LECTURE 16: CALCULATIONAL TECHNIQUES

WARPING FUNCTIONS FOR PATCH $\mathbf{X}$

$$\begin{cases} E = \mathbf{X}_u \cdot \mathbf{X}_u = \|\mathbf{X}_u\|^2 \\ F = \mathbf{X}_u \cdot \mathbf{X}_v = \sqrt{E}\sqrt{G}\cos\theta \\ G = \mathbf{X}_v \cdot \mathbf{X}_v = \|\mathbf{X}_v\|^2 \end{cases}$$

Let $v = v_1 \mathbf{X}_u + v_2 \mathbf{X}_v$ and $w = w_1 \mathbf{X}_u + w_2 \mathbf{X}_v$. Observe,

$$\underline{v \cdot w} = (v_1 \mathbf{X}_u + v_2 \mathbf{X}_v) \cdot (w_1 \mathbf{X}_u + w_2 \mathbf{X}_v)$$

$$= v_1 w_1 \mathbf{X}_u \cdot \mathbf{X}_u + 2(v_1 w_2 + v_2 w_1)\mathbf{X}_u \cdot \mathbf{X}_v + v_2 w_2 \mathbf{X}_v \cdot \mathbf{X}_v$$

$$= \underline{v_1 w_1 E + 2(v_1 w_2 + v_2 w_1) F + v_2 w_2 G}$$

By Lagrange's Identity: $\|\mathbf{X}_u \times \mathbf{X}_v\|^2 = \underbrace{(\mathbf{X}_u \cdot \mathbf{X}_u)}_{E}\underbrace{(\mathbf{X}_v \cdot \mathbf{X}_v)}_{G} - \underbrace{(\mathbf{X}_u \cdot \mathbf{X}_v)^2}_{F^2}$

$$U = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = \frac{1}{\underbrace{\sqrt{EG - F^2}}_{W}}(\mathbf{X}_u \times \mathbf{X}_v) \leftarrow \text{unit-normal on } \mathbf{X}(D).$$

$$U = \frac{\mathbf{X}_u \times \mathbf{X}_v}{W}$$

<u>Notation</u>: for $\Phi = (x, y, z)$

$$\left.\begin{aligned}
\Phi_{uu} &= \frac{\partial^2 x}{\partial u^2} U_1 + \frac{\partial y^2}{\partial u^2} U_2 + \frac{\partial z^2}{\partial u^2} U_3 \\
\Phi_{uv} &= \frac{\partial^2 x}{\partial u \partial v} U_1 + \frac{\partial^2 y}{\partial u \partial v} U_2 + \frac{\partial^2 z}{\partial u \partial v} U_3 \\
\Phi_{vv} &= \frac{\partial^2 x}{\partial v^2} U_1 + \frac{\partial^2 y}{\partial v^2} U_2 + \frac{\partial^2 z}{\partial v^2} U_3
\end{aligned}\right\}$$

vector fields along the coordinate curves.

Φ_{uu}, Φ_{vv} are accelerations of

$u \mapsto \Phi(u, v_0)$

$v \mapsto \Phi(u_0, v)$

Def^{n}/

$$\begin{cases} L = S(\Phi_u) \cdot \Phi_u \\ M = S(\Phi_u) \cdot \Phi_v = S(\Phi_v) \cdot \Phi_u \\ N = S(\Phi_v) \cdot \Phi_v \end{cases}$$

<u>PROPOSITION</u>: for the patch Φ,

$$K = \frac{LN - M^2}{EG - F^2} \quad \& \quad H = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

<u>Proof</u>: set $V = \Phi_u$ and $W = \Phi_v$ for formulas toward end of Lecture 15, note $\|V \times W\|^2 = \|\Phi_u \times \Phi_v\|^2 = EG - F^2$ and $S(V) \cdot V = S(\Phi_u) \cdot \Phi_u = L$ & $S(W) \cdot W = N$ etc... the proposition follows.// (see page 17 of LECTURE 15)

Lemma:
$$\left.\begin{cases} L = S(\mathbf{x}_u)\cdot \mathbf{x}_u = U\cdot \mathbf{x}_{uu} \\ M = S(\mathbf{x}_u)\cdot \mathbf{x}_v = U\cdot \mathbf{x}_{uv} \\ N = S(\mathbf{x}_v)\cdot \mathbf{x}_v = \overset{\text{unit-normal}}{U}\cdot \mathbf{x}_{vv} \end{cases}\right\} \text{nice formulas for } L, M, N$$

