,w)}\end{aligned}$$

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Exterior Algebra over M: $\Omega(M) = \Lambda^0 M \oplus \Lambda^1 M \oplus \Lambda^2 M$ — Λ-product, exterior derivative

$$\boxed{\text{Def}^{n}\ \ \phi, \psi \text{ 1-forms} \implies (\phi\wedge\psi)(v,w) = \phi(v)\psi(w) - \phi(w)\psi(v)}$$

Props:

- $(\phi_1+\phi_2)\wedge\psi = \phi_1\wedge\psi + \phi_2\wedge\psi$
- $\phi\wedge\psi = -\psi\wedge\phi$
- $\alpha\wedge\beta = (-1)^{pq}\,\beta\wedge\alpha \qquad p, q = 0, 1, 2$

Exterior Derivative $d : \Lambda^p M \longrightarrow \Lambda^{p+1} M$

$\boxed{p=0}$ $(df)(V) = V[f]$

$\boxed{p=1}$ $d\phi(V, W) = V[\phi(W)] - W[\phi(V)] = -(W[\phi(V)] - V[\phi(W)])$
$= -d\phi(W, V).$

$\phi \in \Lambda^1(M)$

$\boxed{\text{Th}^m/\ \text{If } f : M \longrightarrow \mathbb{R} \text{ then } d(df) = 0}$

Proof: Let $\psi = df$, $d(df) = d\psi$. Let $V, W \in \Gamma(M)$

$$
\begin{aligned}
(d\psi)(V, W) &= V[\psi(W)] - W[\psi(V)] \\
&= V[df[W]] - W[df(V)] \\
&= V[W[f]] - W[V[f]]
\end{aligned}
$$

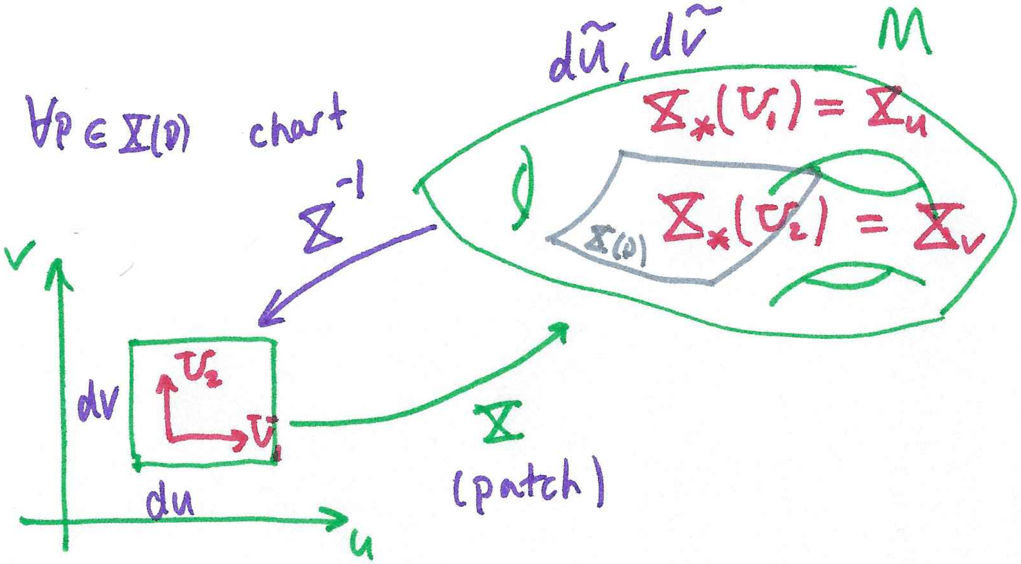
Use Lemma 4.2 use X_u, X_v

$$
\begin{aligned}
d\psi(X_u, X_v) &= X_u(X_v(f)) - X_v(X_u(f)) \\
&= \frac{\partial}{\partial u}\frac{\partial}{\partial v}[f \circ X] - \frac{\partial}{\partial v}\frac{\partial}{\partial u}[f \circ X] \\
&= 0. \quad \Rightarrow d\psi = d(df) = 0 /\!/
\end{aligned}
$$

Recall $\alpha = \sum_I \alpha_I \, dx^I \rightarrow d\alpha = \sum_I d\alpha_I \wedge dx^I$

Discuss CHARTS

$\mathbf{X}^{-1}(p) = (\tilde{u}(p), \tilde{v}(p)) \quad \forall p \in \mathbf{X}(D)$ chart



$\tilde{u}(\mathbf{X}(u,v)) = u$

$\tilde{v}(\mathbf{X}(u,v)) = v$

$\mathbf{X}^{-1}(\mathbf{X}(u,v)) = (u,v)$

$(\tilde{u}(\mathbf{X}(u,v)), \tilde{v}(\mathbf{X}(u,v))) = (u,v)$

using # 4 of p.156

$d\tilde{u}(\mathbf{X}_u) = \frac{\partial}{\partial u}\Big[\underbrace{\tilde{u}(\mathbf{X}(u,v))}_{u}\Big] = \frac{\partial u}{\partial u} = 1 \qquad d\tilde{u}(\mathbf{X}_v) = \frac{\partial u}{\partial v} = 0$

