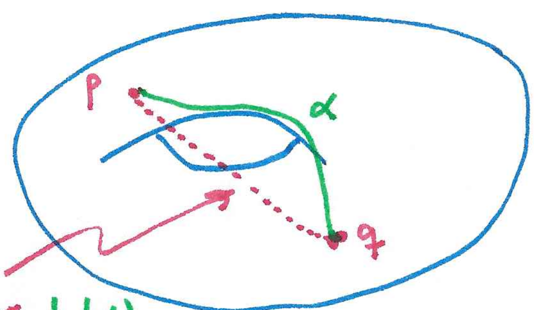
Lecture 20: Isometries and Intrinsic Geometry

Defn/ If $p, q \in M$ then $\rho(p,q) = \inf\{L(\alpha) \mid \alpha$ is curve in M from p to $q\}$

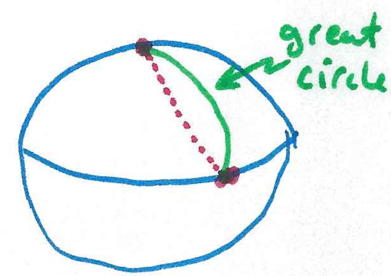
and $L(\alpha) = \int_{t_1}^{t_2} \|\alpha'(t)\|\, dt$

or "curve-segment" if you prefer O'neill's terminology.



$\|p-q\| < L(\alpha)$

- distance between points on surface often larger than straight-line distance.



- if you were stuck on M as a two-dim'l being then $\rho(p,q)$ would be your <u>distance function</u>

Defn/ An <u>isometry</u> $F: M \longrightarrow \overline{M}$ of surfaces in $\mathbb{R}^3$ is an injective map from M to $\overline{M}$ for which $F_*(v) \cdot F_*(w) = v \cdot w$ for all $v, w \in T_pM$ (that is, $d_pF(v) \cdot d_pF(w) = v \cdot w \;\; \forall v, w \in T_pM$)

Thus the differential of an isometry is invertible since $F_*(v) \cdot F_*(w) = v \cdot w$ indicates F_* is orthogonal trans. from T_pM to $T_{F(p)}\overline{M}$, terminology aside, the inverse fnct thm for surfaces applies and we obtain a smooth local inverse at each point. But, we're also given F is 1-1 $\Rightarrow$ F has smooth inverse $\therefore$ F diffeomorphism of M & $\overline{M}$.

Every isometry of surfaces is a diffeomorphism. The converse fails since diffeomorphism merely implies F_* is invertible. The criterion of $F_*(v) \cdot F_*(w) = v \cdot w$ is much stronger.

Prop: If $F : M \to \bar{M}$ and $G : \bar{M} \to N$ are isometries then $G \circ F : M \to N$ is also an isometry of surfaces

Proof: $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$ hence $G \circ F$ is certainly 1-1. Also, by chain rule, $d(G \circ F) = dG \circ dF$. Thus,

$$\begin{aligned} (d(G\circ F)(v)) \cdot (d(G \circ F)(w)) &= dG(dF(v)) \cdot dG(dF(w)) \quad G \text{ an isometry} \\ &= dF(v) \cdot dF(w) \quad F \text{ an isometry} \\ &= v \cdot w \end{aligned}$$

Prop: If $F : M \to \bar{M}$ is an isometry then $F^{-1} : \bar{M} \to M$ is an isometry

Proof: F an isometry $\Rightarrow$ F^{-1} exists and $F_*(v) \cdot F_*(w) = v \cdot w$ $\forall\, v, w \in T_pM$. But, $F_* : T_pM \to T_{F(p)}\bar{M}$ is also invertible hence $v = F_*^{-1}(\bar{v})$ and $w = F_*^{-1}(\bar{w})$ for some $\bar{v}, \bar{w} \in T_{F(p)}\bar{M}$ for any pair $v, w \in T_pM$. Thus,

$$F_*(F_*^{-1}(\bar{v})) \cdot F_*(F_*^{-1}(\bar{w})) = F_*^{-1}(\bar{v}) \cdot F_*^{-1}(\bar{w}) \;\therefore\; F_*^{-1}(\bar{v}) \cdot F_*^{-1}(\bar{w}) = \bar{v} \cdot \bar{w}$$

$\Rightarrow$ F_*^{-1} is an ~~orthogonal~~ isometry from $T_{F(p)}\bar{M}$ to T_pM $\therefore$ F^{-1} an isometry. //

