LECTURE 17: SPECIAL CURVES (§5.6) & SURFACES OF REVOLUTION (§5.7)

1.) **PRINCIPAL CURVE**: α regular in M with α' always pointing in a principal direction

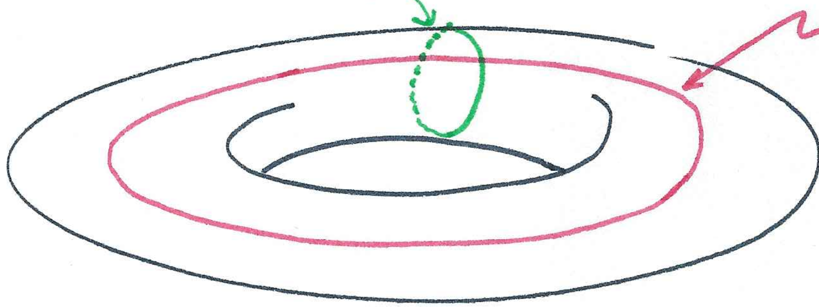
2.) **ASYMPTOTIC CURVE**: α regular curve in M with α' always pointing in an asymptotic direction (normal curvature zero)

3.) **GEODESIC CURVE**: α with α'' always normal to M. again assume α regular.

These are not mutually exclusive. There is much to say, 1st an example or two

principal & geodesic

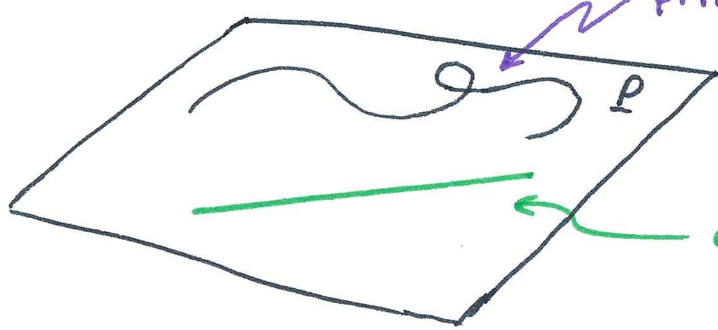
TORUS



top of torus has $k(\alpha') = 0$ this is asymptotic curve

PLANE

principal & asymptotic, not geodesic



$\alpha'' = 0 \leftarrow$ normal to M
lines are geodesics of plane P

Lemma (6.2): Let α be regular curve and U the unit normal of M restricted to α in M

(1.) α principal $\iff$ U' and α' are colinear at each point

(2.) α principal $\Rightarrow$ principal curvature in α'-direction is $\dfrac{\alpha'' \bullet U}{\alpha' \bullet \alpha'}$

(1) Proof: $\S$ α principal then $S(\alpha') = k\alpha'$ along α

But, $S(\alpha') = -\nabla_{\alpha'} U = \underbrace{-U'}_{\text{derivative of } U_\alpha}$ hence U' and α' are colinear.

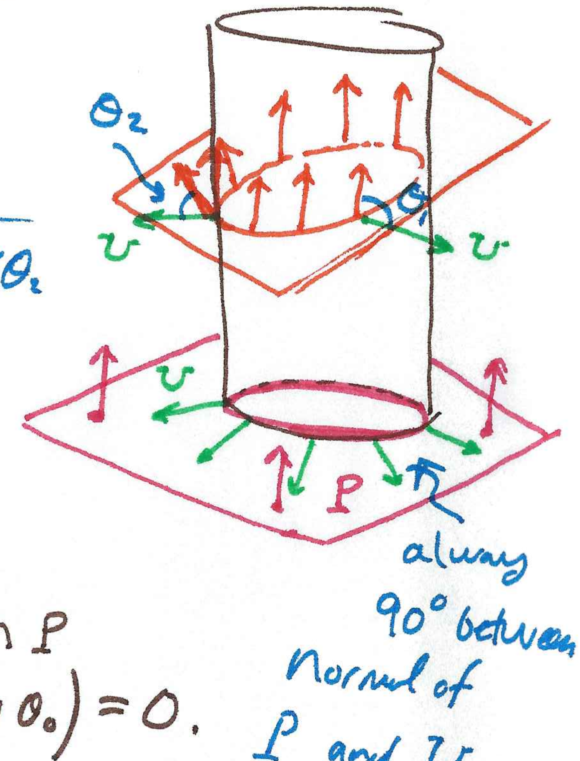
Conversely, if $U' = -k\alpha' \Rightarrow S(\alpha') = k\alpha' \Rightarrow \alpha$ principal.