Proof: See LECTURE 15 or, pg. 226 O'neill. Follows from curve def^{n} of cov. derivative & product rule. (we used these to derive symmetric property for Shape Operator)

Example(1) (Helicoid)

$$\mathbf{x}(u,v) = (u\cos v,\ u\sin v,\ bv)$$

$\mathbf{x}_u = \langle \cos v, \sin v, 0\rangle$

$\mathbf{x}_v = \langle -u\sin v, u\cos v, b\rangle$

$\mathbf{x}_{uu} = \langle 0, 0, 0\rangle$

$\mathbf{x}_{uv} = \langle -\sin v, \cos v, 0\rangle$

$\mathbf{x}_{vv} = \langle -u\cos v, -u\sin v, 0\rangle$

$E = 1$

$F = 0$

$G = u^2 + b^2$

Note $\mathbf{x}_u \times \mathbf{x}_v = \langle b\sin v, -b\cos v, u\rangle$ (can check $\perp$ to $\mathbf{x}_u, \mathbf{x}_v$)

Thus $U = \dfrac{1}{\sqrt{u^2+b^2}}\langle b\sin v, -b\cos v, u\rangle$

Hence $L = 0$, $M = U\cdot \mathbf{x}_{uv} = \dfrac{1}{\sqrt{u^2+b^2}}\overbrace{\langle b\sin v, -b\cos v, u\rangle\cdot\langle -\sin v, \cos v, 0\rangle}^{-b(\sin^2 v + \cos^2 v)}$

$U \perp \mathbf{x}_{vv} \Rightarrow N = 0$ $\qquad \therefore M = \dfrac{-b}{\sqrt{u^2+b^2}}$

continued

Helicoid $\mathbf{x}(u,v) = \langle u\cos v,\ u\sin v,\ bv \rangle$ continued $(b \neq 0)$

Calculated
$$\left\{\begin{array}{ll} L = 0 & E = 1 \\ M = \dfrac{-b}{\sqrt{b^2+u^2}} & F = 0 \\ N = 0 & G = b^2+u^2 \end{array}\right\}$$

Thus, substituting into the alphabet soup f-la's for K & H,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - \frac{b^2}{b^2+u^2}}{b^2+u^2} = \frac{-b^2}{(b^2+u^2)^2}.$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{0+0-0}{2(b^2+u^2)} = 0.$$

- Helicoid is minimal surface with $\underbrace{K = \frac{-b^2}{(b^2+u^2)^2}}$ makes much sense if think about figure 5.23 in O'neill.

Principal Curvatures

$$H \pm \sqrt{H^2 - K} = \pm\sqrt{\frac{b^2}{(b^2+u^2)^2}} = \underbrace{\frac{\pm b}{b^2+u^2}}_{k_1 \,\&\, k_2}$$

Ex 2 $\mathbf{X}(u,v) = (R\cos u,\ R\sin u,\ v)$: find K & H for Cylinder

$\mathbf{X}_u = \langle -R\sin u,\ R\cos u,\ 0\rangle$ $\qquad E = \mathbf{X}_u \cdot \mathbf{X}_u = R^2$

$\mathbf{X}_v = \langle 0,\ 0,\ 1\rangle$ $\qquad F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$

$\qquad G = \mathbf{X}_v \cdot \mathbf{X}_v = 1$

Next we calculate,

$\mathbf{X}_{uu} = \langle -R\cos u,\ -R\sin u,\ 0\rangle$ $\qquad L = \mathbf{U} \cdot \mathbf{X}_{uu} = -R$

$\mathbf{X}_{uv} = \langle 0,\ 0,\ 0\rangle$ $\qquad M = \mathbf{U} \cdot \mathbf{X}_{uv} = 0$

$\mathbf{X}_{vv} = \langle 0,\ 0,\ 0\rangle$ $\qquad N = \mathbf{U} \cdot \mathbf{X}_{vv} = 0$

$$\mathbf{U} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = \frac{\langle R\cos u,\ R\sin u,\ 0\rangle}{R} = \langle \cos u, \sin u, 0\rangle$$