Remark: in local coordinates $F_*(v) \cdot F_*(w) = v \cdot w$

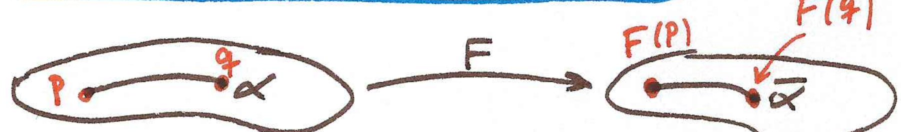
$\Rightarrow [F_*] = R$ with $R^T R = I$ hence $\underbrace{(R^{-1})^T R^{-1} = I}$

$\Longleftrightarrow ((R^{-1})^T R^{-1})^{-1} = I^{-1}$

$\Longleftrightarrow (R^{-1})^{-1} ((R^{-1})^T)^{-1} = I$

$\Longleftrightarrow R R^T = I$

$\boxed{\text{Th}^{\underline{m}}/ \text{ Isometries preserve intrinsic distance: if } F : M \longrightarrow \bar{M} \text{ is an isometry of surfaces, then } \rho(p,q) = \bar{\rho}(F(p), F(q)) \text{ for any two points } p, q \in M.}$

Proof: if 

$\bar{\alpha} = F \circ \alpha$

then $\bar{\alpha}' = F_*(\alpha')$. Therefore, $\bar{\alpha}' \cdot \bar{\alpha}' = F_*(\alpha') \cdot F_*(\alpha') = \alpha' \cdot \alpha'$ (F isometry).

yields $\|\bar{\alpha}'\| = \|\alpha'\|$. If $\mathrm{dom}(\alpha) = [a, b]$ then

$L(\alpha) = \int_a^b \|\alpha'\| dt$ and $L(\bar{\alpha}) = \int_a^b \|\bar{\alpha}'\| dt \quad \therefore L(\alpha) = L(\bar{\alpha})$.

It follows that $\rho(p,q) = \bar{\rho}(F(p), F(q))$. //

[**Remark**: isometry can be thought of as bending surfaces in such a way the distance on M is maintained. Preservation of the dot-product from T_pM to $T_{F(p)}\bar{M}$ also preserves ∡'s.]

Remark: our illustration of geodesic on cylinder obtained from bending paper to coiled paper was an example of an isometry between  (modulo the ⋮) we can't complete the loop as that spoils injectivity.

Sometimes the following concept is liberating,

> **Def^{n}/(4.4)** A <u>LOCAL</u> <u>ISOMETRY</u> $F : M \longrightarrow N$ of surfaces is a mapping that preserves dot-products of tangent vectors under $F_* : T_pM \longrightarrow T_{F(p)}N$ for each $p \in M$.

The inverse function theorem $\Rightarrow$ $F/_{\mathcal{U}} : \mathcal{U} \longrightarrow V$ is invertible thus a local isometry is $\underbrace{\text{locally restrictable to an isometry}}_{\text{or just say "locally an isometry"}}$.

Lemma (4.5): Let $F : M \to N$ be smooth surface map. For each patch $\mathbf{X} : D \to M$, consider the map $\overline{\mathbf{X}} = F \circ \mathbf{X} : D \to N$. Then F is a local isometry $\iff$ for each patch $\mathbf{X}$ we have
$$E = \overline{E}, \quad F = \overline{F}, \quad G = \overline{G}$$

Proof: F local isometry at p $\iff$ $F_*(v) \cdot F_*(w) = v \cdot w$ for frame $v, w \in T_pM$

$$\iff \left.\begin{aligned} F_*(\mathbf{X}_u) \cdot F_*(\mathbf{X}_v) &= \mathbf{X}_u \cdot \mathbf{X}_v \\ F_*(\mathbf{X}_u) \cdot F_*(\mathbf{X}_u) &= \mathbf{X}_u \cdot \mathbf{X}_u \\ F_*(\mathbf{X}_v) \cdot F_*(\mathbf{X}_v) &= \mathbf{X}_v \cdot \mathbf{X}_v \end{aligned}\right\} \text{ for patch } \mathbf{X} \text{ containing } p$$

$$\iff \begin{aligned} \overline{F} &= F \\ \overline{E} &= E \\ \overline{G} &= G \end{aligned} \quad // \qquad (\text{used \#1 of §6.4})$$

Application to construct F:

$$\begin{tikzcd} \mathbf{X}(D) \arrow[rr, "F"] & & \mathbf{Y}(D) \\ & D \arrow[ul, "\mathbf{X}"] \arrow[ur, "\mathbf{Y}"'] & \end{tikzcd}$$

$E = \mathbf{X}_u \cdot \mathbf{X}_u$

$\overline{E} = \mathbf{Y}_u \cdot \mathbf{Y}_u$

etc..

