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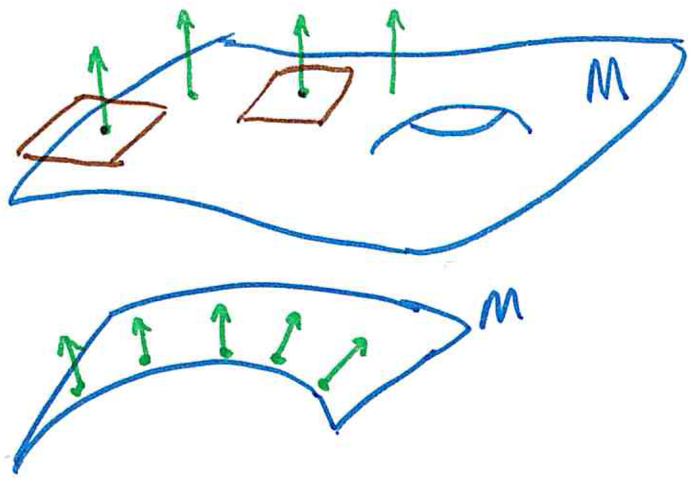
LECTURE 15: SHAPE OPERATOR (§5.1–5.3 O'neill)



Defⁿ/ If $p \in M$ then for each $v \in T_pM$ let $S_p(v) = (-\nabla_v U)(p)$

$$\underbrace{S_p : T_pM \longrightarrow T_pM}_{\text{SHAPE OPERATOR AT } p.}$$

$$\boxed{\begin{aligned} \text{Lemma: } \nabla_v U &: T_pM \longrightarrow T_pM \\ v &\longmapsto \nabla_v U \end{aligned}}$$

Proof: $U \cdot U = 1$

$$v[U \cdot U] = 0 = (\nabla_v U)\cdot U + U \cdot (\nabla_v U) \quad \therefore \quad (\nabla_v U)\cdot U = 0$$

$$\Rightarrow (\nabla_v U)(p) \in T_pM$$

$$(T_pM)^{\perp} = \mathrm{span}\{U(p)\}$$

$$\underline{T_p\mathbb{R}^3 = T_pM \oplus \mathrm{span}\{U(p)\}}$$

Proposition: $S_p : T_pM \to T_pM$ is a linear transformation.

① $\nabla_v Z = \sum_{i=1}^{3} v[z^i]\, U_i$

② $\nabla_v Z = (Z \circ \alpha)'(0)$

Proof $S_p(av + bw) = -\nabla_{av+bw} U$

$= a - \nabla_v U + b \nabla_w U$

$= a\, S_p(v) + b\, S_p(w)$. //

Ex ① Σ : $\|P\| = r$

$\vec{r} = \sum_{i=1}^{3} x^i U_i$

$\|\vec{r}\| = r$

$U = \frac{1}{r} \sum_{i=1}^{3} x^i U_i$

$\nabla_v U = \sum_{i=1}^{3} v\left[\frac{x^i}{r}\right] U_i$

$= \sum_i \frac{1}{r} v[x^i]\, U_i$

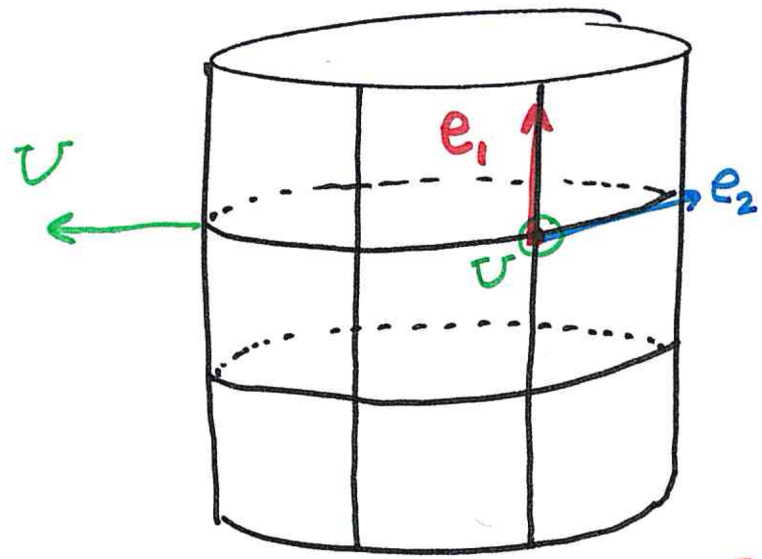
$= \frac{v}{r}$

$\therefore \boxed{S_p(v) = \frac{-v}{r}}$

Ex 2) $\underline{P}$ a plane in $\mathbb{R}^3$; $U = aU_1 + bU_2 + cU_3$

$$\nabla_v U = 0 \hookrightarrow \underline{S_p = 0 \quad \forall p \in \underline{P}}$$