(2) Proof: $\S$ α principal curve $\Rightarrow S(\alpha') = k\alpha'$

However, $\alpha' \bullet S(\alpha') = \alpha'' \bullet U$ by Lemma 2.1 of ($\S$5.2 pg. 209)

Hence $k\,\alpha' \bullet \alpha' = \alpha'' \bullet U \quad \therefore \quad k = \dfrac{\alpha'' \bullet U}{\alpha' \bullet \alpha'}$ //

Lemma 6.3: Let α be curve formed by intersection of surface M and plane P. If the angle between M and P is constant along α then α is principal curve of M



PROOF: Let U be normal to M
Let V be normal to P (V constant vector field)

Both U & V are assumed to be of unit-length; $\|U\| = \|V\| = 1$. If θ_0 is the fixed angle between U & V along $\alpha = M \cap P$ then $U \cdot V = \cos\theta_0$ along α; $\frac{d}{dt}(U \cdot V) = \frac{d}{dt}(\cos\theta_0) = 0$. Product Rule,

$$U_\alpha' \cdot V + U \cdot \cancel{V_\alpha'}^{\,0} = 0 \quad \& \quad U_\alpha \cdot U_\alpha' = 0 \hookrightarrow \underline{U_\alpha' \cdot U = 0}$$

~~[illegible]~~ (no need to reprove Lemma 6.2 again)
~~[illegible]~~

Likewise $\alpha' \cdot V = 0$ and $\alpha' \cdot U = 0$ as α lies in both M and P its tangent is $\perp$ to normals of both. If U & V are LI then it follows α' colinear to U' hence by Lemma 6.2 α' is principal

Remark: 1 if U, V are LI then $U \times V \neq 0$ and

if $U' \perp U, V \Rightarrow U'$ colinear to $U \times V$ }
if $\alpha' \perp U, V \Rightarrow \alpha'$ colinear to $U \times V$ } $\rightarrow$ U' colinear to α'

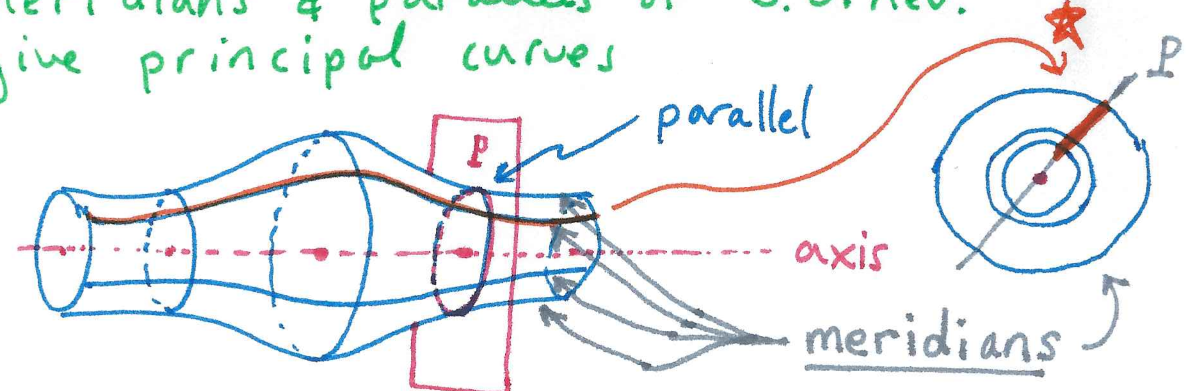
2 On the other hand if $U = \pm V$ then U is also constant since V is plane-normal $\Rightarrow S(\alpha') = 0$

Hence α' is principal with $k(\alpha') = 0$ (α unit-speed) or $k\left(\frac{\alpha'}{\|\alpha'\|}\right) = 0$ if allow nonunit speed... sorry for this boring comment :))

<u>Remark</u>: to apply Lemma 6.3 we need a <u>special</u> type of plane P with respect to M, it must make constant angle θ_0 between the normals of M & P. $\rightarrow$ meridians & parallels of S.O. Rev. give principal curves