$d\tilde{v}(\mathbf{X}_v) = \frac{\partial v}{\partial v} = 1 \qquad d\tilde{v}(\mathbf{X}_u) = \frac{\partial v}{\partial u} = 0$

Why $\mathbf{X}_*(v_1) = \mathbf{X}_u$ and $\mathbf{X}_*(v_2) = \mathbf{X}_v$:

$\mathbf{X} : D \subseteq \mathbb{R}^2 \longrightarrow M \subseteq \mathbb{R}^3$ ← frame in $\mathbb{R}^3$: $\overline{v}_1, \overline{v}_2, \overline{v}_3$

$v_1 = \partial/\partial u \quad v_2 = \partial/\partial v$

$$\mathbf{X}_*(v_1) = dx(v_1)\overline{v}_1 + dy(v_1)\overline{v}_2 + dz(v_1)\overline{v}_3 \qquad \mathbf{X} = (x, y, z)$$

$$= v_1[x]\overline{v}_1 + v_1[y]\overline{v}_2 + v_1[z]\overline{v}_3$$

$$= \frac{\partial x}{\partial u}\overline{v}_1 + \frac{\partial y}{\partial u}\overline{v}_2 + \frac{\partial z}{\partial u}\overline{v}_3$$

$$= \mathbf{X}_u$$

Likewise $\mathbf{X}_*(v_2) = \mathbf{X}_v$

$\mathbf{X}_*(\partial/\partial u) = \mathbf{X}_u$

$\mathbf{X}_*(\partial/\partial v) = \mathbf{X}_v$

Also, $\boxed{\mathbf{X}_u[f] = \frac{\partial}{\partial u}(f \circ \mathbf{X})}$

$$\mathbf{X}_u[f] = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial f}{\partial z} = \frac{\partial}{\partial u}\left[(f \circ \mathbf{X})\right]. //$$

Example ~~When~~ $M = \mathbb{R}^2$

$u = \tilde{u}, \quad v = \tilde{v}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\phi = f_1\,du + f_2\,dv$

$\eta = g\, du \wedge dv$

$g = \eta(v_1, v_2)$

$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

$d\phi = df_1 \wedge du + df_2 \wedge dv = \left(\frac{\partial f_2}{\partial u} - \frac{\partial f_1}{\partial v} \right) du \wedge dv$

$\phi = f_1\,du + f_2\,dv$

$X(u,v) = (u,v)$

$\underbrace{\phi(X_u) = f_1, \quad \phi(X_v) = f_2}_{\phi(v_1) = f_1, \quad \phi(v_2) = f_2}$

Example $\Sigma(u,v) = (1,2,3)u + (-3,0,1)v$ is a plane through origin.

$$\Sigma_u = v_1 + 2v_2 + 3v_3$$
$$\Sigma_v = -3v_1 + v_3$$

$\Sigma(u,v) = (x,y,z)$ find $\Sigma^{-1}(x,y,z)$ just solve for u & v

$x = u - 3v$

$y = 2u \longrightarrow u = y/2 = \tilde{u}$

$z = 3u + v \longrightarrow v = z - 3u = z - \frac{3}{2}y = \tilde{v}$

$$\Sigma^{-1}(x,y,z) = \left(y/2,\ z - \tfrac{3}{2}y \right)$$

$\underline{d\tilde{u} = \frac{1}{2}dy}$ & $\underline{d\tilde{v} = dz - \frac{3}{2}dy}$

$$d\tilde{u}(\Sigma_u) = \tfrac{1}{2}dy(\underbrace{v_1}_{0} + 2v_2 + \underbrace{3v_3}_{0}) = 1 = d\tilde{v}(\Sigma_v)$$