Ex 3) $C: x^2+y^2=r^2$



Geometrically:

$$S_p(e_1) = 0 = -\nabla_{e_1} U$$

$$S_p(e_2) = \frac{-e_2}{r}$$

$$U = \underline{\cos\theta}\, U_1 + \sin\theta\, U_2 \quad \text{unit-normal}$$

$$e_1 = U_3$$

$$e_2 = -\sin\theta\, U_1 + \cos\theta\, U_2$$

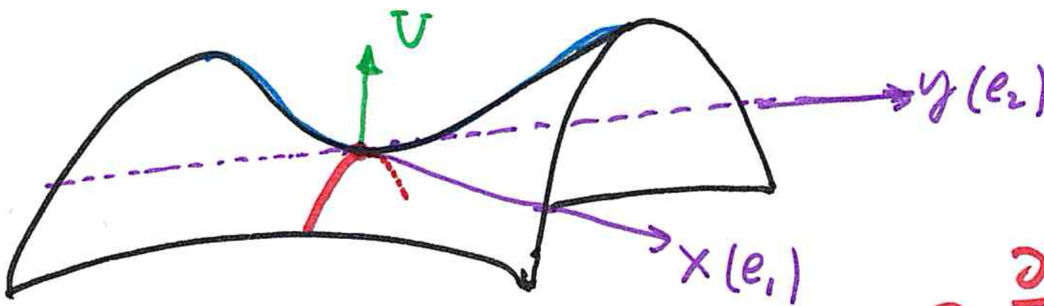
$$\nabla_{e_1} U = \cancelto{0}{U_3[\cos\theta]}\, U_1 + \cancelto{0}{U_3[\sin\theta]}\, U_2 = 0 \therefore S_p(e_1) = 0.$$

$$\nabla_{e_2} U = (-\sin\theta\, U_1 + \cos\theta\, \cancelto{0}{U_2})[x/r]\, U_1 + (-\sin\theta\, \cancelto{0}{U_1} + \cos\theta\, U_2)[y/r]\, U_2$$

$$= \frac{-\sin\theta}{r} U_1 + \frac{\cos\theta}{r} U_2 = \frac{e_2}{r} \quad \therefore \underline{S_p(e_2) = \frac{-e_2}{r}}.$$

(4) Saddle Surface $M: z = xy$

$p = (0,0,0)$



$x = 0, \quad z = 0$ (y-axis)

$y = 0, \quad z = 0$ (x-axis)

$$\mathbf{X}(x,y) = \langle x, y, xy \rangle \quad \longrightarrow \quad \frac{\partial \mathbf{X}}{\partial x} = \langle 1, 0, y \rangle, \quad \frac{\partial \mathbf{X}}{\partial y} = \langle 0, 1, x \rangle$$

$$\underline{U}(x,y) = \frac{\partial \mathbf{X}}{\partial x} \times \frac{\partial \mathbf{X}}{\partial y} = \frac{1}{\sqrt{x^2+y^2+1}} \langle -y, -x, 1 \rangle$$

<u>Calculate</u>: $S_p(e_1)$: $\alpha'(0) = e_1$ $\quad \alpha$ a curve on M.

$$\alpha(t) = (t, 0, 0)$$

$$(U \circ \alpha)(t) = \frac{1}{\sqrt{t^2+1}} \langle 0, -t, 1 \rangle \hookrightarrow \underline{U_\alpha'(0) = \langle 0, -1, 0 \rangle}. = -e_2$$

$$S_p(e_1) = -\nabla_{e_1} U = -U_\alpha'(0) = + e_2$$

$$\alpha(t) = (0, t, 0)$$

$$(U \circ \alpha)(t) = \frac{1}{\sqrt{1+t^2}} \langle -t, 0, 1 \rangle \hookrightarrow U_\alpha'(0) = \langle -1, 0, 0 \rangle$$

$$\boxed{S_p(e_2) = e_1}$$

Ex 1 SPHERE

$$S_p(v) = \frac{-v}{r}$$

Ex 2 PLANE ~~CYLINDER~~

$$S_p(v) = 0$$

Ex 3 CYLINDER

$$S_p(e_1) = 0$$

$$S_p(e_2) = \frac{-e_2}{r}$$

Ex 4 SADDLE

$$S_p(e_1) = e_2$$

$$S_p(e_2) = e_1$$

The matrix $S_p : T_pM \longrightarrow T_pM$ was thus,

Sphere:

$$[S_p] = \begin{bmatrix} -1/r & 0 \\ 0 & -1/r \end{bmatrix}$$

$$K = \det[S_p] = \frac{1}{r^2}$$

$$H = \frac{1}{2}\operatorname{trace}[S_p] = \frac{-1}{r}$$

Plane:

$$[S_p] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$K = 0$$

$$H = 0$$

Cylinder:

$$[S_p] = \begin{bmatrix} 0 & \\ & -1/r \end{bmatrix}$$

$$K = 0$$

$$H = \frac{-1}{2r}$$

Saddle:

$$[S_p] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K = -1$$

$$H = 0$$

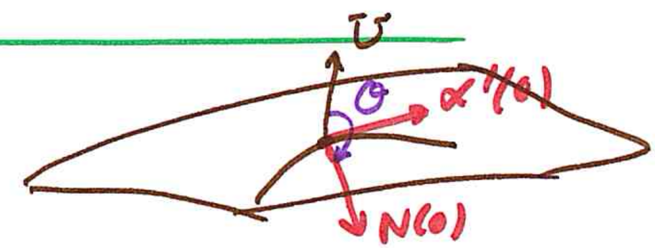
§5.2 NORMAL CURVATURE

Lemma 2.1: If α a curve in M then $\alpha'' \cdot U = S(\alpha') \cdot \alpha'$

<u>Proof</u>: $\alpha' \cdot U = 0 \hookrightarrow \alpha'' \cdot U + \alpha' \cdot U_\alpha' = 0$

$\alpha'(t) \cdot U(\alpha(t)) = 0 \qquad \alpha'' \cdot U = -U_\alpha' \cdot \alpha' = S(\alpha') \cdot \alpha'$ //

Def^n/ Let u be a unit-tangent to $M \subseteq \mathbb{R}^3$ at p then $k(u) = S(u) \cdot u$ is the <u>NORMAL</u> <u>CURVATURE</u> of M in the u-direction at p.



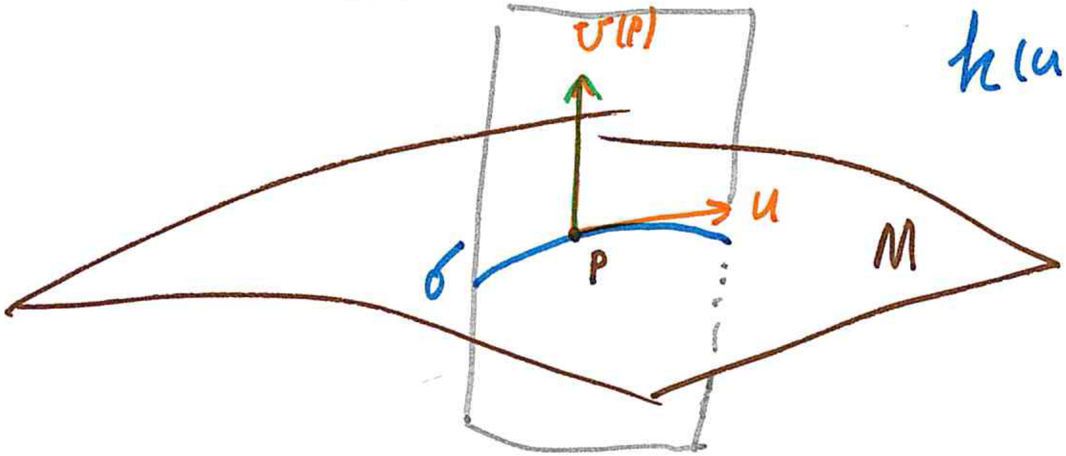
<u>OBSERVATIONS</u>

1.) $k(u) = k(-u)$

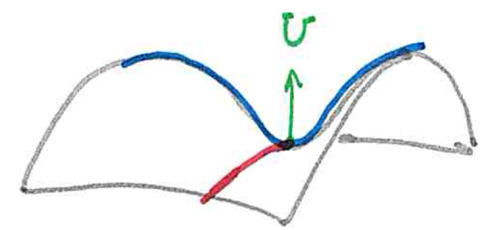
2.) If α unit-speed then $\alpha''(0) = \kappa(0) N(0)$
thus $k(u) = \alpha''(0) \cdot U(p) = \kappa(0) N(0) \cdot U(p) = \kappa(0) \cos\theta$
where θ is $\measuredangle$ between $N(0)$ & $U(p)$

3.) can fix $\theta = 0, \pi$ by using <u>normal</u> <u>section</u> whose curve of intersection σ has $N(0) = \pm U(p)$.

$$k(u) = \kappa_\sigma(0)\underbrace{N(0)\cdot v(p)}_{\pm 1} = \pm\kappa_\sigma$$