I'm not attempting the P to obtain meridian

well, I give some idea from the other perspective ☆

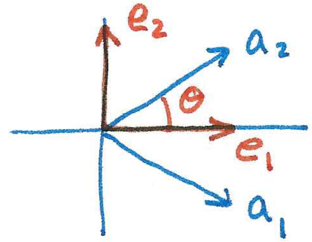


Asymptotic Curves

Lemma (6.4): Let $p \in M \subset \mathbb{R}^3$ and $K(p) = \det S_p$ the Gaussian Curvature,

(1.) If $K(p) > 0$ then $\nexists$ asymptotic directions at p

(2.) If $K(p) < 0$ then $\exists$ exactly two asymptotic directions at p (a_1 & a_2)



which are bisected by the principal ~~dir~~ directions at angle θ as shown with $\tan^2\theta = \dfrac{-k_1(p)}{k_2(p)}$

(3.) If $K(p) = 0$ then all directions are asymptotic if p is planar pt. otherwise $\exists$ exactly one asymptotic direction which is also principal

Proof: Asymptotic α has $k(\alpha') = 0$ along unit-speed α. Recall $K = k_1 k_2$ and Euler's formula $k(u) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta$ where $k_1 = k(e_1)$, $k_2 = k(e_2)$

(1.) $K(p) > 0 \Rightarrow k_1, k_2 > 0$ or $k_1, k_2 < 0 \Rightarrow k(u) \neq 0 \ \forall \theta$.

(2.) $K(p) < 0 \Rightarrow 0 = k_1\cos^2\theta + k_2\sin^2\theta \Rightarrow \tan^2\theta = \frac{-k_1}{k_2} \therefore \theta = \pm\tan^{-1}\sqrt{\frac{-k_1}{k_2}}$.

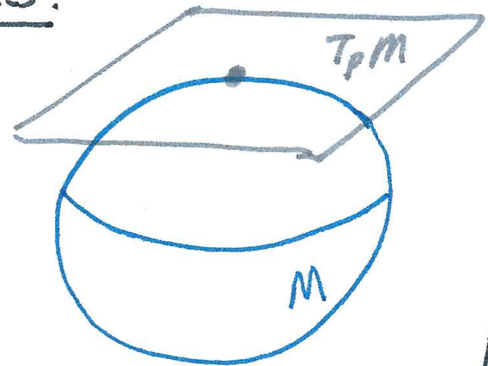
the domain of $\tan^{-1}$ places θ as shown a_1 & a_2 above & below $\underbrace{\theta = 0}_{e_1\text{-direction}}$.

(3.) p planar $\Rightarrow k_1 = k_2 = 0 \therefore k(u) = 0 \ \forall u$.

otherwise, say $k_2(p) = 0 \Rightarrow k(u) = k_1(p)\cos^2\theta = 0$ only for $\underbrace{\theta = \frac{\pi}{2}}_{e_2\text{-direction}}$.

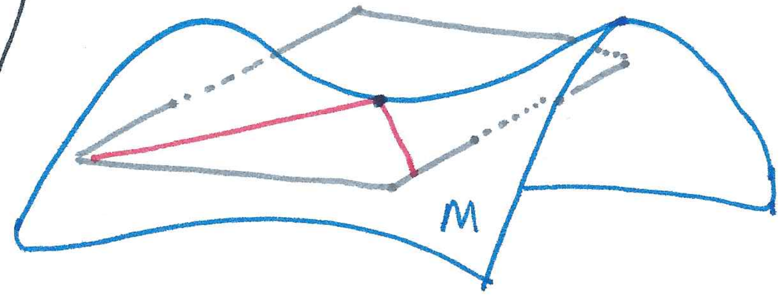
EXAMPLES:

SPHERE



(no intersection, no asymptotic directions.)

