LECTURE 14: TOPOLOGICAL PROP. OF SURFACES & MANIFOLDS:

Def^n/ A surface M is <u>connected</u> if for each $p, q \in M$ there exists a map $\underbrace{\alpha : [a,b] \longrightarrow M}_{\text{curve-segment}}$ s.t. $\alpha(a) = p$ & $\alpha(b) = q$

(path-connected)

Def^n/ M is <u>compact</u> if every open cover of M admits a finite subcover

$\left.\begin{array}{l} [a,b] \\ [a,b] \times [c,d] \end{array}\right\}$ examples of compact sets.

<u>Lemma 7.2</u>: A SURFACE M IS COMPACT IFF IT CAN BE COVERED BY THE IMAGES OF FINITELY MANY TWO-SEGMENTS

<u>Btw</u>, MASSEY's Algebraic Topology: any compact surface is the image of a single 2-segment

$\boxed{\text{**Lemma (7.3)** A continuous fnct on a compact } \mathcal{R} \subseteq M \text{ attains a maximum value at some point } p \in \mathcal{R}.}$

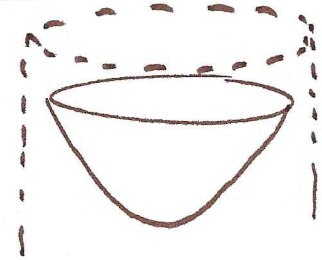
Ex $C : x^2 + y^2 = 1$

$f(x,y,z) = z$ ← continuous & unbounded on C

$\therefore$ C is <u>NOT</u> COMPACT

Ex OPEN DISK : $D : x^2+y^2 < 1$

$f(x,y) = \dfrac{1}{1-x^2-y^2}$ unbounded on D



$\therefore$ <u>D NOT COMPACT</u>.

<u>COMPACT</u>: smoothly closed up with no fuzzy edges & finite in size.

Def^n/ A surface M is <u>orientable</u> if $\exists$ a diff. 2-form μ on M which is everywhere nonzero.

Ex/ $\mathbb{R}^2_{uv}$: $\mu = du \wedge dv$

Ex/ $\Phi(D)$: $\mu = d\tilde{u} \wedge d\tilde{v}$

Prop. 7.5/ A SURFACE $M \subset \mathbb{R}^3$ is orientable $\Longleftrightarrow$ $\exists\, \mathcal{U}$ a unit-normal vector field on M. Moreover, if M is connected then $\pm\mathcal{U}$ are the only two normal.

Proof/ $\Rightarrow$/ S $\exists\, \mu \in \Lambda^2 M$ and $\mu(P) \neq 0$ $\forall P \in M$. Define, for v, w a basis of T_pM

$$Z(P) = \frac{v \times w}{\mu(P)(v,w)} = \frac{v' \times w'}{\mu(P)(v',w')} \in (T_pM)^\perp$$

Observe $\in(v,w) = v \times w$. $\quad \in(\underbrace{av+bw}_{v'}, \underbrace{cv+dw}_{w'}) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (v,w)$

Likewise, $\mu(P)(av+bw, cv+dw) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mu(P)(v,w)$.

Let $\mathcal{U}(P) = \dfrac{Z(P)}{\|Z(P)\|}$ is a unit-normal. Consider fnct $f(P) = \mathcal{V}(P) \cdot \mathcal{U}(P)$ where $\mathcal{V}(P)$ is any other unit-normal

$\Rightarrow f(P) = \pm 1 \underset{\text{connected}}{\hookrightarrow} f(P) = 1$ or $f(P) = -1$ for all $P \in M$

$\therefore\ \mathcal{V} = \mathcal{U}$ or $\mathcal{V} = -\mathcal{U}$.

<u>Proof</u>: <u>$\Leftarrow$</u>) S $\exists$ a unit-normal U on M. Let $v, w \in T_pM$

$$\mu(p)(v,w) = U(p) \cdot (v \times w)$$

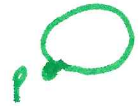
Since $\exists \{v, w\}$ LI at each p, $v \times w \neq 0$ and as $U(p) \neq 0$ and by $\perp$ theorems $U(p)$ is colinear to $v \times w \Rightarrow \mu(p) \neq 0$. //

<u>Examples</u>
- spheres
- cylinders
- surfaces of revolution
- quadric surfaces

$\boxed{\text{Thm/ A compact surface in } \mathbb{R}^3 \text{ is orientable}}$

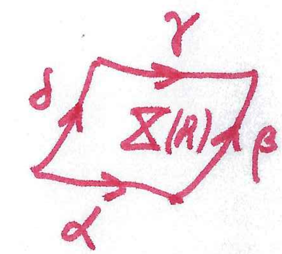
$\boxed{\text{Thm/ A simply-connected is orientable}}$