Likewise $d\tilde{u}(\Sigma_v) = d\tilde{v}(\Sigma_u) = 0$

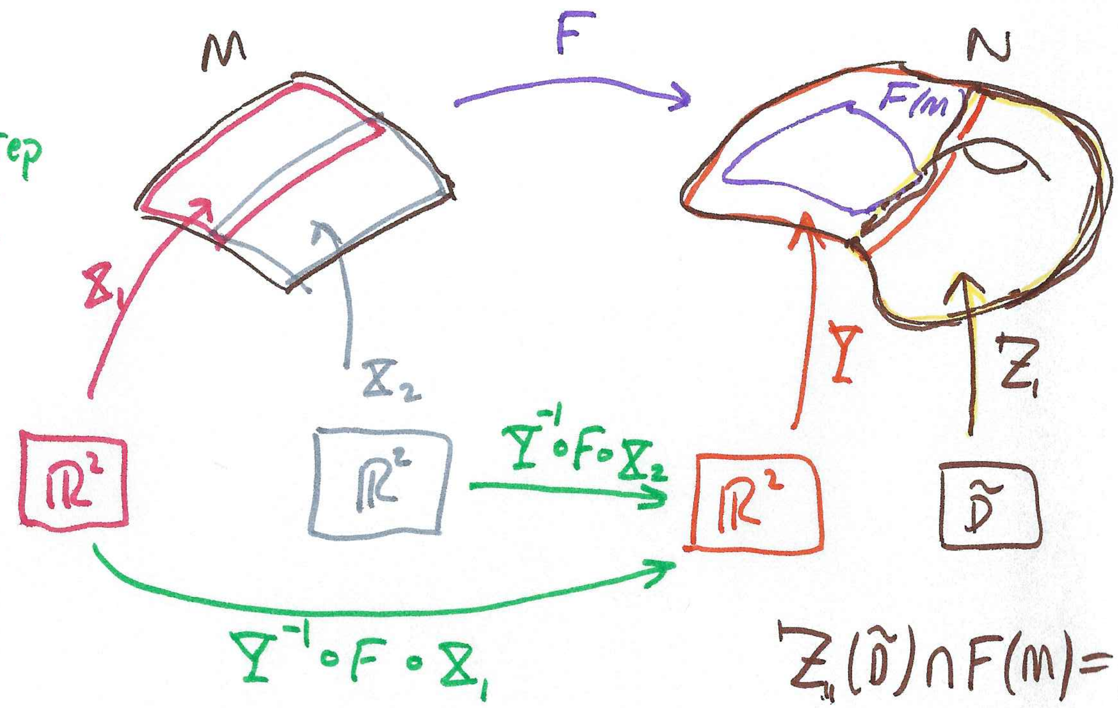
Def^{n}/ a form ϕ is closed if $d\phi = 0 \not\Rightarrow \phi = d\psi$

a form ϕ is exact if $\phi = d\psi \Rightarrow d\phi = d^2\psi = 0$.

Maps between surfaces in $\mathbb{R}^3$ (§4.5)

Defn/ $F: M \to N$ is <u>smooth</u> iff every local coord. rep. of F is smooth. Where local coord. rep is a composite of the patch $\mathbf{Y}$ on N and patch $\mathbf{X}$ on M

$$\mathbf{Y}^{-1} \circ F \circ \mathbf{X}$$



$\mathbf{Z}_1(\tilde{D}) \cap F(M) = \phi$

Ex 1) $\Sigma - \{N, S\} \longrightarrow$ cylinder

$$\mathbf{X}(u,v) = (\cos v \cos u, \cos v \sin u, \sin v)$$

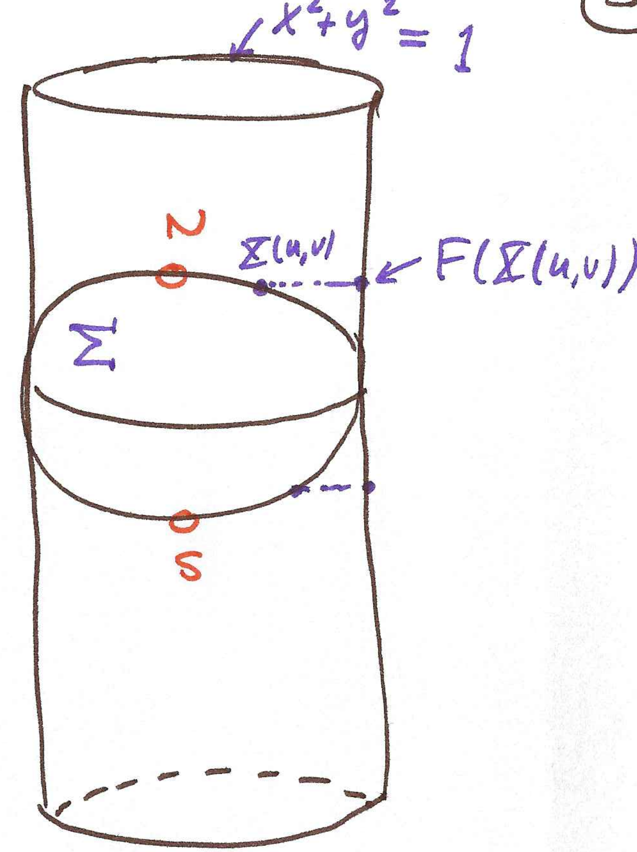
$$F(\mathbf{X}(u,v)) = (\cos u, \sin u, \sin v)$$

