LECTURE 21: Orthogonal Coordinates (uses charts!) (§6.6)
Integration and Orientation (§6.7)
Total Curvature (§6.8)

- I'm using my notes for Chapter 6 from the previous time I taught. Sorry for the break in format, but, I think they'll do.

§6.6 ORTHOGONAL COORDINATES

The central eq^{ns} of last section,

$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$
$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

Now we develop a technique for intrinsic calculation of K.

Def^{n}/ An orthogonal coordinate patch $\Phi : D \to M$ is one for which $F = \Phi_u \cdot \Phi_v = 0$. Moreover, the associated frame field of an orthogonal patch $\Phi : D \to M$ consists of E_1, E_2 on $\Phi(D)$ defined by

$$E_1 = \frac{\Phi_u(u,v)}{\sqrt{E}(u,v)} \qquad E_2 = \frac{\Phi_v(u,v)}{\sqrt{G}(u,v)}$$

or in my usual calc III. notation, $E_1 = \widehat{\Phi}_u$, $E_2 = \widehat{\Phi}_v$

We now introduce some notation at odds with the notation of the last 2 or 3 chapters.

$$u : \Phi(D) \longrightarrow D$$
$$v : \Phi(D) \longrightarrow D$$

Technically, $(u,v) = \Phi^{-1}$ and we had used $\tilde{u}, \tilde{v}$ to denote these previously. Anyway,

$$E_1 = \frac{\Phi_u}{\sqrt{E}}, \quad E_2 = \frac{\Phi_v}{\sqrt{G}}$$

dual forms θ_1, θ_2 have $\theta_i(E_j) = \delta_{ij}$

$$\left.\begin{matrix} du(\Phi_u) = 1 & dv\cdot(\Phi_u) = 0 \\ du(\Phi_v) = 0 & dv(\Phi_v) = 1 \end{matrix}\right\}$$ see Ex. 7 of §4.4 with $\tilde{u}, \tilde{v}$ etc...

$$du(X_u) = 1 \qquad dv(X_u) = 0$$
$$du(X_v) = 0 \qquad dv(X_v) = 0$$

$$E_1 = \frac{X_u}{\sqrt{E}} \qquad E_2 = \frac{X_v}{\sqrt{G}} \qquad \begin{array}{l} E = X_u \cdot X_u \\ G = X_v \cdot X_v \end{array}$$

<u>Claim</u>: $\theta_1 = \sqrt{E}\, du$, $\theta_2 = \sqrt{G}\, dv$ for orthogonal coordinates (u,v) on M.

<u>Proof</u>: $E_1(\theta_1) = \frac{1}{\sqrt{E}} X_u(\sqrt{E}\, du) = \underbrace{\frac{1}{\sqrt{E}}\sqrt{E}}_{1}\, \underbrace{du(X_u)}_{1} = 1.$

$E_1(\theta_2) = \frac{1}{\sqrt{E}} X_u(\sqrt{G}\, dv) = \sqrt{\frac{G}{E}}\, dv(X_u) = 0.$

$E_2(\theta_2) = \frac{1}{\sqrt{G}} X_v(\sqrt{G}\, dv) = \frac{\sqrt{G}}{\sqrt{G}}\, dv(X_v) = 1.$

$E_2(\theta_1) = \frac{1}{\sqrt{G}} X_v(\sqrt{E}\, du) = \sqrt{\frac{E}{G}}\, du(X_v) = 0.$ //

Calculate exterior derivatives towards finding f-la for ω_{12},

$$\begin{aligned} d\theta_1 &= d(\sqrt{E}\, du) \\ &= d\sqrt{E} \wedge du + \sqrt{E}\, d(du) \\ &= \left(\frac{\partial\sqrt{E}}{\partial u} du + \frac{\partial \sqrt{E}}{\partial v} dv\right)\wedge du \\ &= \frac{-(\sqrt{E})_v}{\sqrt{G}}\, du \wedge \theta_2 \end{aligned}$$

$$d\theta^1 = \omega_{12} \wedge \theta^2$$

$$d\theta_2 = d(\sqrt{G}\, dv) = (\sqrt{G})_{,u}\, du \wedge dv = \frac{-(\sqrt{G})_{,u}}{\sqrt{E}}\, dv \wedge \theta_1$$

However, 1st struct. eq^{ns} state
$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$

Note, $\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$

and $d\theta_1 = \omega_{12} \wedge \theta_2 = \omega_{12}(E_1)\theta_1 \wedge \theta_2$

whereas $d\theta_2 = -\omega_{12}(E_2)\theta_2 \wedge \theta_1$

Thus,

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$$
$$= \left(+\frac{(\sqrt{G})_{,u}}{\sqrt{E}}\, dv\right) - \left(\frac{(\sqrt{E})_{,v}}{\sqrt{G}}\right) du$$