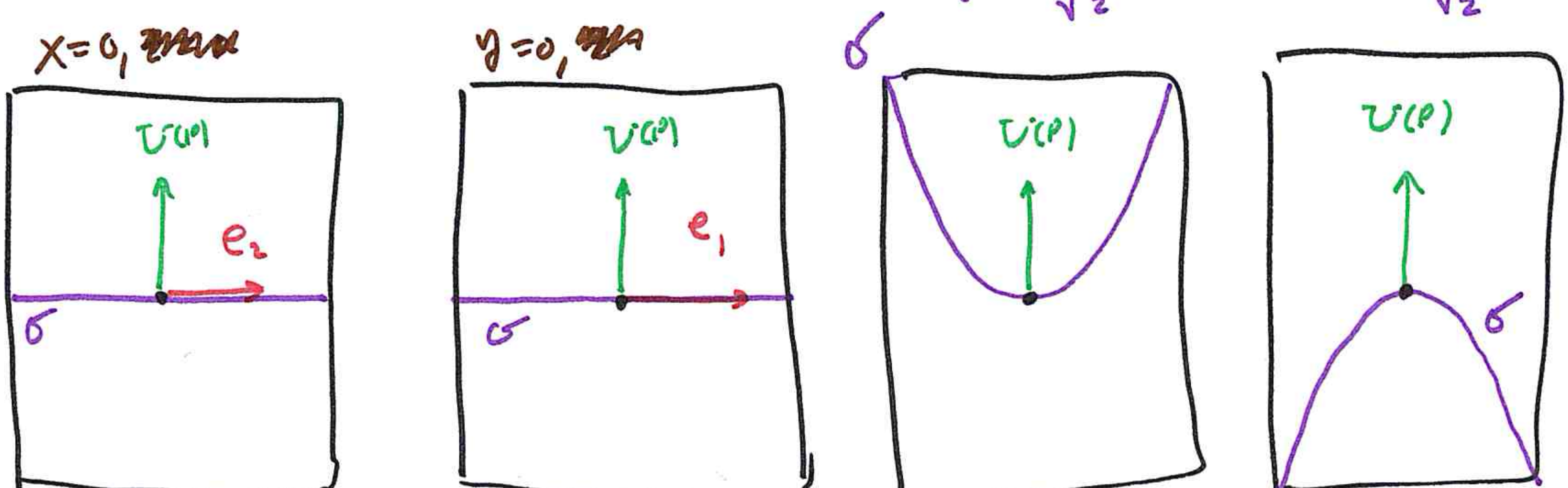


$$k(u) = \pm\kappa_\sigma$$



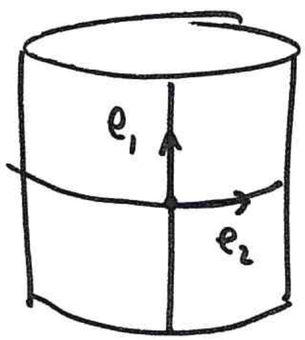
<u>Example</u>: $z = xy$ study $p = (0,0,0)$

$x=0$, | $y=0$, | $y=x$, $u=\frac{v_1+v_2}{\sqrt{2}}$ | $y=-x$, $u=\frac{v_1-v_2}{\sqrt{2}}$



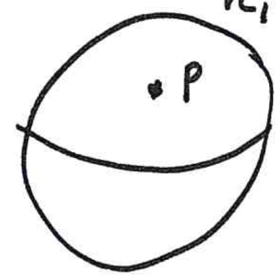
$k(e_2)=0$ | $k(e_1)=0$ | $k(u)>0$ | $k<0$

Def^n/ The max/min values of $k(u)$ of M at p are denoted k_1, k_2 ($k_1 \geq k_2$) for which the principal (directions) vectors u_1, u_2 have $k(u_1) = k_1$, $k(u_2) = k_2$ (principal curvatures)



$k(e_1) = 0 = k_1$
$k(e_2) = -1/r = k_2$

$k_1 = k_2 = -1/r$



Def^n/ $p \in M$ is an umbilic pt. if $k(u)$ is constant for all $u \in T_pM$ with $\|u\| = 1$.

S_p is SYMMETRIC!