$$\mathbf{Y}(u,v) = (\cos u, \sin u, v)$$

$$\mathbf{Y}^{-1}(p^1, p^2, p^3) = (\tan^{-1}(p^2/p^1), p^3)$$

$$\mathbf{Y}^{-1}(F(\mathbf{X}(u,v))) = (u, \sin v)$$

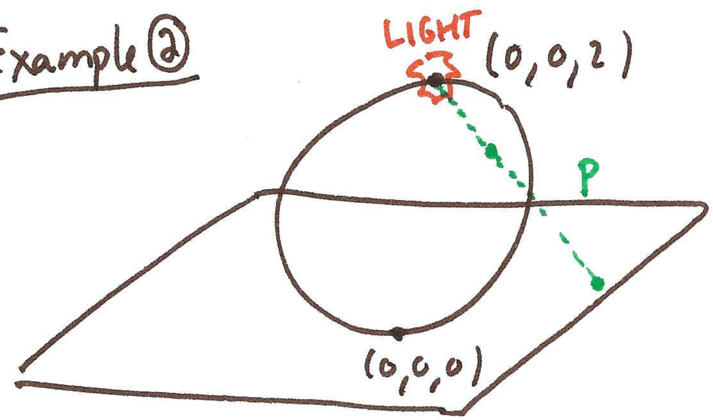
$$\left[\tan^{-1}\left(\frac{\sin u}{\cos u}\right) = \tan^{-1}(\tan(u)) = u.\right]$$



$$(u,v) \xmapsto{h} (u, \sin(v))$$

$$J_h = \begin{bmatrix} 1 & 0 \\ 0 & \cos(v) \end{bmatrix} \qquad \det(J_h) = \cos(v)$$

Example 2



$$P(p^1, p^2, p^3) = \left(\frac{2p^1}{2-p^3}, \frac{2p^2}{2-p^3} \right)$$

$$P : \underset{\substack{\uparrow \\ \text{sphere} \\ \text{modulo} \\ N}}{M} \longrightarrow \underset{\substack{\uparrow \\ \text{plane}}}{N}$$

$$\mathbf{X}(u,v) = (\underbrace{\cos v \cos u}_{p^1}, \underbrace{\cos v \sin u}_{p^2}, \underbrace{\sin v + 1}_{p^3})$$

$$P(\mathbf{X}(u,v)) = \left(\frac{2\cos v \cos u}{1-\sin v}, \frac{2\cos v \sin u}{1-\sin v} \right)$$

$$h(u,v) = \frac{2\cos v}{1-\sin v}(\cos u, \sin u) = f(v)(\cos u, \sin u)$$

$$J_h = \left[\frac{\partial h}{\partial u} \middle| \frac{\partial h}{\partial v} \right] = \left[\begin{array}{c|c} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \end{array} \right]$$

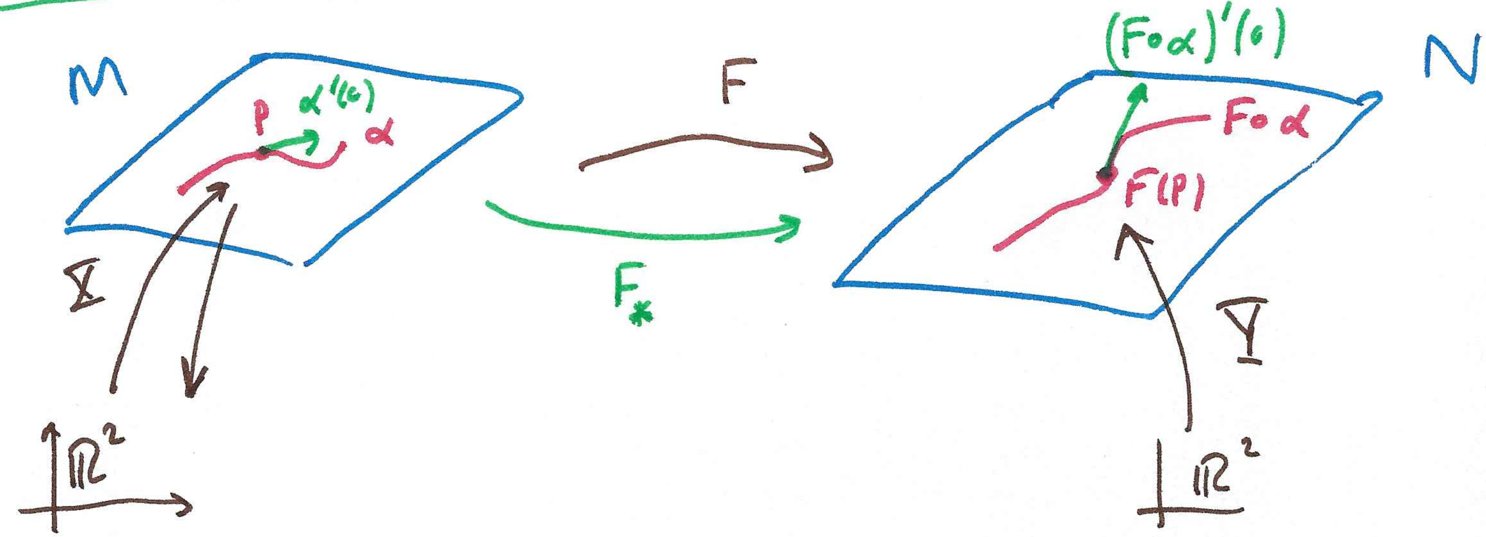
$$\det(J_h) = -f(v)f'(v)\left[\sin^2 u + \cos^2 u\right]$$