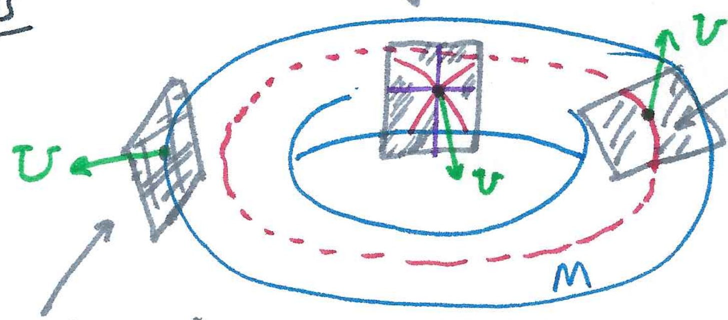
SADDLE SURFACE



$z = xy$ intersects $z = 0$ $T_{(0,0,0)}M$ along x & y axes which are the asymptotic directions at $p = (0,0,0)$

TORUS

two asymptotic directions where $K < 0$
bisected princ. directions.



Gives one asymptotic direction at $K = 0$ non-planar point
M & T_pM intersect along the top curve.

no intersection of M & T_pM

Def

Def$^{\underline{n}}$/ A regular curve α in $M \subset \mathbb{R}^3$ is an <u>asymptotic</u> <u>curve</u> if α' is always pointing in an asymptotic direction (that is, $k(\alpha') = 0$)

Recall, $S(\alpha') \cdot \alpha' = \alpha'' \cdot U$ (we learned this in §5.2)

However, $k(\alpha') = S(\alpha') \cdot \alpha'$ by def$^{\underline{n}}$ of normal curvature

Thus α asymptotic $\Rightarrow k(\alpha') = 0 \Rightarrow \underline{S(\alpha') \cdot \alpha' = \alpha'' \cdot U = 0}$.

<u>Remark:</u> α asymptotic $\Rightarrow$ acceleration α'' is <u>tangent</u> to M ($\alpha'' \cdot U = 0$)

Also, as $S(\alpha') = -\underbrace{U'}_{\text{along } \alpha}$ we have the criterion $U' \cdot \alpha' = 0$

see Helicoid Example from LECTURE 16 $L = 0$ and $N = 0$ show coord. curves have acceler. tangent to M.

<u>Proposition:</u> M is minimal iff there exist <u>two</u> <u>orthogonal</u> asymptotic directions at each point on M

<u>Proof:</u> If M minimal then $k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$ thus $k(u) = k_1(\cos^2\theta - \sin^2\theta) \Rightarrow \theta = \pm \pi/4$ gives $a_1, a_2 = u$ with $\frac{\pi}{2}\angle$ between thus $a_1 \cdot a_2 = 0$

Conversely, if there exist a_1, a_2 with $a_1 \cdot a_2 = 0$ and $k(a_1) = k(a_2) = 0$. Let a_1 be at θ_1 and $\underline{a_2 \text{ at } \theta_2}$, $\theta_2 = \theta_1 + \frac{\pi}{2}$

$$0 = k_1 \cos\theta_1 + k_2 \sin\theta_1$$
$$0 = k_1 \cos\theta_2 + k_2 \sin\theta_2 = -k_1 \sin\theta_1 + k_2 \cos\theta_1 \Rightarrow k_1 = -k_2$$
$$\therefore H = 0. //$$

Def^n/ A **RULED** surface is swept out by a line moving in $\mathbb{R}^3$/

→ Helicoid

→ Saddle Surface

I'd be pretty to have the line sweeping animated.

Lemma 6.6: A RULED surface M has Gaussian curvature $K \leq 0$. Furthermore, $K = 0$ iff the unit-normal U is parallel along each ruling of M

Proof: Since ruled surfaces contain lines $\alpha(t) = p + tq$ and $\alpha'' = 0 \Rightarrow \alpha$ asymptotic $\Rightarrow K \leq 0$ by Lemma 6.4 (see 5)

Suppose $\alpha(t) = p + tq$ is line (ruling) in M and $U \parallel$ along α

then $S(\alpha') = -U'_\alpha = 0 \Rightarrow k(\alpha') = 0$ and α principal $\therefore K = 0$.

Likewise, if $K = 0$ then by Lemma 6.4 case (3) asymptotic curves are principal $\Rightarrow S(\alpha') = -U' = 0$

thus U is parallel to ruling α. (see pg. 245 for uncluttered statement, sorry.)