$\underbrace{F(\mathbf{X}(u,v)) = \mathbf{Y}(u,v)}$ implicitly defines F as local isom. provided
$\overline{E} = E$
$\overline{F} = F$
$\overline{G} = G$

Example (4.6) Plane $\longrightarrow$ Cylinder

① $\mathbf{X} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $\mathbf{X}(u,v) = (u,v)$ patch on plane by identity map

Oh, add a zero $\mathbf{X}(u,v) = (u,v,0)$ if you must.

② $\mathbf{Y} : D \subset \mathbb{R}^2 \longrightarrow C \subset \mathbb{R}^3$

$$\mathbf{Y}(u,v) = (R\cos(u/R), R\sin(u/R), v)$$

Then for plane, $\mathbf{X}_u = \mathbf{v}_1$, $\mathbf{X}_v = \mathbf{v}_2$ and we have $E = 1, F = 0, G = 1$.

For cylinder, $\left.\begin{array}{l} \mathbf{Y}_u = \langle -\sin(u/R), \cos(u/R), 0\rangle \\ \mathbf{Y}_v = \langle 0, 0, 1\rangle \end{array}\right\}$ $\bar{E} = 1, \bar{F} = 0, \bar{G} = 1$

Thus F implicitly defined by $F(\mathbf{X}(u,v)) = \mathbf{Y}(u,v)$ that is, $\boxed{F(u,v) = (R\cos(u/R), R\sin(u/R), v)}$ is local isometry of $\mathbb{R}^2$ and right circular cylinder.

a.k.a. $\longrightarrow$ $F(x,y) = (R\cos(x/R), R\sin(x/R), y)$.

Ex (4.6.2) Helicoid onto a catenoid (If any example sells this course this is it! See the amazing animation of www.cogsci.indiana.edu/farg/harry/mat/helicoid.htm)

(shows smooth interpolation from Helicoid to catenoid and back)

$$\mathbf{X}(u,v) = (u\cos v,\ u\sin v,\ v) \ : \text{Helicoid}$$

$$\mathbf{Y}(u,v) = \underbrace{(g,\ h\cos v,\ h\sin v)}_{\substack{g(u) = \sinh^{-1}(u) \\ h(u) = \sqrt{1+u^2}}} \ : \text{Catenoid}$$

We can calculate, for example, $\mathbf{Y}_v = \langle 0, -h\sin v, h\cos v\rangle \therefore \|\mathbf{Y}_v\|^2 = h^2 = 1+u^2 = \bar{G}$

$$E = \bar{E} = 1, \quad F = \bar{F} = 0, \quad G = \bar{G} = 1+u^2$$

Thus, $F(\mathbf{X}(u,v)) = \mathbf{Y}(u,v)$ implicitly defines F as local isom.

$$\boxed{F(u\cos v,\ u\sin v,\ v) = \left(\sinh^{-1}(u),\ \sqrt{1+u^2}\cos v,\ \sqrt{1+u^2}\sin v\right)}$$

(IMPLICIT)

This may be futile, but I'll give it a shot,

$$\left.\begin{aligned} x &= u\cos v \\ y &= u\sin v \end{aligned}\right\} \tan v = y/x \ \ \& \ \ x^2+y^2 = u^2 \ \therefore \ u = \pm\sqrt{x^2+y^2}, \quad v = z$$

$$z = v \ \Big\} \ \underbrace{\tan z = y/x}_{\text{Helicoid}}$$

oops.