$$= -\frac{2\cos v}{1-\sin v}\left[\frac{-2\sin v(1-\sin v) - 2\cos(v)(-\cos v)}{(1-\sin v)^2}\right]$$

$$= \frac{-2\cos v\,[2 - 2\sin v]}{(1-\sin v)^3} = \frac{-4\cos v}{(1-\sin v)^2} \neq 0$$

Def^n/ $F : M \longrightarrow N$ then $(F_*)_p : T_pM \longrightarrow T_{F(p)}N$

$$F_*(\alpha'(0)) = (F \circ \alpha)'(0)$$



Prop./ $F_*(\mathbf{X}_u) = \mathbf{Y}_u$ & $F_*(\mathbf{X}_v) = \mathbf{Y}_v$

Def^n/ $F : M \longrightarrow N$ is <u>regular</u> if $(F_*)_p$ is injective for each $p \in M$

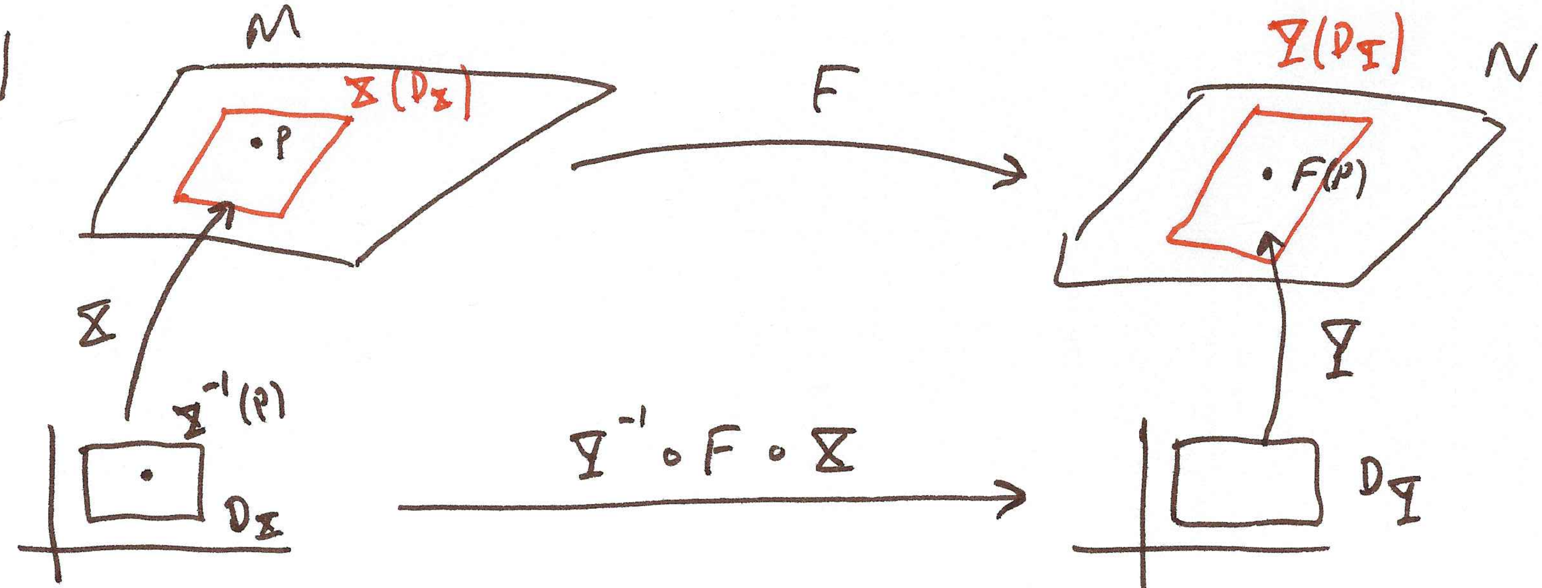
$(F_*)_p$ is linear from $T_pM \longrightarrow T_{F(p)}N$

$\mathrm{Ker}\,(F_*)_p = \{0\}$

Def^n/ A smooth map $F: M \to N$ with smooth inverse $F^{-1}: N \to M$ is called a <u>DIFFEOMORPHISM</u>.

