LECTURE 18: adapted frame fields and connection forms on M

A frame on $\mathbb{R}^3$ was a set of 3 orthonormal vector fields $E_1, E_2, E_3 \in \Gamma(\mathbb{R}^3)$ now we adapt such a frame to <u>fit</u> with a given surface $M \subset \mathbb{R}^3$,

Def/ An <u>adapted</u> <u>frame</u> <u>field</u> on $\mathcal{O} \subset M \subset \mathbb{R}^3$ is a Euclidean frame field such that $E_3(p) \in (T_pM)^\perp \; \forall p \in \mathcal{O}$.

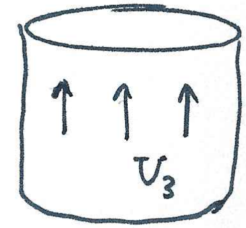
We have $E_i \cdot E_j = \delta_{ij}$ so ~~[illegible]~~ notation! $E_1(p), E_2(p) \in T_pM \; \forall p \in \mathcal{O}$. If details of $\mathcal{O}$ don't matter much to the discussion we'll just say <u>E_1, E_2, E_3 is frame on M</u>

<u>Lemma 1.2</u>: there is an adapted frame field on $\mathcal{O} \subset M$ $\iff$ $\mathcal{O}$ is orientable and $\exists$ a nonvanishing tangent-vect field on $\mathcal{O}$.

<u>Proof</u>: <u>$\Rightarrow$</u>] $\oint$ E_1, E_2, E_3 is adapted frame field on $\mathcal{O} \subset M$. then E_3 serves as nonvanishing normal to $\mathcal{O}$ $\therefore$ $\mathcal{O}$ oriented. Moreover, E_1 & E_2 are nonvanishing tangent fields.

<u>$\Leftarrow$</u>] $\mathcal{O}$ orientable $\Rightarrow \exists\, U$ a unit-normal on $\mathcal{O}$. Let $E_3 = U$. Also we assume $\exists\, E_2$ a nonvanishing tang. field on $\mathcal{O}$. Define $E_1 = E_2 \times E_3$ then it follows E_1, E_2, E_3 forms the desired frame field on $\mathcal{O} \subset M \subset \mathbb{R}^3$.

Remark: the technique of the proof is sometimes useful to construct frame fields on a given M.



Ex (1): CYLINDER $M : x^2 + y^2 = R^2$

$\nabla(x^2+y^2) = \langle 2x, 2y, 0 \rangle \longrightarrow E_3 = \frac{1}{R}(x U_1 + y U_2)$.

Notice $U_3 \cdot E_3 = 0 \therefore E_1 = U_3$ gets us tang.-field to M

Finally $E_2 = E_3 \times E_1 = \frac{1}{R}(x U_1 \times U_3 + y U_2 \times U_3) = \frac{-x}{R} U_2 + \frac{y}{R} U_1$

Thus $\underbrace{E_1 = U_3}_{\hat{z}}, \quad \underbrace{E_2 = \frac{y}{R} U_1 - \frac{x}{R} U_2}_{\hat{\theta}}, \quad \underbrace{E_3 = \frac{1}{R}(x U_1 + y U_2)}_{\hat{r}}$

multivariable calculus I'd use notation for this frame field.
(oh, this is the cylindrical frame restricted to $r = R$)

Remark: setting $E_2 = E_3 \times E_1$ for given E_1 & E_3 forces $E_1 \times E_2 = E_3$ and $E_2 \times E_3 = E_1$. In short, it's a — RIGHT-HANDED FRAME —

Ex (2) SPHERE Σ : $x^2 + y^2 + z^2 = R^2$

$$E_3 = \frac{1}{R}(x U_1 + y U_2 + z U_3) \leftarrow \boxed{\frac{\nabla(x^2+y^2+z^2)}{\|\nabla(x^2+y^2+z^2)\|} \quad \text{normalized gradient}}$$

"$\hat{\theta}$" : $E_1 = \dfrac{-y U_1 + x U_2}{\sqrt{x^2+y^2}}$: clearly $E_1 \cdot E_3$ and $\|E_1\| = 1$ except at N/S poles where $x = y = 0$.

$$E_2 = \underbrace{E_3 \times E_1}_{\hat{\rho} \times \hat{\theta}} = \underbrace{\frac{1}{R\sqrt{x^2+y^2}}\left(-xz U_1 - yz U_2 + (x^2+y^2) U_3\right)}_{\hat{\phi}}$$

(in my usual vector calculus notation I can't help but comment)

Remark: in spherical coord. these are far prettier, but, I shall not digress farther..