(EXPLICIT)

Probably can clean-up, but,

$$\boxed{F(x,y,z) = \left(\sinh^{-1}\left(\pm\sqrt{x^2+y^2}\right),\ \cos(z)\sqrt{1+x^2+y^2},\ \sin(z)\sqrt{1+x^2+y^2}\right)}$$

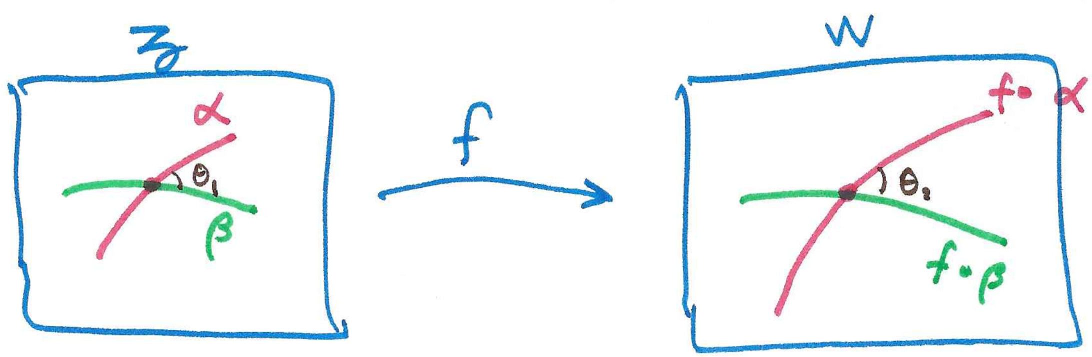
<u>Remark</u>: FIG 6.10 O'neill is excellent! (p. 285)

Def$^{\underline{n}}$/ A mapping of surfaces $F : M \longrightarrow N$ is <u>CONFORMAL</u> provided there exists $\lambda : M \longrightarrow (0, \infty)$ such that $\| F_*(v_p) \| = \underbrace{\lambda(p)}_{\text{scale-factor}} \| v_p \|$ for all $v_p \in T_pM$.

<u>Remark</u>: every local isometry is a conformal map with scale factor $\lambda = 1$.

<u>Comment</u>: working Exercises 1 & 8 of §6.4 gives you some sense of the similarity & distinction between isometry and conformal.

We also saw conformal mapping in study of complex analysis for $f : \mathbb{C} \longrightarrow \mathbb{C}$ with $f'(z_0) \neq 0$. The $\angle$ between curves is preserved at such points



$\theta_1 = \theta_2$

§6.5: Intrinsic Geometry of Surfaces

An <u>isometric</u> <u>invariant</u> is some concept preserved by isometry. Here, the isometry $F : M \longrightarrow \bar{M}$ is that of <u>surfaces</u>. In other words, of the concepts S, ω_{ij}, θ^i, E_i, U etc... which of these belongs to the geometry of M alone? Given what we saw in §4.5, for example Helicoid $\longleftrightarrow$ Catenoid, this concept of geometry is not the naive "shape" as captured by S or U. In retrospect, E_1, E_2 & θ^1, θ^2 & ω_{12} seem like good candidates as they do not get shaped by the shape operator. Anyway, fuzzy intuition aside, we <u>prove</u> the isometric nature of $E_1, E_2, \theta^1, \theta^2, \omega_{12}$ in what follows $\swarrow$

Lemma (5.1) The connection form $\omega_{12} = -\omega_{21}$ is the only one-form that satisfies $d\theta^1 = \omega_{12} \wedge \theta^2$ & $d\theta^2 = \omega_{21} \wedge \theta^1$

Proof: $\omega_{12} = \omega_{12}(E_1)\theta^1 + \omega_{12}(E_2)\theta^2$. However,

$$d\theta^1 = \omega_{12} \wedge \theta^2 \Rightarrow d\theta^1 = \left(\omega_{12}(E_1)\theta^1 + \omega_{12}(E_2)\theta^2\right) \wedge \theta^2 = \omega_{12}(E_1)\,\theta^1 \wedge \theta^2$$

Also,

$$d\theta^2 = \omega_{21} \wedge \theta^1 \Rightarrow d\theta^2 = \left(-\omega_{12}(E_1)\theta^1 - \omega_{12}(E_2)\theta^2\right) \wedge \theta^1 = \omega_{12}(E_2)\,\theta^1 \wedge \theta^2$$