Th^m/ Let $F: M \to N$ be smooth map of surfaces and if $\exists\, p \in M$ such that $(F_*)_p : T_pM \to T_{F(p)}N$ is injective then $\exists$ nbhd $\mathcal{U}$ of p in M and $\mathcal{V}$ of $F(p)$ in N such that $F|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$

<u>Proof</u>



<u>Claim</u>: injectivity of $(F_*)_p \Rightarrow h = \mathbf{Y}^{-1} \circ F \circ \mathbf{X}$ has invertible J_h at $\mathbf{X}^{-1}(p)$.

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Corollary: a bijection $F: M \to N$ which is smooth and regular has smooth F^{-1}. That is, a regular bijection of M & N is a diffeomorphism.

Ex ①: $F(u,v) = (\tan u, \tan v)$ $\quad$ $F: (-\pi/2, \pi/2)^2 \longrightarrow \mathbb{R}^2$

$F^{-1}(u,v) = (\tan^{-1}(u), \tan^{-1}(v))$

$J_F = \left[\begin{array}{c|c} \sec^2 u & 0 \\ \hline 0 & \sec^2 v \end{array}\right]$ invertible on dom F.

Ex ②: $\Sigma - N \approx \mathbb{R}^2$

O'neill argues $\left.\begin{array}{l} P_*(\Sigma_u) = Y_u \\ P_*(\Sigma_v) = Y_v \end{array}\right\}$ LI hence P_* injective.

Ex ③ A $\underbrace{\text{cylinder } C}_{x^2+y^2=1}$ over a <u>closed</u> is <u>diffeomorphic</u> to $\mathbb{R} - \{0\}$

$F(x,y,z) = e^z(x,y)$

$\boxed{\Sigma(u,v) = (\cos u, \sin u, v) \qquad F(\Sigma(u,v)) = e^v(\cos u, \sin u)}$ — Not wrong, but ignore for what follows

$F(x,y,z) = (u,v)$

$u = e^z x \to x = ue^{-z}$

$v = e^z y \to y = ve^{-z}$

$x^2+y^2=1$ $\qquad$ $u^2+v^2 = e^{2z}$, $\quad e^z = \sqrt{u^2+v^2}$

$\to 2z = \ln(u^2+v^2)$

$\left.\begin{array}{l} z = \ln\sqrt{u^2+v^2} \\ x = u/\sqrt{u^2+v^2} \\ y = v/\sqrt{u^2+v^2} \end{array}\right\} F^{-1}(u,v)$

THE PULL-BACK

Def^n/ Let $F : M \to N$ be smooth map (F need not be invertible etc.) of surfaces M & N then define:

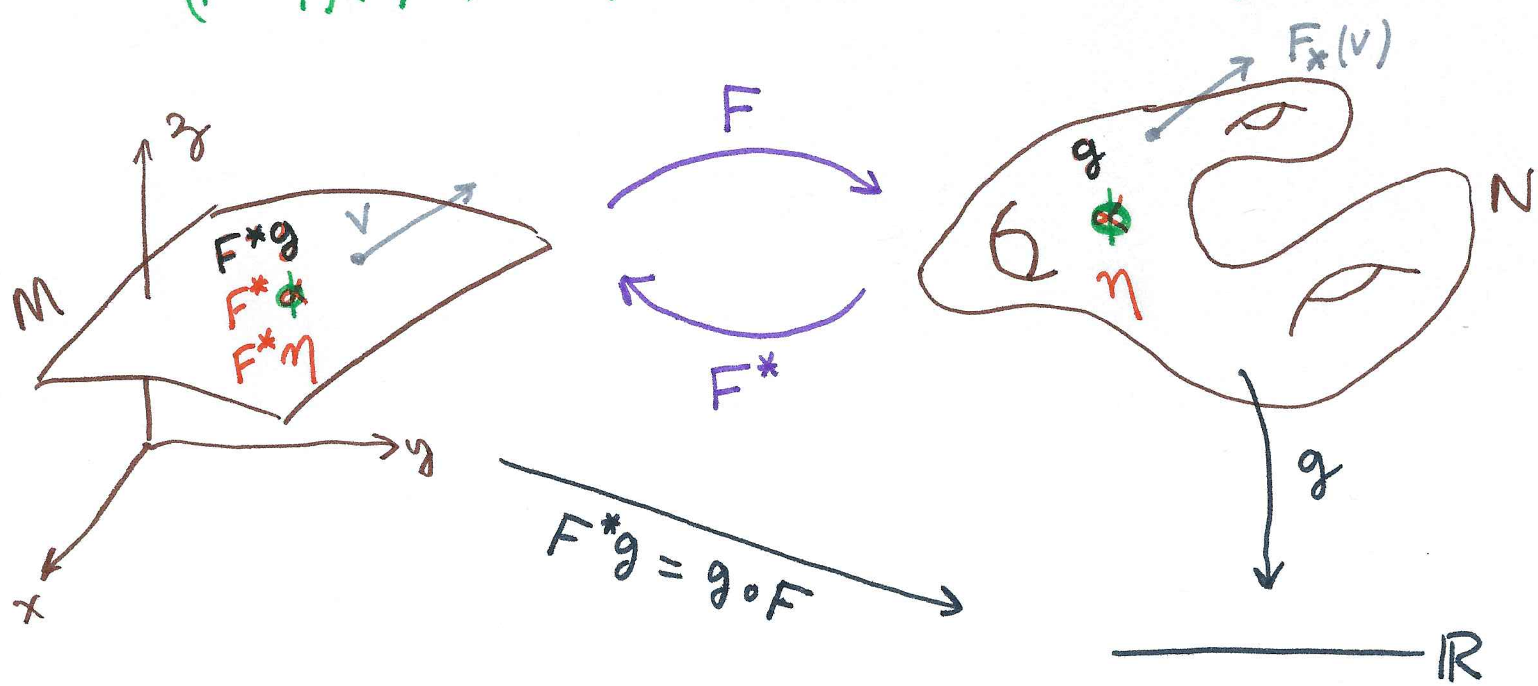
$$\left(F^*\phi\right)_p(v) = \phi_{F(p)}\left((F_*)_p(v)\right) \quad \forall v \in T_pM$$