By Lemma 5.1, the above must be ω_{12} as it is found by solving 1st structural eqs.

$$\boxed{\omega_{12} = \frac{-1}{\sqrt{G}} \frac{\partial}{\partial v}\left(\sqrt{E}\right) du + \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}\left(\sqrt{G}\right) dv}$$

Example 6.2 p. 296 (NOTE: this is way easier than shape operator methods)

Geographical coord. on sphere

$$E = r^2 \cos^2 v, \quad F = 0, \quad G = r^2$$

from $X(u,v) = (r\cos v \cos u,\ r\cos v \sin u,\ r\sin v)$ this is calculated by $E = X_u \cdot X_u$ and $G = X_v \cdot X_v$.

Therefore,

$$\theta_1 = \sqrt{E}\, du = r\cos(v)\, du$$
$$\theta_2 = \sqrt{G}\, dv = r\, dv$$

Clearly, $\frac{\partial}{\partial v}\left(\sqrt{E}\right) = \frac{\partial}{\partial v}(r\cos v) = -r\sin v$

whereas $\frac{\partial}{\partial u}\left(\sqrt{G}\right) = \frac{\partial}{\partial u}(r) = 0$ hence,

$$\omega_{12} = \frac{-1}{r}(-r\sin v)\, du = \sin(v)\, du$$

Thus $\boxed{\omega_{12} = \sin(v)\, du}$

Gaussian Curvature from intrinsic calculus

$$\omega_{12} = \frac{-(\sqrt{E})_v}{\sqrt{G}}\,du + \frac{(\sqrt{G})_u}{\sqrt{E}}\,dv$$

Recall, $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$ and we know $\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$ hence $\theta_1 \wedge \theta_2 = \sqrt{EG}\,du \wedge dv$. It remains to exterior differentiate ω_{12},

$$d\omega_{12} = -\frac{\partial}{\partial v}\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right] dv \wedge du + \frac{\partial}{\partial u}\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right] du \wedge dv$$

$$= \left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right) du \wedge dv$$

$$= \frac{1}{\sqrt{EG}}\left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right) \theta_1 \wedge \theta_2$$

The proposition below follows from $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$,

Proposition 6.3: If $X : D \to M$ is orthogonal patch with $E = X_u \cdot X_u$ and $G = X_v \cdot X_v$ then the Gaussian curvature K is given by:

$$\boxed{K = \frac{-1}{\sqrt{EG}}\left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right)}$$

Again the isometric invariance of K is made manifest (see Lemma 4.5 where $F : M \to \bar{M}$ is shown to give $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ for isometry F)

Def^{n}/(7.1 p.298) The interior R° of rectangle R: $a \le u \le b,\ c \le v \le d$ is the open set $a<u<b,\ c<v<d$. A 2-segment $\vec{X}: R \longrightarrow M$ is <u>patchlike</u> provided $\vec{X}|_{R^{\circ}}: R^{\circ} \longrightarrow M$ is a patch in M.

To calculate area we integrate $\sqrt{EG-F^2}$ over M.

$$\text{AREA} = \iint_D \sqrt{EG-F^2}\, du\, dv$$

Could be improper as D is open... so instead integrate over 2-segments (slight mod. of patch)

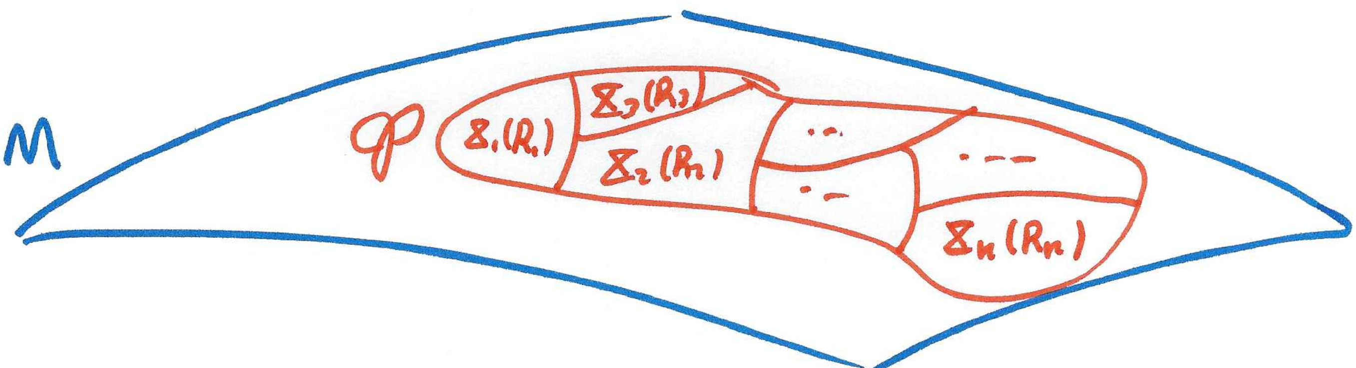
<u>Example:</u>

<u>SPHERE</u>: $\left.\begin{array}{l} E = r^2\cos^2 v \\ F = 0 \\ G = r^2 \end{array}\right\}$ following O'neill's f-la from p.296