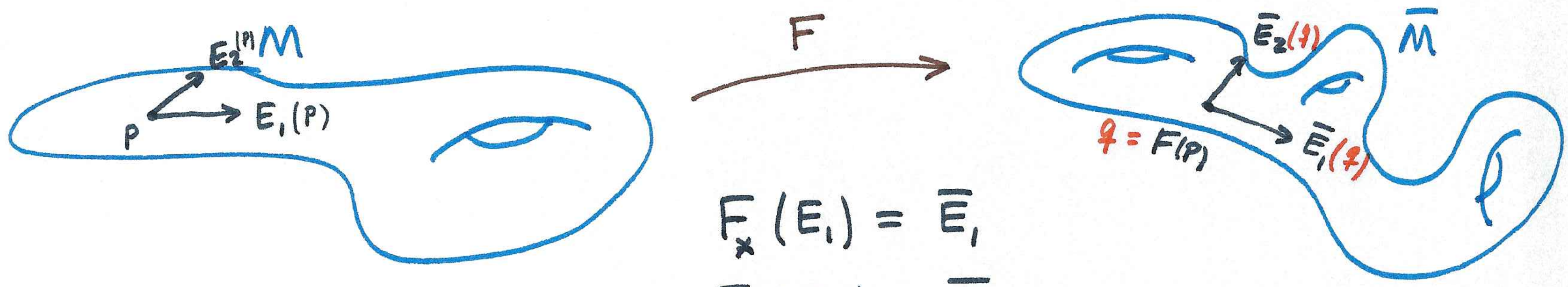
Therefore, $d\theta^1$ fixes the value $\omega_{12}(E_1) = d\theta^1(E_1, E_2)$ and by the same token, $\omega_{12}(E_2) = d\theta^2(E_1, E_2)$ $\therefore$ ω_{12} is completely specified by the values of $d\theta^1$ and $d\theta^2$ $\therefore$ ω_{12} is the <u>unique</u> one-form satisfying $d\theta^1 = \omega_{12} \wedge \theta^2$ and $d\theta^2 = \omega_{21} \wedge \theta^1$. //

Remark: I assume E_1, E_2 is frame on M with coframe θ^1, θ^2 and all the rights & priviliges granted therein. (I'm using Chapter 4 CALCULUS)

<u>FREE YOUR MIND</u>: of $\omega_{ij}(v) = (\nabla_v E_i)\cdot E_j$ we can define ω_{12} w/o the use of ∇. In particular,

$$\boxed{\text{Def}^{n}/\ \ \omega_{12}(E_1) = d\theta^1(E_1, E_2) \quad \omega_{12}(E_2) = d\theta^2(E_1,E_2) \quad \text{and } \underbrace{\text{extend linearly}}_{\boxed{\omega_{12}(V) = V^1\omega_{12}(E_1) + V^2\omega_{12}(E_2)}}\text{. Then } \omega_{21} = -\omega_{12}.}$$

<u>Remark</u>: clearly ω_{12} satisfies structural eq^{n}'s by Lemma 5.1 (see ⑩)



$F_*(E_1) = \bar{E}_1$

$F_*(E_2) = \bar{E}_2$

push-forward $\underbrace{\text{frame at } p}_{E_1, E_2}$ to frame at $\underbrace{F(p) = q}_{\bar{E}_1, \bar{E}_2}$

F ISOMETRY $\underbrace{\text{(LOCALLY)}}_{\substack{\text{could look}\\ \text{at subsets of}\\ M \text{ if 1-1 fails globally}}}$ $\longrightarrow$

<u>Lemma</u>

$$\bar{E}_i \cdot \bar{E}_j = F_*(E_i)\cdot F_*(E_j) = E_i \cdot E_j = \delta_{ij}$$

Lemma (5.3): Let $F: M \to \bar{M}$ be an isometry, and let E_1, E_2 be a tangent frame field on M. If $\bar{E}_1, \bar{E}_2$ is push-forward of E_1, E_2 ($F_*(E_j) = \bar{E}_j$ for $j = 1, 2$) then

(1.) $\theta^1 = F^*(\bar{\theta}^1)$ & $\theta^2 = F^*(\bar{\theta}^2)$

(2.) $\omega_{12} = F^*(\bar{\omega}_{12})$ (F^* is the pull-back)

Proof (1): $\left(F^*(\bar{\theta}^j)\right)(V) = \bar{\theta}^j\left(F_*(V)\right)$ def$^{\underline{n}}$ of pull-back.