- Our results for frame fields on $\mathbb{R}^3$ or open sets in $\mathbb{R}^3$ don't directly apply since M is not an open set in $\mathbb{R}^3$. HOWEVER, as Schlomo Sternberg illustrates in his text on CURVATURE we can thicken M to a RIBBON $\bigcup_{c \in (-\varepsilon, \varepsilon)} (M + c E_3)$ roughly.

then connection equations hold for RIBBON and we can restrict those back to M once again,

CONNECTION EQUATIONS ON SURFACE

Th$^{\underline{m}}$/ If E_1, E_2, E_3 is an adapted frame on M then $\nabla_v E_i = \sum_{j=1}^{3} w_{ij}(v)\, E_j(p)$ for $i = 1,2,3$ and all $v \in T_pM$

Recall, from § 2.7 p. 88, $w_{ij}(v) = (\nabla_v E_i) \cdot E_j$

In view of the def$^{\underline{n}}$ of w_{ij} this theorem just expresses orthonormal expansion of $\nabla_v E_i$ in terms of E_1, E_2, E_3 frame.

[Comment: $w_{ij}(v)$ tells us the rate at which E_i (initially) rotates toward E_j as p moves in the v-direction.]

Proposition: for the shape operator S at p and $v \in T_pM$,
$$S(v) = w_{13}(v)\, E_1(p) + w_{23}(v)\, E_2(p)$$

Proof: $U = E_3$ for M thus $S(v) = -\nabla_v U = -\nabla_v E_3$.

Recall $w_{ij} = -w_{ji}$ thus $\nabla_v E_3 = w_{31}(v) E_1 + w_{32}(v) E_2 + \underbrace{\cancel{w_{33}(v)}}_{0} E_3$

$$\therefore \quad S(v) = w_{13}(v) E_1 + w_{23}(v) E_2$$

where we've used $w_{13} = -w_{31}$ & $w_{23} = -w_{32}$ to absorb $(-)$. //

Coframe on M

E_1, E_2, E_3 adapted to M, only E_1, E_2 are tangent to M

$\underbrace{\theta^1 \quad \theta^2 \quad \theta^3}_{\text{coframe}}$ $\quad \theta^i(E_j) = \delta_{ij}$

θ^1, θ^2 one-forms on M

$\underbrace{\theta^3 \equiv 0}$ on M.

$\boxed{\text{$\theta^3$ kills any lin. combo of E_1 & E_2 hence all of T_pM for each p.}}$

We study the following

θ^1, θ^2 dual to E_1, E_2

ω_{12} gives how E_1 rotates to E_2

ω_{13}, ω_{23} capture shape operator as given by E_3 (~~this revealed in pages to come~~...)

Example: dual to cylindrical frame (Ex 2.7.6 my notes)

$$\begin{aligned} \theta^1 &= dr \\ \theta^2 &= r\,d\theta \\ \theta^3 &= dz \end{aligned} \quad \xrightarrow{r=R} \quad \left.\begin{aligned} \theta^1 &\equiv 0 \\ \theta^2 &= R\,d\theta \\ \theta^3 &= dz \end{aligned}\right\}$$

So, the label doesn't fit with our E_3 is normal... this is why we sometimes begin by relabeling old example

Ex: Coframe to Cylinder

$$\underbrace{\begin{aligned}\theta^1 &= r\,d\theta\\ \theta^2 &= dz\\ \theta^3 &= dr\end{aligned}}_{\text{adjusted Ex. 2.7.6 my notes.}} \xrightarrow{r=R} \boxed{\begin{aligned}\theta^1 &= R\,d\theta\\ \theta^2 &= dz\end{aligned}}$$