$$\underbrace{U \cdot \Phi_u}_{\text{function of } v} = 0$$

$\Phi : D \longrightarrow M$ a patch

U the unit-normal to $M = \Phi(D)$

$\Phi_u(u_0, v_0) \in T_{\Phi(u_0,v_0)} M$

$$0 = \frac{\partial}{\partial v}\left[U \cdot \Phi_u \right] = \underbrace{\frac{\partial U}{\partial v}}_{-S(\Phi_v)} \cdot \Phi_u + U \cdot \Phi_{uv}$$

$$-S(\Phi_v) \cdot \Phi_u = -U \cdot \Phi_{uv}$$

$$S(\Phi_v) \cdot \Phi_u = U \cdot \Phi_{uv}$$

$$S(\Phi_u) \cdot \Phi_v = U \cdot \Phi_{vu}$$

$$S(\Phi_u) \cdot \Phi_v = S(\Phi_v) \cdot \Phi_u$$

But, $\{\Phi_u, \Phi_v\}$ forms a basis for T_pM hence

$S_p : T_pM \longrightarrow T_pM$ has $\underbrace{S_p(v) \cdot w = v \cdot S_p(w)}_{S_p \text{ is symmetric linear trans.}}$ $\forall v, w \in T_pM$

$$\Rightarrow \boxed{[S_p]^T = [S_p]}$$

Th^m/ $[S_p] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ w.r.t a <u>principal</u> <u>basis</u>
for p non-umbilic the basis is $\{e_1, e_2\}$ where
$k(e_1) = e_1 k_1$ and $k(e_2) = k_2 e_2$ with $k_1 \geq k_2$
For p umbilic, ~~~~ any orthonormal basis will do.

<u>Proof</u>: $k_1 = k(e_1) = S(e_1) \cdot e_1$. Choose $e_2 \in T_pM$ with $e_2 \cdot e_1 = 0$
$\{e_1, e_2\}$ is orthonormal basis for T_pM. If $u \in T_pM$
with $\|u\| = 1$ then $u = (\underline{u \cdot e_1}) e_1 + (\underline{u \cdot e_2}) e_2$. Let
θ be $\mathbb{R}$-# s.t. $u = (\cos\theta) e_1 + (\sin\theta) e_2$

$$g(\theta) = k(u) = S_p(\cos\theta\, e_1 + \sin\theta\, e_2) \cdot (\cos\theta\, e_1 + \sin\theta\, e_2)$$

Let $[S_p] = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \longleftrightarrow \begin{array}{l} S_p(e_1) = a e_1 + b e_2 \\ S_p(e_2) = b e_1 + c e_2 \end{array}$

$$g(\theta) = \cos^2\theta \underbrace{S_p(e_1) \cdot e_1}_{a} + 2\cos\theta\sin\theta \underbrace{S_p(e_1) \cdot e_2}_{b} + \sin^2\theta \underbrace{S_p(e_2) \cdot e_2}_{c}$$

$$\underline{g(\theta) = a\cos^2\theta + c\sin^2\theta + 2\cos\theta\sin\theta\, b}$$

$k_1 = k(e_1)$ is max-normal curvature

$u = \cos\theta\, e_1 + \sin\theta\, e_2$: $\theta = 0 \hookrightarrow u = e_1 \hookrightarrow g(0)$ is maximum

$$g'(\theta) = 2a(-\sin\theta\cos\theta) + 2c\sin\theta\cos\theta + 2(\cos^2\theta - \sin^2\theta)b$$

$$g'(0) = 0 = 2b \quad \therefore \quad \underline{b = 0}$$

$$g''(\theta) = 2(c - a)\frac{d}{d\theta}(\sin\theta\cos\theta) = 2(c-a)\left[\cos^2\theta - \sin^2\theta\right]$$

$g(0)$ is max $\hookrightarrow g''(0) \leq 0 \rightarrow 2(c-a) \leq 0$

$c - a \leq 0$

$c \leq a$

$\underline{a \geq c}$.

Thus $[S_p] = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ with $a \geq c$.

and $S_p(e_1) = a e_1$ & $S_p(e_2) = c e_2$

thus identify $a = k_1$ and $c = k_2$. //

Cor 2.6 $k(u) = k_1 \cos^2\theta + k_2 \sin^2\theta$ Euler's Formula for normal curvature.

SPECIAL CASE:

$P = (0,0,0)$

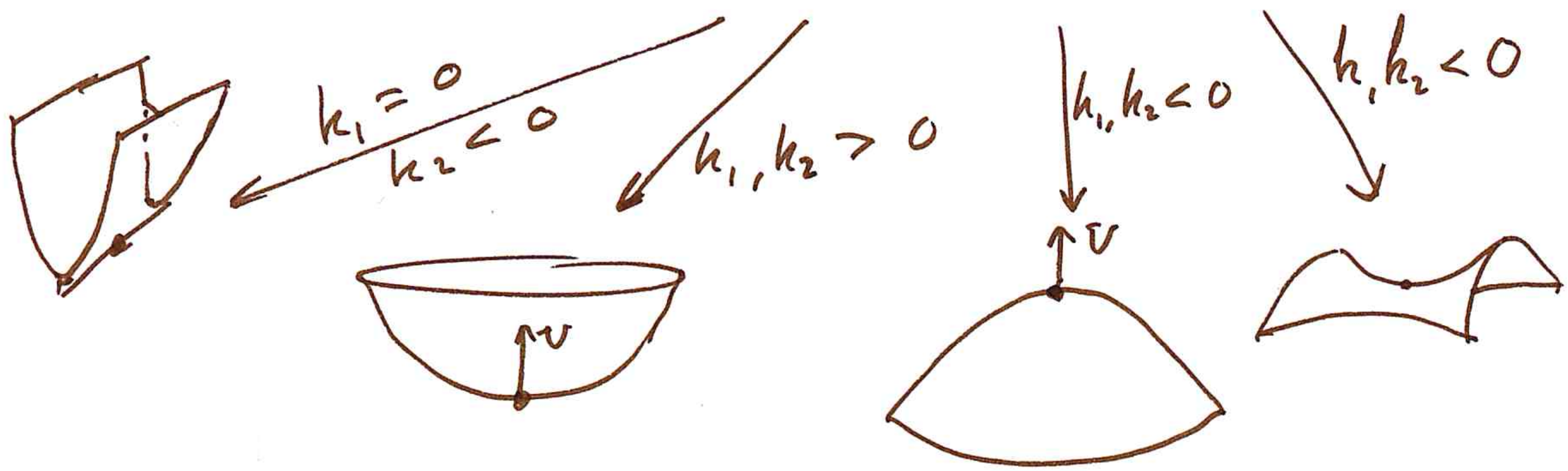
$T_pM = (xy)\text{-plane in } \mathbb{R}^3$

$f(0,0) = f_x(0,0) = f_y(0,0) = 0$

Taylor Expansion: $f(x,y) \approx \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) + \cdots$

Exercise in Text: for graph $z = f(x,y)$ can calculate $k_1 = f_{xx}(0,0)$ and $k_2 = f_{yy}(0,0)$

$$M \approx z = \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right)$$



§5.3 GAUSSIAN CURVATURE

Defⁿ/ for $M \subset \mathbb{R}^3$ with shape operator S

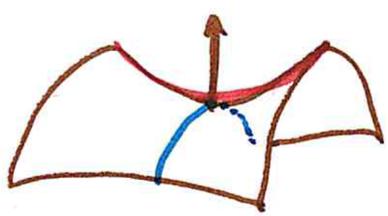
$$K(p) = \det(S_p) = \det[S_p] \quad \longleftarrow \text{GAUSSIAN CURVATURE}$$

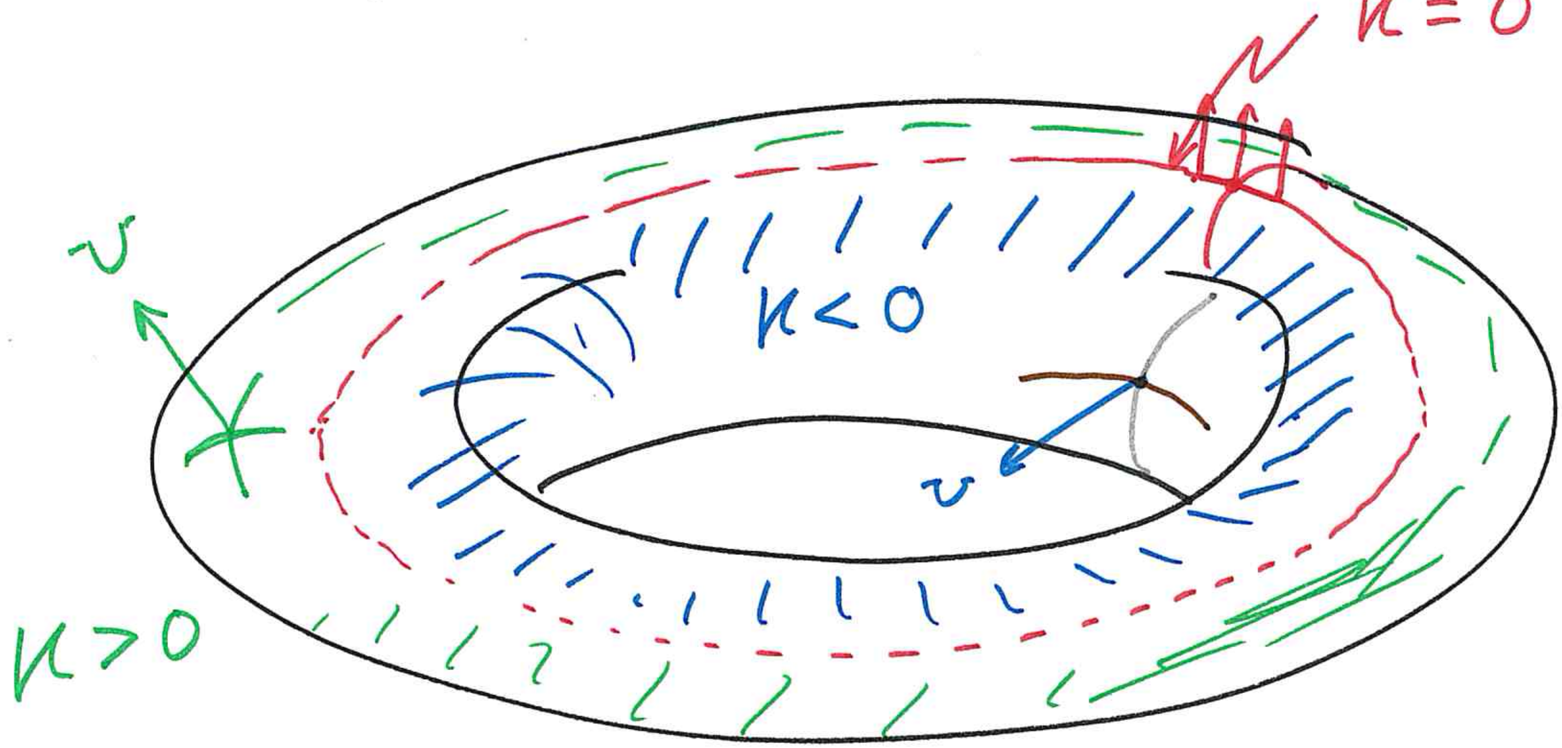
$$H(p) = \frac{1}{2}\operatorname{trace}(S_p) = \frac{1}{2}\operatorname{trace}[S_p] \quad \longleftarrow \text{MEAN CURVATURE}$$