Thus,

$$\underbrace{K = \frac{LN - M^2}{EG - F^2} = 0}_{\boxed{K = 0}} \quad \& \quad \underbrace{H = \frac{1(-R) + R^2(0) - 2(0)(0)}{2R^2}}_{\boxed{H = \frac{-1}{2R}}}$$

(flat)

Ex 3 Saddle Surface $M: z = xy$

Following O'neill's Ex (2) on 229-230. Use Monge Patch $\mathbf{X}(u,v) = (u, v, uv)$

$$\left.\begin{aligned}
\mathbf{X}_u &= \langle 1, 0, v \rangle \\
\mathbf{X}_v &= \langle 0, 1, u \rangle \\
U &= \frac{1}{\sqrt{1+u^2+v^2}} \langle -v, -u, 1 \rangle \\
\mathbf{X}_{uu} &= \mathbf{X}_{vv} = \langle 0, 0, 0 \rangle \\
\mathbf{X}_{uv} &= \langle 0, 0, 1 \rangle
\end{aligned}\right\}
\qquad
\begin{aligned}
E &= \mathbf{X}_u \cdot \mathbf{X}_u = 1 + v^2 \\
F &= \mathbf{X}_u \cdot \mathbf{X}_v = uv \\
G &= \mathbf{X}_v \cdot \mathbf{X}_v = 1 + u^2 \\
L &= \mathbf{X}_{uu} \cdot U = 0 \\
M &= \mathbf{X}_{uv} \cdot U = \frac{1}{\sqrt{1+u^2+v^2}} \\
N &= \mathbf{X}_{vv} \cdot U = 0
\end{aligned}$$

Calculate $EG - F^2 = (1+v^2)(1+u^2) - (uv)^2 = 1 + u^2 + v^2$. Hence,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - \frac{1}{1+u^2+v^2}}{1+u^2+v^2} = \frac{-1}{(1+u^2+v^2)^2}$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{-2uv/\sqrt{1+u^2+v^2}}{2(1+u^2+v^2)} = \frac{-uv}{(1+u^2+v^2)^{3/2}}$$

Can convert to Cartesian formulas

$$\boxed{K = \frac{-1}{(1+x^2+y^2)^2}} \quad \& \quad \boxed{H = \frac{-z}{(1+x^2+y^2)^{3/2}}}$$

#8, #9, #10 of §5.4 look interesting.

#8) A PATCH X IS ORTHOGONAL IF $F = X_u \cdot X_v = 0$

(a) Show $S(X_u) = \frac{L}{E} X_u + \frac{M}{G} X_v$ and $S(X_v) = \frac{M}{E} X_u + \frac{N}{G} X_v$

Orthogonality of basis $\{X_u, X_v\}$ of T_pM is very nice

$$S(X_u) = \frac{(S(X_u)\cdot X_u)\,X_u}{X_u\cdot X_u} + \frac{(S(X_u)\cdot X_v)\,X_v}{X_v\cdot X_v} = \frac{L}{E} X_u + \frac{M}{G} X_v.$$

$$S(X_v) = \frac{(S(X_v)\cdot X_u)\,X_u}{X_u\cdot X_u} + \frac{(S(X_v)\cdot X_v)\,X_v}{X_v\cdot X_v} = \frac{M}{E} X_u + \frac{N}{G} X_v.$$

(#8b)

PATCH IS PRINCIPAL IF $F = M = 0$, SHOW X_u, X_v principal for a principal patch

Using the formula's derived for part (a) with $M = 0$,

$$S(X_u) = \frac{L}{E} X_u \quad \text{and} \quad S(X_v) = \frac{N}{G} X_v$$

Hence $\underline{k_1, k_2 = \frac{L}{E}, \frac{N}{G}}$. (not sure which is max. w/o further info.)

Also, $K = \frac{LN}{EG}$ and $H = \left(\frac{L}{E} + \frac{N}{G}\right)\left(\frac{1}{2}\right)$.

