LECTURE 19: SURFACE THEOREMS

We follow §6.3 of O'neill where we find a selection of interesting theorems about surfaces. We assume M <u>connected</u> in this LECTURE.

$\boxed{\text{Th}^{\underline{m}}/(3.1) \text{ If its shape operator is zero on } M \text{ then } M \text{ is part of a plane in } \mathbb{R}^3}$

<u>Proof</u>: If $S_p(v) = -\nabla_v E_3 = 0 \quad \forall p \in M$ and $v \in T_pM$ then $E_3' = 0$ hence E_3 is constant vector field on M. Consider for ~~any~~ curve α in M, and $p \in M$, with $\alpha(0) = p$ & $\alpha(1) = q$

$$f(t) = (\alpha(t) - p) \cdot E_3$$

$$\Rightarrow \frac{df}{dt} = \alpha'(t) \cdot E_3 = 0 \quad \forall t.$$

However, $f(0) = 0 \Rightarrow f(t) = 0 \Rightarrow f(1) = (q - p) \cdot E_3 = 0$

But, as $q \in M$ was arbitrary, this shows E_3 is constant normal to plane M.

<u>Remark</u>: the existence of a planar point $p \in M$ ($S_p(v) = 0 \ \forall v$) does not suffice to say M is planar near p. There are examples with planar points near complicated behaving M.

$\boxed{\textbf{Lemma (3.2): } \text{If } M \text{ is an all-umbilic surface in } \mathbb{R}^3 \text{, then } M \text{ has } K \text{ constant with } K \geq 0}$

Proof: if E_1, E_2, E_3 is an adapted frame field and M is all umbilic then $k(u)$ is constant $\Rightarrow$ E_1, E_2, E_3 is principal frame field with $k_1 = k_2 = k$. Also, by last LECTURE, or Th^m 2.6 on p. 272,

$$\left.\begin{aligned} E_1[k_2] &= (k_1 - k_2)\omega_{12}[E_2] \\ E_2[k_1] &= (k_1 - k_2)\omega_{12}(E_1) \end{aligned}\right\} \quad \begin{aligned} &E_1[k] = E_2[k] = 0 \\ &\Rightarrow dk[E_1] = dk[E_2] = 0 \end{aligned}$$

Hence $dk = 0$ on M. But, $K = k_1 k_2 = k^2$ thus $\longrightarrow K = k^2 \geq 0.$

$dK = 2k\,dk = 0 \Rightarrow K = \text{constant}$ as M is connected. //

$\boxed{\text{Th}^m\text{/ If } M \subset \mathbb{R}^3 \text{ is all-umbilic and } K > 0 \text{, then } M \text{ is part of sphere with radius } 1/\sqrt{K}}$

Proof: see 275 – 276 of O'neill, based on clever curve construction paired with the Lemma above. //

$\boxed{\textbf{Corollary: } \text{A compact all-umbilic surface } M \subset \mathbb{R}^3 \text{ is an entire sphere}}$

PROOF: all-umbilic $\Rightarrow K$ constant $\Rightarrow K = 0$ or $K > 0$. But, all-umbilic and $K = 0$ gives $S = 0$ on M hence M part of plane which is not compact $\therefore$ M has $K > 0$ and is part of sphere hence M is whole sphere as M compact. //

Th^{m} (3.5) On every compact surface $M \subset \mathbb{R}^3$ there is a point at which the Gaussian curvature $K > 0$

<u>Proof</u>: see p. 277 – 278 of O'neill.

<u>Application of Th^{m} 3.5</u>: there are <u>no</u> compact surfaces in $\mathbb{R}^3$ with $K \leq 0$.

<u>Lemma (3.6) [HILBERT]</u>: Let m be point of $M \subset \mathbb{R}^3$ such that

(1) k_1 has local max. at m

(2) k_2 has local min at m

(3) $k_1(m) > k_2(m)$

Then $K(m) \leq 0$.

<u>Proof</u>: see 279 – 280 of O'neill. Uses lots of neat ideas just like the last one.

Th^{m} (3.7) (Liebmann) If M is a compact surface in $\mathbb{R}^3$ with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$

<u>Proof</u>: $K > 0$ by Th^{m} (3.5). Can argue k_1, k_2 continuous with $k_1 \geq k_2 \geq 0$. But as M compact $\Rightarrow k_1$ has max. at some $p \in M$. Yet with $K = k_1 k_2$ constant $\Rightarrow k_2$ has min. at p. Cannot allow $k_1(p) > k_2(p)$ as Hilbert's Lemma would force $K \leq 0$. Hence $k_1 = k_2$ at $p \Rightarrow M$ all umbilic $\therefore$ sphere of radius $1/\sqrt{K}$. //