Observation: $[S_p]$ has e-values k_1 & k_2

$$\left.\begin{aligned} \det[S_p] &= k_1 k_2 = K(p) \\ \operatorname{trace}[S_p] &= k_1 + k_2 = 2H(p) \end{aligned}\right\}$$

check indeed this is the relation of e-values of 2x2 matrix to det & trace.

$K > 0$	$K < 0$	$K = 0$
$k_1, k_2 > 0$ $k_1, k_2 < 0$	$k_1 k_2 < 0$	$k_1 = 0$, $k_2 \neq 0$; $k_1 = 0$, $k_2 = 0$ ← planar point
		



$\kappa = 0$
v
$\kappa < 0$
v
$\kappa > 0$

Lemma: $\{v, w\}$ LI in T_pM

$$\underline{S(v) \times S(w) = K(P)\, v \times w} \; *$$

$$\underline{S(v) \times w + v \times S(w) = 2H(P)\, v \times w} \; **$$

Proof $S(v) = av + bw$ $\quad [S] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$S(w) = cv + dw$

$\det(S) = ad - bc$ & $\operatorname{trace}(S) = a + d$

$$\begin{aligned} S(v) \times S(w) &= (av + bw) \times (cv + dw) \\ &= (ad - bc)\, v \times w \\ &= \det(S)\, v \times w \\ &= \underline{K\, v \times w}. \end{aligned}$$

$$\begin{aligned} S(v) \times w + v \times S(w) &= a\, v \times w + v \times (dw) = (a+d)\, v \times w \\ &= 2H\, v \times w. \end{aligned}$$

$$(\vec{x} \times \vec{y}) \cdot (\vec{a} \times \vec{b}) = \det \begin{bmatrix} \vec{x} \cdot \vec{a} & \vec{y} \cdot \vec{a} \\ \vec{x} \cdot \vec{b} & \vec{y} \cdot \vec{b} \end{bmatrix}$$

Take dot-product of * with $v \times w$

$$(\underbrace{S(v)}_{x} \times \underbrace{S(w)}_{y}) \cdot (\underbrace{v}_{a} \times \underbrace{w}_{b}) = \det \begin{bmatrix} S(v) \cdot v & S(w) \cdot v \\ S(v) \cdot w & S(w) \cdot w \end{bmatrix} = K \underbrace{(v \times w) \cdot (v \times w)}_{\|v \times w\|^2}$$

$$(\vec{x}\times\vec{y})\cdot(\vec{a}\times\vec{b}) = \sum_i (\vec{x}\times\vec{y})_i\,(\vec{a}\times\vec{b})_i$$

$$= \sum_{i,j,k,l,m} x_j\, y_k\, a_l\, b_m\, \underbrace{\epsilon_{jki}\,\epsilon_{lmi}}_{\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

$$= \sum_{j,k,l,m} \delta_{jl}\delta_{km}\, x_j y_k a_l b_m - \sum_{j,k,l,m} \delta_{jm}\delta_{kl}\, x_j y_k a_l b_m$$