Silly application: spheres and torus' are not ruled. (they have $K > 0$ somewhere or everywhere)

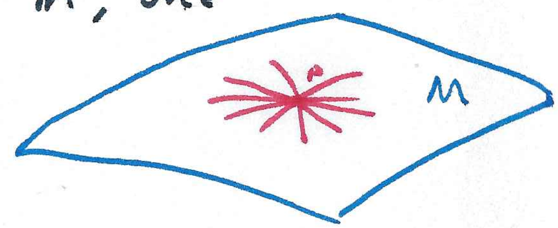
GEODESICS OF M

Def^{n}/ A curve α in $M \subset \mathbb{R}^3$ is <u>geodesic</u> of M provided its acceleration α'' is always normal to M

Comments

1. • "inhabitants" of M "feel" no acceleration along geodesic α as α'' is off-world.

2. • later we'll learn geodesics, like lines in plane, give length minimizing paths on M. Also there are <u>many</u> geodesics through a point on M, one for each choice of initial velocity.



3. • $\alpha'' \perp T_pM \rightarrow \alpha'' \cdot \alpha' = 0$

$$\rightarrow \frac{d}{dt}(\alpha' \cdot \alpha') = 2\alpha'' \cdot \alpha' = 0$$

$\therefore \|\alpha'\|$ is constant.

<u>geodesics</u> <u>have</u> <u>constant</u> <u>speed</u>.

4. • constant curves are trivially geodesic, usually omit to reduce clutter.

5. • lines have $\alpha'' = 0$ hence line in M will be geodesic

EXAMPLES OF GEODESICS

- Planes: if α is geodesic to plane with normal U $\Rightarrow \alpha' \cdot U = 0$

$$\Rightarrow \alpha'' \cdot U = 0$$

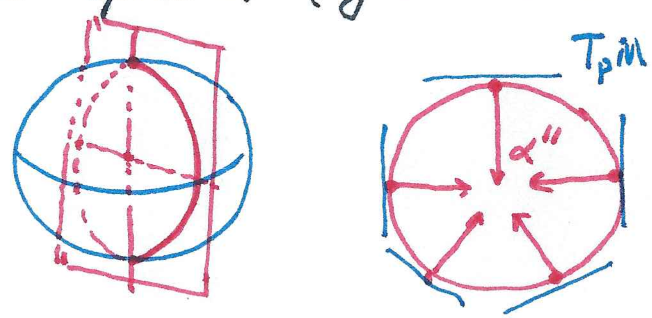
$$\Rightarrow \alpha'' = 0$$

$$\Rightarrow \alpha \text{ a line}$$

also know α'' colinear with U

- SPHERES: the geodesics are great circles.

a great circle is formed by intersection of plane through origin and the sphere. (given constant speed parametrization)



- CYLINDERS: for $x^2+y^2=R^2$ the geodesics have form

$$\alpha(t) = (R\cos(at+b), R\sin(at+b), ct+d) \quad \text{(derived ?)}$$

$$\alpha''(t) = -a^2R\langle \cos(at+b), \sin(at+b), 0\rangle = -a^2R\underbrace{U(\alpha(t))}_{\text{unit-normal to cylinder}}$$

$a=0$ — RULING — $\alpha(t) = (R\cos b, R\sin b, ct+d)$

— $a, c \neq 0$ gives helix —

$c=0$ — CIRCLE — $\alpha(t) = (R\cos(at+b), R\sin(at+b), d)$

Derivation of Cylinder Geodesic

$$\alpha(t) = (R\cos\theta, R\sin\theta, H) \leftarrow \theta, H \text{ functions of } t$$

$$\underbrace{\alpha''(t) = ((R\cos\theta)'', (R\sin\theta)'', H'')}_{}$$

$$U = \cos\tilde{\theta}\, U_1 + \sin\tilde{\theta}\, U_2 \implies H'' = 0 \therefore \underline{H = ct + d}.$$

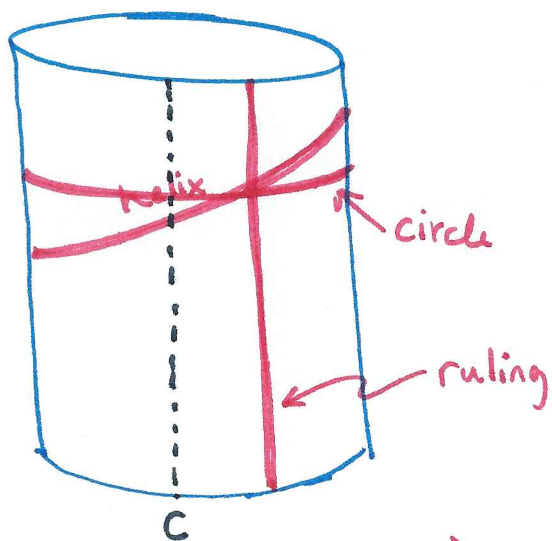
Also, $\alpha'(t) = \langle -R\sin\theta\,\theta', R\cos\theta\,\theta', H' \rangle$