$$\sqrt{EG-F^2} = r^2|\cos v| \quad \text{for } -\tfrac{\pi}{2} < v < \tfrac{\pi}{2}.$$
$$= r^2 \cos v \qquad \cos v \ge 0.$$

$$\therefore \text{Area} = \iint_{[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]*} \sqrt{EG-F^2}\, du\, dv$$
$$= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} r^2\cos v\, dv\, du$$
$$= 2\pi r^2 \sin v \Big|_{-\pi/2}^{\pi/2}$$
$$= \boxed{4\pi r^2}$$

Def$^{\underline{n}}$/ A <u>paving</u> of a region $\mathcal{P}$ in a surface M consists of finitely many <u>patch-like</u> 2-segments $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k$ whose images fill $\mathcal{P}$ such that $\bigcup_{i=1}^{k} \mathbf{x}_i(R_i) = \mathcal{P}$ and $p \in \mathcal{P} \Rightarrow p \in \mathbf{x}_i(R_i^\circ)$ for just one $i \in \mathbb{N}_k$.



<u>Remark</u>: an entire <u>compact</u> surface is always pavable.

Def$^{\underline{n}}$/ An <u>area</u> <u>form</u> on a surface M is a differentiable 2-form μ whose value on any pair $v, w \in T_pM$

$$\mu(v,w) = \pm\sqrt{(v\cdot v)(w\cdot w)-(v\cdot w)^2} = \pm\|v\times w\|$$

The $\pm$ is not avoidable. $\exists$ two orientations locally for M.

<u>Remark</u>: $\mu(E_1, E_2) = \pm 1$ $\forall$ frames $\{E_1, E_2\}$ on M also suffices.

Lemma (7.5)(p. 301): A surface M has an area form iff it is orientable. On a connected surface there are exactly two area forms. We denote these by $\pm dM$.

This is nearly the definition of orientable. Recall M orientable iff $\exists \eta$ - 2-form on M where $\eta|_p \neq 0$ $\forall p \in M$. See Prop. 7.5 in Chpt. 4 for the rest, let's see, (p. 186 - 187 if time permits)

Prop. 7.5 says M orientable $\iff \exists\, U$ unit normal vect. field on M

The fundamental identity there is:

$$\mu(v,w) = U \cdot (v \times w)$$

In this way U induces 2-form on M.

Def^{n}/ If $\mathbf{x}$ is patchlike 2-segment in surface oriented by dM then,

$$\iint_{\mathbf{x}} dM = \iint_R dM(\mathbf{x}_u, \mathbf{x}_v)\, du\, dv$$

(+) $\mathbf{x}$ is positively oriented (gives area of $\mathbf{x}(R)$)

(−) $\mathbf{x}$ is negatively oriented. (gives − area of $\mathbf{x}(R)$)

Area of paving $\mathcal{P}$ given by positively oriented paving

$$\text{AREA}(\mathcal{P}) = \sum_{i=1}^{k} \iint_{\mathbf{x}_i} dM.$$

Def$^{\underline{n}}$ (7.6) Let ν be a 2-form on a pavable oriented region $\mathcal{P} \subset M$ then the integral of ν over $\mathcal{P}$

$$\iint_{\mathcal{P}} \nu = \sum_{i=1}^{k} \iint_{\mathbf{x}_i} \nu$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is positively oriented paving of $\mathcal{P}$.

To integrate $f : M \longrightarrow \mathbb{R}$ on $\mathcal{P}$ simply calculate

$$\iint_{\mathcal{P}} f \, dM$$

This we soon use to calculate $\iint K \, dM$ in next §.

Def$^{\underline{n}}$/(8.1)(p. 304) Let K be the Gaussian curvature of a compact surface M oriented by area form dM.

$$\iint_M K \, dM = \text{total Gaussian curvature of } M.$$

We can calculate the total curvature of any pavable region $\mathcal{P}$ in the same way.
Thus to calculate total curvature of M we

1.) find paving of M
2.) find total curvature for each 2-segment (which is assumed patchlike)

Then,

$$\iint_{\mathbf{x}} K \, dM = \iint_R \mathbf{x}^*(K \, dM) = \iint_R K(\mathbf{x}) \, \mathbf{x}^*(dM)$$

$$= \underbrace{\int_a^b \int_c^d}_{} K(\mathbf{x}) \sqrt{EG - F^2} \, du \, dv$$

(Typo here) $\rightarrow$ these should swap if $a \leq u \leq b$.