NON-EXAMPLE : MOBIUS BAND



Def$^{\underline{n}}$/ A closed curve α is <u>homotopic</u> to a <u>constant</u> provided $\exists$ 2-segment $\Phi : \underbrace{[a,b]\times[0,1]}_{R} \longrightarrow M$ (homotopy) s.t. α is the base curve $\alpha(u) = \Phi(u,0)$ and the other three edge curves are constant at $p = \alpha(a) = \alpha(b)$

$$\beta(v) = \Phi(b,v)$$
$$\gamma(u) = \Phi(u,1)$$
$$\delta(v) = \Phi(a,v)$$



$$\partial\Phi(R) = \alpha+\beta-\gamma-\delta$$

Def$^{\underline{n}}$/ M is simply connected provided it is connected and every loop in M is homotopic to a constant

$$\alpha_v(u) = \Phi(u,v) \quad \xrightarrow{v=0} \quad \alpha_0(u) = \Phi(u,0) = \alpha(u)$$
$$\xrightarrow{v=1} \quad \alpha_1(u) = \Phi(u,1) = \gamma(u) = p$$

Lemma (7.8): ϕ is a closed 1-form on surface M and if a loop α in M is homotopic to a constant then $\int_\alpha \phi = 0$

Proof: $d\phi = 0$ $\quad 0 = \iint_\Phi d\phi = \int_{\partial\Phi}\phi = \int_{\alpha+\beta-\gamma-\delta}\phi = \int_\alpha \phi \quad \therefore \int_\alpha \phi = 0$ //

Application : punctured plane P

$$\psi = \frac{x\,dy - y\,dx}{x^2+y^2} \qquad \alpha \quad \underline{\text{loop}}\ \underline{\text{around}}\ \underline{\text{origin}}$$

$$\int_\alpha \psi = 2\pi \quad \therefore \ \alpha \text{ is } \underline{\text{NOT}} \text{ HOMOTOPIC TO CONSTANT}$$

$\therefore$ <u>P NOT SIMPLY CONNECTED</u>.

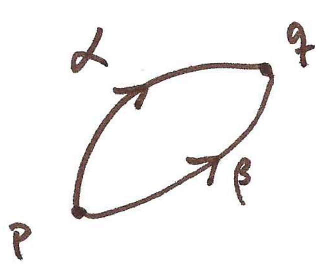
Lemma (7.4) (Poincaré) On simply connected surface every closed form is exact.

$$\phi \text{ s.t. } d\phi = 0 \implies \exists \psi \text{ s.t. } d\psi = \phi$$

ψ a <u>potential</u>

<u>Proof</u>: M simply connected surface

α, β are two curve-segments both from p to q. Let ϕ have $d\phi = 0$



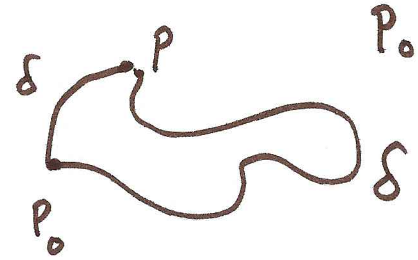
$\alpha - \beta = \text{loop}$

$$0 = \int_{\text{loop}} \phi = \int_{\alpha-\beta} \phi = \int_\alpha \phi + \int_{-\beta} \phi = \int_\alpha \phi - \int_\beta \phi = 0$$

PATH - INDEPENDENCE!

Proof continued: pick $P_0 \in M$ then let δ be curve-seg. from P_0 to P

$$f(P) = \int_\delta \phi$$



We wish $\phi = df \longrightarrow \phi(v) = df(v)$

Consider $\alpha : [a,b] \longrightarrow M$ with $\alpha'(a) = v$

the path β is formed by $\underbrace{\delta}_{\text{starts at } P_0}$ then $\alpha|_{[a,t]}$ ends at $\alpha(t)$.

$$f(\alpha(t)) = \int_\beta \phi = \int_\delta \phi + \int_{\alpha|_{[a,t]}} \phi = f(P) + \int_a^t \phi(\alpha'(u))du$$

$$\alpha'(t)[f] = (f \circ \alpha)'(t) = \frac{d}{dt}\left[\int_a^t \phi(\alpha'(u))du\right] = \phi(\alpha'(t))$$

$t=0$: $\alpha'(0)[f] = \phi(\alpha'(0))$

$$\underbrace{v[f]}_{df(v)} = \phi(v)$$

$df(v) = \phi(v) \therefore df = \phi$ hence ϕ is exact. //

Manifolds

§4.8: O'neill describes an axiomatic geometric surface.

M an abstract set — cover of it by patches
— smoothly overlap
— Hausdorff Axiom
— $\alpha'(t)[f] = \frac{d(f\circ\alpha)}{dt}(t)$

§4.1 – 4.7 : CALCULUS FOR $M \subseteq \mathbb{R}^3$ where topology given by subspace topology of Euclidean $\mathbb{R}^3$ topology.

Def^n/ A n-dim'l manifold is a set M with a collection $\mathcal{P}$ of patches (one-to-one fncts. $\mathbf{X} : D \to M$, $\underset{\text{open}}{D} \subseteq \mathbb{R}^n$) satisfying — (1) the patches cover M

(2.) the patches smoothly overlap ($\mathbf{Y}^{-1}\circ\mathbf{X}$, $\mathbf{X}^{-1}\circ\mathbf{Y}$)

(3.) the Hausdorff property

$p \neq q \in M$ then $\exists$ disjoint patches $\mathbf{X}$ & $\mathbf{Y}$ with $p \in \mathbf{X}(D)$, $q \in \mathbf{Y}(E)$

SURFACE IS JUST $n=2$ dim'l manifold.