$$\|\alpha'(t)\|^2 = R^2(\sin^2\theta + \cos^2\theta)(\theta')^2 + c^2 = \underline{\text{constant}}$$

$$R^2(\theta')^2 = \text{constant}$$

$$\theta' = \text{constant}$$

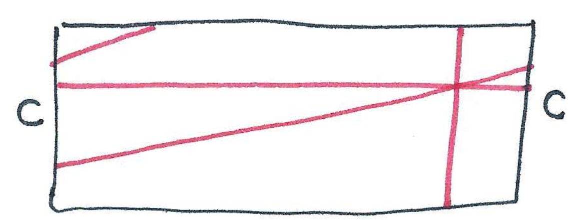
$$\underline{\theta = at + b}. \quad \therefore \underline{\alpha(t) = (R\cos(at+b), R\sin(at+b), ct+d)}.$$



not quite

I'll try again in video.

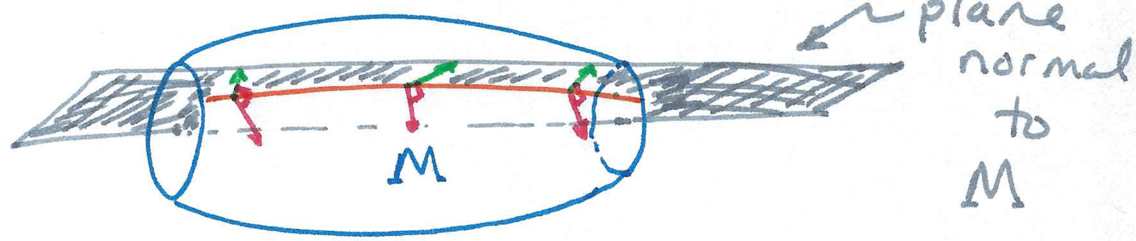
Remark: if we cut the cylinder along C then lay it flat then the ruling, circle and helix are just <u>lines</u>



Proposition: If a unit-speed curve α in M lies in a plane P everywhere orthogonal to M along α then α is geodesic to M

Proof: $\alpha' \cdot \alpha' = 1$ and
thus $\alpha' \cdot \alpha'' = 0$ and both α', α'' lie in P
and as α' is tangent to $M \Rightarrow \alpha''$ is orthogonal to M
$\Rightarrow \alpha$ geodesic. //

Application: meridians are geodesics to surface of revolution.



§5.7: SURFACES OF REVOLUTION:

Study patch $\mathbf{X}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$ for $h(u) > 0$

$$\mathbf{X}_u = \langle g', h'\cos v, h'\sin v \rangle \qquad \xrightarrow{\mathbf{X}_u \cdot \mathbf{X}_u} E = (g')^2 + (h')^2$$

Square of speed of profile curve

$$\mathbf{X}_v = \langle 0, -h\sin v, h\cos v \rangle \qquad \xrightarrow{\mathbf{X}_u \cdot \mathbf{X}_v} F = 0 \qquad \xrightarrow{\mathbf{X}_v \cdot \mathbf{X}_v} G = h^2$$

Likewise calculate,

$$\mathbf{X}_u \times \mathbf{X}_v = \langle hh', -hg'\cos v, -hg'\sin v \rangle$$

$$\| \mathbf{X}_u \times \mathbf{X}_v \| = h\sqrt{(g')^2 + (h')^2} \quad \left(\text{using } \sqrt{EG - F^2}\right)$$

$$U = \frac{1}{\sqrt{(g')^2 + (h')^2}} \langle h', -g'\cos v, -g'\sin v \rangle \quad (\text{the } h\text{'s cancel})$$

$$\mathbf{X}_{uu} = \langle g'', h''\cos v, h''\sin v \rangle \qquad \xrightarrow{\mathbf{X}_{uu} \cdot U} L = \frac{-g'h'' + g''h'}{\sqrt{(g')^2 + (h')^2}}$$

$$\mathbf{X}_{uv} = \langle 0, -h'\sin v, h'\cos v \rangle \qquad \xrightarrow{\mathbf{X}_{uv} \cdot U} M = 0$$

$$\mathbf{X}_{vv} = \langle 0, -h\cos v, -h\sin v \rangle \qquad \xrightarrow{\mathbf{X}_{vv} \cdot U} N = \frac{g'h}{\sqrt{(g')^2 + (h')^2}}$$