Ex: coframe to sphere

$$\underbrace{\begin{aligned}\theta^1 &= \rho\,d\phi\\ \theta^2 &= \rho\sin\phi\,d\theta\\ \theta^3 &= d\rho\end{aligned}}_{\text{adjusted Ex. 2.7.7 my notes.}} \xrightarrow{\rho=R} \boxed{\begin{aligned}\theta^1 &= R\,d\phi\\ \theta^2 &= R\sin\phi\,d\theta\end{aligned}}$$

Th$^{\underline{m}}$ (CARTAN'S STRUCTURE EQ$^{\underline{n}}$'S ADAPTED TO M)

Suppose E_1, E_2, E_3 is an adapted frame on $M \subset \mathbb{R}^3$ then its dual-forms θ^1, θ^2 and connection-forms ω_{ij} on M satisfy:

(1) $\begin{bmatrix} d\theta^1 = \omega_{12} \wedge \theta^2 \\ d\theta^2 = \omega_{21} \wedge \theta^1 \end{bmatrix}$ — 1$^{\underline{st}}$ structural eq$^{\underline{n}}$'s ← $d\theta = \omega \wedge \theta$

(2.) $[\omega_{31} \wedge \theta^1 + \omega_{32} \wedge \theta^2 = 0]$ — symmetry eq$^{\underline{n}}$. $\cancel{d\theta^3} 0 = \omega_{31} \wedge \theta^1 + \omega_{32} \wedge \theta^2$

(3.) $[d\omega_{12} = \omega_{13} \wedge \omega_{32}]$ — Gauss' Eq$^{\underline{n}}$

(4.) $\begin{bmatrix} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{bmatrix}$ — Codazzi Eq$^{\underline{n}}$'s. } tells us how shape of M changes from point to point since S is determined by ω_{13} & ω_{23}

Proof: these are just $d\theta = \omega \wedge \theta$ and $d\omega = \omega \wedge \omega$ with $\theta^3 \equiv 0$ from our previous work in Chapter 2. //

Application: following ⑥
$\theta^1 = R\,d\phi \longrightarrow d\theta^1 = 0$
$\theta^2 = R\sin\phi\, d\theta \longrightarrow d\theta^2 = R\cos\phi\, d\phi \wedge d\theta$

Now, $d\theta^2 = \omega_{21} \wedge \theta^1 = \omega_{21} \wedge (R\,d\phi) = R\cos\phi\, d\phi \wedge d\theta$
$d\theta^1 = \omega_{12} \wedge \theta^2$
$0 = \omega_{12} \wedge (R\sin\phi\, d\theta) \Longrightarrow \omega_{21} = -\cos\phi\, d\theta \quad \therefore \underline{\omega_{12} = \cos\phi\, d\theta}.$

Remark: this is based on Ex. 2.7.7 of my notes with $\begin{bmatrix} 1 \longrightarrow 3 \\ 2 \longrightarrow 1 \\ 3 \longrightarrow 2 \end{bmatrix}$, this is ω_{23} in Ex. 2.7.7. as it ought.

§ 6.2 FORM COMPUTATIONS

VECTOR FIELD V ON M ; $\quad V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2$

ONE-FORM α ON M ; $\quad \alpha = \alpha(E_1)\theta^1 + \alpha(E_2)\theta^2$

TWO-FORM μ on M ; $\quad \mu = \mu(E_1, E_2)\theta^1 \wedge \theta^2$

The claims above follow from def$^{\text{n}}$'s we gave in Chapter 4 as well as the duality condition $\theta^i(E_j) = \delta_{ij}$.

Example Proof: θ^1, θ^2 basis hence $\alpha = c_1\theta^1 + c_2\theta^2$. But,

$$\alpha(E_1) = c_1\cancelto{1}{\theta^1(E_1)} + c_2\cancelto{0}{\theta^2(E_1)} = c_1 \quad \text{and} \quad \alpha(E_2) = c_1\cancelto{0}{\theta^1(E_2)} + c_2\cancelto{1}{\theta^2(E_2)} = c_2$$

Thus $\alpha = \alpha(E_1)\theta^1 + \alpha(E_2)\theta^2$ as claimed.