Apply to E_i, $\left(F^*(\bar{\theta}^j)\right)(E_i) = \bar{\theta}^j\left(F_*(E_i)\right) = \bar{\theta}^j(\bar{E}_i) = \delta_{ij} = \theta^j(E_i)$

Yet $\theta^j(E_i) = \delta_{ij}$ hence θ^j and $F^*(\bar{\theta}^j)$ have values which agree on the basis E_1, E_2 $\therefore$ $\theta^j = F^*(\bar{\theta}^j)$ as hoped.//

Proof (2): notice $\bar{\omega}_{12}$ is defined by $\bar{\omega}_{12}(\bar{E}_j) = d\bar{\theta}^j(\bar{E}_1, \bar{E}_2)$. (see 11)

Consider then

$$\left(F^*(\bar{\omega}_{12})\right)(E_j) = \bar{\omega}_{12}\left(F_*(E_j)\right) = \bar{\omega}_{12}(\bar{E}_j) = d\bar{\theta}^j(\bar{E}_1, \bar{E}_2)$$

Notice $d\bar{\theta}^j = d\left(F^{-1*}\theta^j\right)$ by (1) hence $d\bar{\theta}^j = F^{-1*}(d\theta^j)$ by properties of pull-back (commutes with d) $\Rightarrow$ $d\bar{\theta}^j(\bar{E}_1, \bar{E}_2) = F^{-1*}(d\theta^j)(\bar{E}_1, \bar{E}_2)$

$$= d\theta^j\left((F^{-1})_*(F_*(E_1)), (F^{-1})_*(F_*(E_2))\right)$$

$$= d\theta^j(E_1, E_2) = \omega_{12}(E_j)$$

continued

We've shown, for arbitrary j,

$$\left(F^*(\bar{\omega}_{12})\right)(E_j) = \omega_{12}(E_j) \quad \therefore \quad \underline{F^*(\bar{\omega}_{12}) = \omega_{12}}. //$$

Gauss' Awesome Theorem: The Gaussian curvature is an isometric invariant. Explicitly, if $F: M \to \bar{M}$ is an isometry then $K(p) = \bar{K}(F(p))$ for each $p \in M$. $\boxed{\text{Def}^{n}/ \ K \text{ is \underline{defined} by } d\omega_{12} = -K\theta^1 \wedge \theta^2}$

Proof: we define K and $\bar{K}$ implicitly by $d\omega_{12} = -K\theta^1 \wedge \theta^2$ and $d\bar{\omega}_{12} = -\bar{K}\bar{\theta}^1 \wedge \bar{\theta}^2$. [Once more, F an isometry with $F_*(E_j) = \bar{E}_j$ and by the previous Lemma $F^*(\bar{\theta}^j) = \theta^j$ and $F^*(\bar{\omega}_{12}) = \omega_{12}$.]. Notice,

$$d\omega_{12} = d(F^*(\bar{\omega}_{12})) = F^*(d\bar{\omega}_{12}) = F^*(-\bar{K}\bar{\theta}^1 \wedge \bar{\theta}^2)$$

$$\Rightarrow -K\theta^1 \wedge \theta^2 = -F^*(\bar{K})F^*(\bar{\theta}^1) \wedge F^*(\bar{\theta}^2) \quad \text{prop. of pull-back}$$

$$= -\bar{K} \circ F \ \theta^1 \wedge \theta^2 \quad \text{Lemma 5.3}$$

Thus $K = \bar{K} \circ F$ or $K(p) = \bar{K}(F(p))$ for each $p \in M$. //

Comments about intrinsic nature of Gaussian Curvature

(1) • the eq$^{\underline{n}}$ $d\omega_{12} = -K\,\theta^1 \wedge \theta^2$ shows us that K can be calculated w/o knowledge of the shape operator.

(2) • $K = k_1 k_2$ where k_1 & k_2 are tied to the <u>shape</u> of M in $\mathbb{R}^3$. So, while K is intrinsic, neith k_1 nor k_2 are. Yet, for isometric surfaces M & $\bar{M}$ certainly k_1, k_2 are related to $\bar{k}_1, \bar{k}_2$ as $k_1 k_2 = \bar{k}_1 \bar{k}_2$.

(3) • the term <u>flat</u> is from the <u>intrinsic perspective</u>. To the inhabitants of the cylinder the geometry is same as if they lived on a plane, well at least locally! On the cylinder you can walk from home and back again w/o ever turning.

(4) • SPHERE WITH $K = \frac{1}{R^2}$ <u>CANNOT</u> BE ISOMETRICALLY mapped to the plane. It's not possible to make a flat map which faithfully preserves the sphere's geometric data, somewhere $\angle$ and/or lengths will be distorted.