(0.) $F^*g = g \circ F$ for each $g : N \to \mathbb{R}$

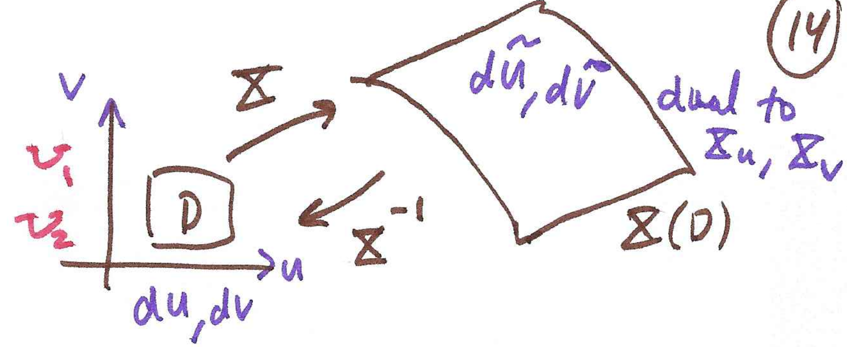
(1.) for $\phi \in \Lambda^1(N)$ we define $F^*\phi \in \Lambda^1(M)$ by $\underbrace{(F^*\phi)(v) = \phi(F_*(v))}_{\text{for each tangent } v \text{ to } \underline{\underline{\underline{M}}}}$

(2.) for $\eta \in \Lambda^2(N)$ define $F^*\eta \in \Lambda^2(M)$ by

$$(F^*\eta)(v, w) = \eta(F_*v, F_*w) \quad \text{for all tangents } v, w \text{ to } M$$



Example: $(\Sigma^{-1})^*(du) = d\tilde{u}$



$$\left((\Sigma^{-1})^*(du)\right)(\Sigma_u) = du\left((\Sigma^{-1})_*(\Sigma_u)\right)$$
$$= du(v_1)$$
$$= \frac{\partial u}{\partial u}$$
$$= 1.$$

$\Sigma_*(v_1) = \Sigma_u \longrightarrow v_1 = (\Sigma^{-1})_*(\Sigma_u)$

$\Sigma^{-1} \circ \Sigma = Id$

$(\Sigma^{-1})_* \circ \Sigma_* = Id$

Likewise, $\left((\Sigma^{-1})^*(du)\right)(\Sigma_v) = \frac{\partial u}{\partial v} = 0$ $\quad \therefore \underline{(\Sigma^{-1})^*(du) = d\tilde{u}}$.

$\underline{dg \circ dF = d(g \circ F)}$

$F_* = dF$

Properties:

$$\boxed{\begin{array}{ll} (1) & F^*(\alpha+\beta) = F^*\alpha + F^*\beta \\ (2) & F^*(\alpha \wedge \beta) = (F^*\alpha)\wedge(F^*\beta) \\ (3) & F^*(d\alpha) = d(F^*\alpha) \end{array}}$$

Proof: Let g be 0-form

$$\left(F^*(dg)\right)(\Sigma_u) = dg\left(F_*(\Sigma_u)\right)$$
$$= d(g \circ F)(\Sigma_u)$$
$$= d(F^*g)(\Sigma_u)$$