Lemma 2.2:
(1) $\omega_{13} \wedge \omega_{23} = K\,\theta_1 \wedge \theta_2$
(2) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\,\theta_1 \wedge \theta_2$

Proof:

$$\begin{aligned}
\omega_{13} \wedge \omega_{23} &= \left(\omega_{13}(E_1)\theta^1 + \omega_{13}(E_2)\theta^2\right) \wedge \left(\omega_{23}(E_1)\theta^1 + \omega_{23}(E_2)\theta^2\right) \\
&= \left[\omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)\right]\theta^1 \wedge \theta^2 \\
&= (\det S)\,\theta^1 \wedge \theta^2 \quad : \text{By Lemma} \\
&= K\,\theta^1 \wedge \theta^2
\end{aligned}$$

$$\boxed{\textbf{Lemma: } \det \tilde{S} = \omega_{13}(E_1)\,\omega_{23}(E_2) - \omega_{13}(E_2)\,\omega_{23}(E_1)}$$

Proof: we wish to express $\det \tilde{S}$ in terms of frame E_1, E_2.
We have $S(V) = \omega_{13}(V)\,E_1 + \omega_{23}(V)\,E_2$ hence

$$\begin{bmatrix} S(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2 \\ S(E_2) = \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2 \end{bmatrix} \longrightarrow [\tilde{S}] = \begin{bmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{bmatrix}$$

Therefore, $\det(\tilde{S}) = \omega_{13}(E_1)\,\omega_{23}(E_2) - \omega_{13}(E_2)\,\omega_{23}(E_1)$ as claimed. //

Continue to prove (2): $\omega_{13} \wedge \theta^2 + \theta^1 \wedge \omega_{23} = 2H\,\theta^1 \wedge \theta^2$

$$\omega_{13} \wedge \theta^2 + \theta^1 \wedge \omega_{23} = \left(\omega_{13}(E_1)\theta^1 + \underbrace{\cancel{\omega_{13}(E_2)\theta^2}}_{0}\right) \wedge \theta^2 + \theta^1 \wedge \left(\underbrace{\cancel{\omega_{23}(E_1)\theta^1}}_{0} + \omega_{23}(E_2)\theta^2\right)$$

$$= \left[\omega_{13}(E_1) + \omega_{23}(E_2)\right]\theta^1 \wedge \theta^2$$

$$= \operatorname{trace}(\tilde{S})\,\theta^1 \wedge \theta^2$$

$$= \underline{2H\,\theta^1 \wedge \theta^2}. //$$

$$\boxed{\text{Corollary (2.7): } d\omega_{12} = -K\,\theta^1 \wedge \theta^2} \longleftarrow \text{2nd structural Eq}^{\underline{n}}.$$

Proof: $\underbrace{d\omega_{12} = \omega_{13} \wedge \omega_{32}}_{\text{Gauss Eq}^{\underline{n}}} = -\omega_{13} \wedge \omega_{23} = -K\,\theta^1 \wedge \theta^2. //$

Application: calculate K for sphere.

$$\left.\begin{aligned}\theta^1 &= R\,d\phi \\ \theta^2 &= R\sin\phi\,d\theta\end{aligned}\right\} \longrightarrow \theta^1\wedge\theta^2 = R^2\sin\phi\,d\phi\wedge d\theta$$

(no need for this)

$$\omega_{12} = \cos\phi\,d\theta \longrightarrow d\omega_{12} = -\sin\phi\,d\phi\wedge d\theta$$

Then as $\underbrace{d\omega_{12} = -K\theta^1\wedge\theta^2}_{\text{"2nd Structural Eq"}} \longrightarrow -\sin\phi\,d\phi\wedge d\theta = -KR^2\sin\phi\,d\phi\wedge d\theta$

$\therefore$ $\boxed{K = 1/R^2}$

Def/ A PRINCIPAL FRAME FIELD on $M \subset \mathbb{R}^3$ is an adapted frame field E_1, E_2, E_3 such that E_1, E_2 are everywhere principal vectors on M.

For nonumbilic points this determines a particular choice of frame (modulo sign choice). As O'neill points out for S.O.Rev. the tangents to meridians and parallels.

Lemma (2.5): If p is a nonumbilic point of $M \subset \mathbb{R}^3$ then $\exists$ a principal frame field on some nbhd of p in M.