$$= \Big(\sum_k y_k b_k\Big)\Big(\sum_l x_l a_l\Big) - \Big(\sum_k y_k a_k\Big)\Big(\sum_j x_j b_j\Big)$$

$$= (\vec{y}\cdot\vec{b})(\vec{x}\cdot\vec{a}) - (\vec{y}\cdot\vec{a})(\vec{x}\cdot\vec{b})$$

$$= \det\begin{bmatrix} \vec{x}\cdot\vec{a} & \vec{y}\cdot\vec{a} \\ \vec{x}\cdot\vec{b} & \vec{y}\cdot\vec{b} \end{bmatrix}$$

$$K = \frac{\det \begin{bmatrix} S(v)\cdot v & S(w)\cdot v \\ S(v)\cdot w & S(w)\cdot w \end{bmatrix}}{\det \begin{bmatrix} v\cdot v & v\cdot w \\ v\cdot w & w\cdot w \end{bmatrix}}$$

By almost same calculation, take $\cdot$-product with $v \times w$ of **

$$H = \frac{\det \begin{bmatrix} S(v)\cdot v & S(v)\cdot w \\ w\cdot v & w\cdot w \end{bmatrix} + \det \begin{bmatrix} v\cdot v & v\cdot w \\ S(w)\cdot v & S(w)\cdot w \end{bmatrix}}{2 \det \begin{bmatrix} v\cdot v & w\cdot w \\ v\cdot w & w\cdot w \end{bmatrix}}$$

Cor 3.5 $\boxed{k_1, k_2 = H \pm \sqrt{H^2 - K}}$

See: $K = k_1 k_2$

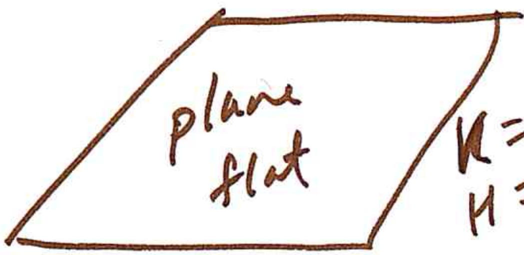
$H = \dfrac{k_1 + k_2}{2}$

$$H^2 - K = \frac{(k_1+k_2)^2}{4} - k_1 k_2 = \frac{k_1^2}{4} + \frac{k_1 k_2}{2} - k_1 k_2 + \frac{k_2^2}{4}$$

$$= \frac{k_1 + 2k_1 k_2 + k_2^2}{4}$$

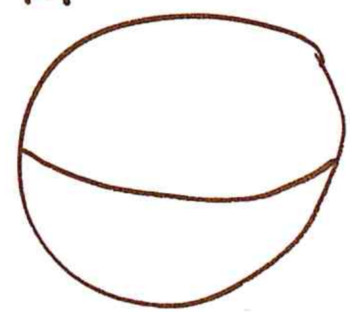
$$= \left(\frac{k_1 + k_2}{2}\right)^2$$

Def^{n}/ A surface M is <u>flat</u> if $K = 0$ on M.
A surface M is <u>minimal</u> if $H = 0$ on M.



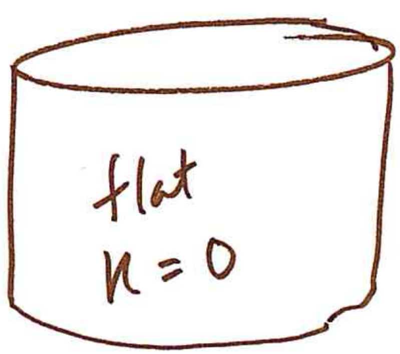
$K = 0$
$H = 0$

$h_1 = h_2 = \frac{-1}{r}$



$K = \frac{1}{r^2}$

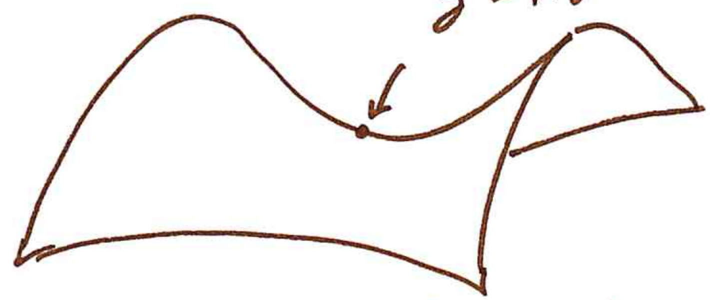
$H = \frac{1}{2}\left(\frac{-1}{r} + \frac{-1}{r}\right) = \frac{-1}{r}$



$h_1 = 0, \; h_2 = \frac{-1}{r}$

$H = \frac{-1}{2r}$

$z = xy$



$[S] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$K = -1, \quad H_{(0,0,0)} = 0$