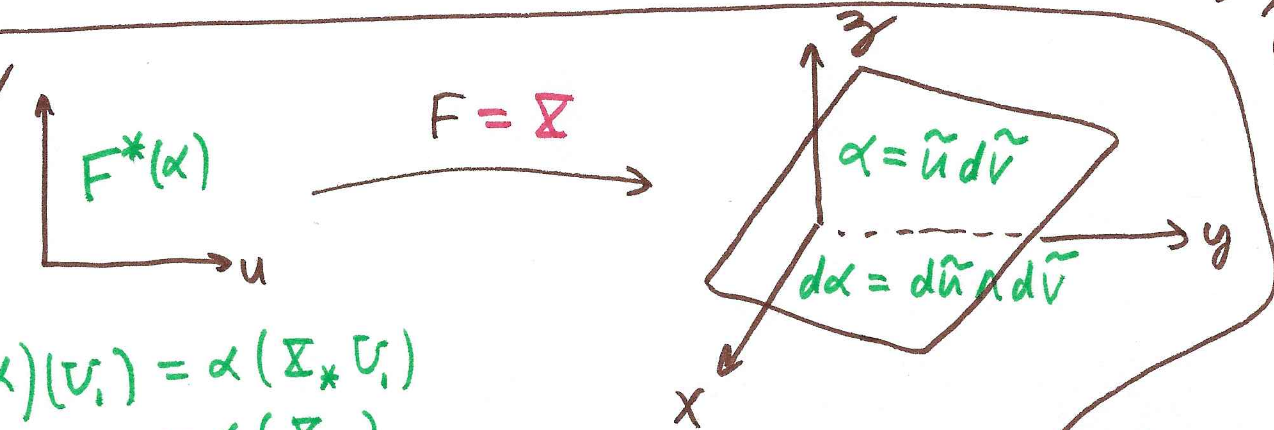
$\therefore \underline{F^*(dg) = d(F^*g)}$ //

Example: $F(u,v) = (u+v,\ u,\ u-v) = (x,y,z)$

$x = u+v$
$y = u$
$z = u-v$

$y = u$
$x - z = 2v$
$v = \frac{1}{2}(x-z)$

$F^{-1}(x,y,z) = \left(y,\ \frac{1}{2}(x-z)\right)$



$$(F^*\alpha)(v_1) = \alpha(X_* v_1) = \alpha(X_u)$$

$$(F^*\alpha)(v_2) = \alpha(X_v)$$

$$F^*\alpha = \cancelto{0}{\alpha(X_u)}\,du + \cancelto{u}{\alpha(X_v)}\,dv = u\,dv \longrightarrow d(F^*\alpha) = d(u\,dv) = \underline{du \wedge dv}.$$

$$\alpha(X_u) = (\tilde{u}\,d\tilde{v})(X_u) = \tilde{u}\,d\tilde{v}(X_u) = 0$$
$$\alpha(X_v) = (\tilde{u}\,d\tilde{v})(X_v) = \tilde{u} \longrightarrow u$$

$$F^*(d\alpha) = F^*(d\tilde{u} \wedge d\tilde{v}) = (F^* d\tilde{u}) \wedge F^*(d\tilde{v}) = \underline{du \wedge dv}$$

$$\alpha = \tilde{u}\,d\tilde{v} = \underbrace{y/_2\,(dx - dz)} \longrightarrow d\alpha = \underline{\tfrac{1}{2}\,dy \wedge dx - \tfrac{1}{2}\,dy \wedge dz}$$

subst. u, v for x, y, z

$$F^*\alpha = \frac{u}{2}\big(du + dv - (du - dv)\big) = u\,dv$$

$$F^* d\alpha = \tfrac{1}{2}\,du \wedge (\cancelto{0}{du} + dv) - \tfrac{1}{2}\,du \wedge (\cancelto{0}{du} - dv) = du \wedge dv = d(u\,dv). \checkmark$$

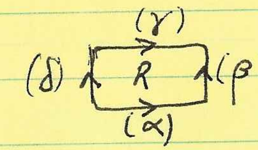
Integration of Forms

① 1-forms on curve-segments

2-forms on two-segments

① $\int_\alpha \phi = \int_{[a,b]} \alpha^* \phi = \int_a^b \phi(\alpha'(t))\,dt$

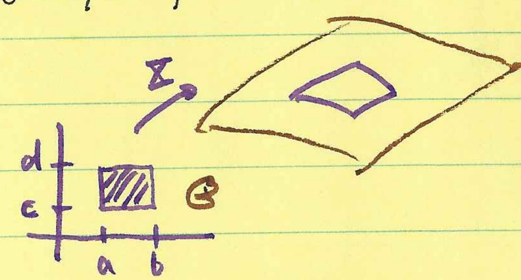
Thm (6.2) $\alpha : [a,b] \to M$ a curve segment from $p = \alpha(a)$ to $q = \alpha(b)$ then $\int_\alpha df = f(q) - f(p)$

② 2-segment is image of 

$$\iint_{\mathbf{x}} \eta = \iint \mathbf{x}^* \eta = \int_a^b \int_c^d \eta(\mathbf{x}_u, \mathbf{x}_v)\,dv\,du$$

$$\partial \mathbf{x} = \alpha + \beta - \gamma - \delta$$

$$\iint_{\mathbf{x}} d\phi = \int_{\partial \mathbf{x}} \phi$$



- oriented integrals, switch signs under a reversal of direction of α.