Proof: if P is nonumbilic $\Rightarrow k(u)$ is non-constant at $P \Rightarrow k_1(P) \neq k_2(P)$ Or, $(k_1 - k_2)(P) \neq 0$. By continuity, $\exists$ nbhd of P for which all $q \in$ nbhd have $(k_1 - k_2)(q) \neq 0$. Then the normalized e-vect. of S on nbhd give the frame.
(skipped some detail here see 271-272)

Let E_1, E_2, E_3 be a principal frame field, thus $\exists$ functions k_1 and k_2 for which $S(E_1) = k_1 E_1$ & $S(E_2) = k_2 E_2$

$$S(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2 = k_1 E_1 + 0 \cdot E_2$$

$$S(E_2) = \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2 = k_2 E_2 + 0 \cdot E_1$$

Thus we derive,

$$\omega_{13}(E_1) = k_1 \ \ \& \ \ \omega_{13}(E_2) = 0 \ \ \therefore \ \ \boxed{\omega_{13} = k_1 \theta^1}$$

$$\omega_{23}(E_1) = 0 \ \ \& \ \ \omega_{23}(E_2) = k_2 \ \ \therefore \ \ \boxed{\omega_{23} = k_2 \theta^2}$$

Th^m/ If E_1, E_2, E_3 is principal frame on M then $E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2)$ and $E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1)$

<u>Proof</u>: We have $d\omega_{13} = \omega_{12} \wedge \omega_{23} \Rightarrow d(k_1\theta^1) = \omega_{12} \wedge (k_2\theta^2)$

$\therefore \ dk_1 \wedge \theta^1 + k_1 d\theta^1 = \omega_{12} \wedge (k_2 \theta^2)$ and as $d\theta^1 = \omega_{12} \wedge \theta^2$

$\Rightarrow dk_1 \wedge \theta^1 + k_1 \omega_{12} \wedge \theta^2 = k_2 \omega_{12} \wedge \theta^2$

$\Rightarrow dk_1 \wedge \theta^1 = (k_2 - k_1)\omega_{12} \wedge \theta^2$

could have omitted this ☺

$\Rightarrow dk_1[E_2]\theta^2 \wedge \theta^1 = (k_2 - k_1)\big(\omega_{12}[E_1]\theta^1 + \underbrace{\omega_{12}[E_2]\theta^2}\big) \wedge \theta^2$

$dk_1 = \cancel{dk_1[E_1]\theta^1}^{\,0} + dk_1(E_2)\theta^2$

$\Rightarrow -dk_1(E_2)\theta^1 \wedge \theta^2 = (k_2 - k_1)\omega_{12}(E_1)\theta^1 \wedge \theta^2 \quad \boxed{\star}$

Continuing, comparing coeff. of $\theta^1 \wedge \theta^2$ yields,

$$-dh_1(E_2) = (h_2 - h_1)\,\omega_{12}(E_1)$$

$$\Rightarrow \underline{E_2[h_1] = (h_1 - h_2)\,\omega_{12}(E_1)} \; //$$

Likewise, $d\omega_{23} = \omega_{21} \wedge \omega_{13}$

$$d(h_2\theta^2) = -\omega_{12} \wedge (h_1\theta^1)$$

$$dh_2 \wedge \theta^2 + h_2\, d\theta^2 = -h_1\omega_{12} \wedge \theta^1$$

$$dh_2 \wedge \theta^2 + h_2\, \omega_{21} \wedge \theta^1 = -h_1\, \omega_{12} \wedge \theta^1$$

$$dh_2 \wedge \theta^2 = (-h_1 + h_2)\,\omega_{12} \wedge \theta^1$$

$$dh_2[E_1]\theta^1 \wedge \theta^2 = (k_2 - h_1)\,\omega_{12}[E_2]\,\theta^2 \wedge \theta^1$$

$$\therefore\; dh_2[E_1] = (h_1 - h_2)\,\omega_{12}[E_2]$$

$$\underline{E_1[k_2] = (h_1 - k_2)\,\omega_{12}[E_2]} \; //$$

Remark: this theorem shows for principal frame ω_{12} controls how principal curvatures h_1 & k_2 change in principal directions E_1 & E_2.