§5.5 THE IMPLICIT CASE

Suppose Z is nonvanishing normal vector field to $M \subseteq \mathbb{R}^3$ for example, $g(x,y,z) = 0$ defining M has $Z = \nabla g = \sum_i \frac{\partial g}{\partial x^i} U_i$

We wish to derive formulas for Z to derive K & H etc.

$$U = \frac{1}{\|Z\|} Z \quad \text{is unit-normal}$$

$$\nabla_V U = \nabla_V \left(\frac{1}{\|Z\|} Z \right) = \frac{1}{\|Z\|} (\nabla_V Z) + V\left[\frac{1}{\|Z\|}\right] Z$$

$$\boxed{S(V) = \frac{-1}{\|Z\|} \nabla_V Z - V\left[\frac{1}{\|Z\|}\right] Z}$$

We saw, $S(V) \times S(W) = K\, V \times W$ & $S(V) \times W + V \times S(W) \mp 2H\, V \times W$

Let V, W be vector fields tangent to M with $Z = V \times W$

$$K (V \times W) \cdot (V \times W) = (S(V) \times S(W)) \cdot (V \times W)$$

$$K \|Z\|^2 = \left(\frac{-1}{\|Z\|} \nabla_V Z + b_V Z \right) \times \left(\frac{-1}{\|Z\|} \nabla_W Z + b_W Z \right) \cdot Z$$

$$= \left(\frac{1}{\|Z\|^2} (\nabla_V Z) \times (\nabla_W Z) \right) \cdot Z \qquad \underbrace{(A \times Z) \cdot Z = 0}_{\text{other terms vanish.}}$$

$$\therefore \quad \boxed{K = \frac{[(\nabla_V Z) \times (\nabla_W Z)] \cdot Z}{\|Z\|^4}}$$

Similarly we also derive,

$$\boxed{H = \frac{-1}{2\|Z\|^3} Z \cdot \left(\nabla_V Z \times W + V \times \nabla_W Z \right)}$$

to use these formulas for H & K we need a normal field Z AND $V, W \in \Gamma(M)$ for which $V \times W = Z$ on M.

Example (5.2 on pg. 237 of O'neill)

$$M : g = x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

$$Z = \frac{1}{2} \nabla g = \frac{x}{a^2} U_1 + \frac{y}{b^2} U_2 + \frac{z}{c^2} U_3 = \sum_{i=1}^{3} \frac{x^i}{(a^i)^2} U_i \qquad \begin{pmatrix} a^1 = a \\ a^2 = b \\ a^3 = c \end{pmatrix}$$

$$V\left[\frac{x}{a^2}\right] = \frac{1}{a^2} V[x] = \frac{V^1}{a^2}$$

$$\nabla_V Z = V\left[\frac{x}{a^2}\right] U_1 + V\left[\frac{y}{b^2}\right] U_2 + V\left[\frac{z}{c^2}\right] U_3 = \sum_{i=1}^{3} \frac{V^i}{(a_i)^2} U_i$$

Likewise for W, thus calculate

$$Z \cdot (\nabla_V Z) \times (\nabla_W Z) = \det \begin{bmatrix} x/a^2 & y/b^2 & z/c^2 \\ V^1/a^2 & V^2/b^2 & V^3/c^2 \\ W^1/a^2 & W^2/b^2 & W^3/c^2 \end{bmatrix} = \frac{1}{a^2 b^2 c^2} \underbrace{\det \begin{bmatrix} x & y & z \\ V^1 & V^2 & V^3 \\ W^1 & W^2 & W^3 \end{bmatrix}}_{Z \cdot (V \times W)}$$

$$Z = \langle x, y, z \rangle$$

$$K = \frac{Z \cdot (\nabla_V Z) \times (\nabla_W Z)}{\|Z\|^4} = \frac{x^2/a^2 + y^2/b^2 + z^2/c^2}{a^2 b^2 c^2 \|Z\|^4}$$

Continuing, as $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$,

$$K = \frac{1}{a^2 b^2 c^2 \|Z\|^4} \qquad Z = \frac{x}{a^2}U_1 + \frac{y}{b^2}U_2 + \frac{z}{c^2}U_3$$

$$\boxed{K = \frac{1}{a^2 b^2 c^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}}$$

Remark: we never found explicit V & W for which $V \times W = Z$. Interesting.

Set $a = b = c = R$ and $M : x^2 + y^2 + z^2 = R^2$

$$K = \frac{1}{R^6\left(\frac{x^2+y^2+z^2}{R^4}\right)^2} = \frac{1}{R^6\left(\frac{1}{R^2}\right)^2} = \boxed{\frac{1}{R^2} = K}$$

Remark: for $M : F(x,y,z) = 0$ there is also a known formula

Hessian Matrix.

$$K = \frac{-1}{\|\nabla F\|^4} \det \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{bmatrix}$$

WHY?