THUS $\mathbf{X}$ IS PRINCIPAL PATCH SO WE HAVE: (see LECTURE 16)

$$S(\mathbf{X}_u) = \frac{L}{E}\mathbf{X}_u \quad \text{and} \quad S(\mathbf{X}_v) = \frac{N}{G}\mathbf{X}_v$$

But, $u \mapsto \mathbf{X}(u, v_0)$ and $v \mapsto \mathbf{X}(u_0, v)$ are meredians & circles of M and we can identify $k_\mu = \frac{L}{E}$ & $k_\pi = \frac{N}{G}$ ↲

We found,

$$\underbrace{k_\mu = \frac{L}{E} = \frac{-g'h'' + g''h'}{((g')^2 + (h')^2)^{3/2}}}_{\text{principal curvature along } \mathbb{X}_u} \quad \& \quad \underbrace{k_\pi = \frac{N}{G} = \frac{g'}{h((g')^2 + (h')^2)^{1/2}}}_{\text{principal curvature along } \mathbb{X}_v}$$

Then as $K = k_\mu k_\pi$,

$$K = \frac{-g' \begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h((g')^2 + (h')^2)^2}$$

determinant.

function of u alone, K constant as v-varies over parallels.

If we specialize to case $g(u) = u$ then,

$$K = \frac{-h''}{h(1 + (h')^2)^2}$$

Example: TORUS: for $0 < r < R$

$$g(u) = r\sin u$$
$$h(u) = R + r\cos u$$

$$E = \dot{g}^2 + \dot{h}^2 = r^2$$
$$F = 0$$
$$G = (R + r\cos u)^2$$
$$L = \frac{-(r\cos u)(-r\cos u) - (r\sin u)(-r\sin u)}{r} = r$$
$$M = 0$$
$$N = \frac{(r\cos u)(R + r\cos u)}{r} = R\cos u + r\cos^2 u$$

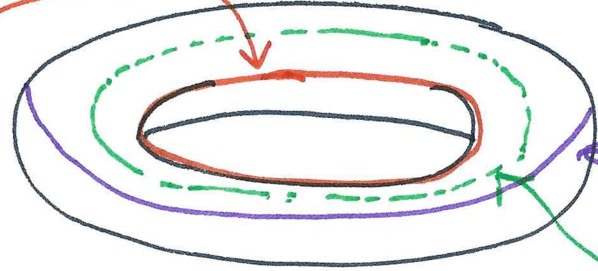
Then,

$$k_u = \frac{L}{E} = \frac{1}{r} \quad \text{and} \quad k_v = \frac{N}{G} = \frac{R\cos u + r\cos^2 u}{(R + r\cos u)^2} = \frac{\cos u}{R + r\cos u}$$

$$\therefore \boxed{K = \frac{\cos u}{r(R + r\cos u)}}$$

$$\mathbf{X}(u,v) = \big(r\sin u,\ (R + r\cos u)\cos v,\ (R + r\cos u)\sin v\big)$$

$\boxed{K = \dfrac{-1}{r(R-r)}}$ $u = \pi$ inside equator



$u = 0$ outside equator : $\boxed{K = \dfrac{1}{r(R+r)}}$

top circle, $u = \frac{\pi}{2}$, $\boxed{K = 0}$

($K = 0$ on bottom circle $u = -\frac{\pi}{2}$ also)

Example: Catenoid

$$y = c\cosh(x/c) \longrightarrow \begin{cases} H = 0 \\ K = \dfrac{-1}{c^2\cosh^4(x/c)} \end{cases}$$

$\boxed{\text{Th}^{\text{m}}\text{/ If a surface of revolution } M \text{ is a minimal surface, then } M \text{ is contained either in a plane or a catenoid.}}$

Proof: pg. 255, O'neill. //