Example (8.2)(p. 305)

(1.) constant curvature:

$$K_{TOTAL} = \iint_M K\, dM = K \iint_M dM = K \text{ AREA}(M)$$

$$K_{TOTAL}(\text{sphere}) = \frac{1}{r^2}(4\pi r^2) = \boxed{4\pi.}$$

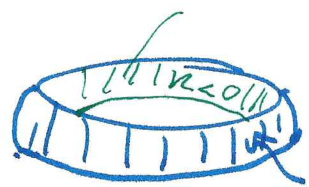
$$K_{TOTAL}(\text{bugle}) = \frac{-1}{c^2}(2\pi c^2) = \boxed{-2\pi}$$

(2.) TORUS: Let X be 2-segment which covers T

$$X^*(dT) = \sqrt{EG - F^2}\, du\, dv = r(R + r\cos(u))\, du\, dv$$

However, we can calculate,

$$K(X) = \frac{\cos(u)}{r(R + r\cos u)}$$



$$\therefore \iint_T K\, dT = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(u)\, du\, dv = \boxed{0.}$$

Apparently $K > 0$, $K < 0$ balance out on TORUS..

(3.) CATENOID: (based on Ex 7.1 of Chpt. 5)(p. 254-255)

$$\iint_X K\, dM = -\int_0^a \int_0^{2\pi} \frac{du\, dv}{\cosh^2(u/c)} = -4\pi \tanh\left(\frac{a}{c}\right)$$

As $a \to \infty$ find $K_{TOTAL} \to \boxed{-4\pi}$

All integer multiples of 2π

CURIOUS...

Def^{n}/(8.3) (p. 306) Let M, N be surfaces oriented by area forms dM and dN. Then the <u>Jacobian</u> of $F: M \longrightarrow N$ is the $\mathbb{R}$-valued function J_F on M such that

$$F^*(dN) = J_F \, dM$$

Let's calculate, def^{n} of pull-back of two-form.

$$J_F(p)\, dM(v,w) = F^*(dN)(v,w) = dN(F_*(v), F_*(w))$$

Note, F regular iff $J_F(p) \neq 0 \quad \forall p \in M$.

F is orientation preserving at p if $J_F(p) > 0$.

F is orientation reversing at p if $J_F(p) < 0$.

Moreover,

$$\underbrace{|J_F(p)|}_{\substack{\text{rate at}\\ \text{which } F \text{ is}\\ \text{expanding area at } P.}} |dM(v,w)| = |dN(F_*(v), F_*(w))|$$

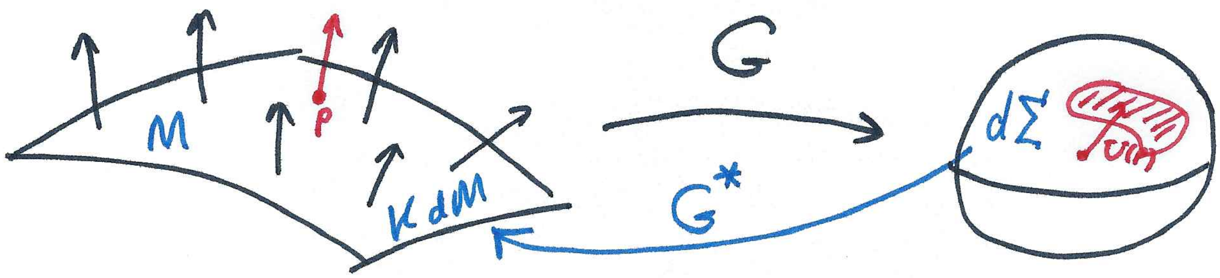
$$\iint_M J_F \, dm = \iint_M F^*(dN) = \text{signed area of } F(M) \text{ a.k.a. algebraic area.}$$

Can be (+), (−) depending on orientation of F ($J_F < 0$ vs. $J_F > 0$)

GAUSS MAP

$$G : M \longrightarrow \Sigma = \text{unit-sphere.}$$

$$G(P) = U(P) \leftarrow \text{unit-normal to } M \text{ at } P.$$



Th^{m} (8.4) THE GAUSSIAN CURVATURE K OF AN ORIENTED SURFACE $M \subset \mathbb{R}^3$ IS THE JACOBIAN OF ITS GAUSS MAP

Proof: If $U = \sum g_i U_i$ then $G = (g_1, g_2, g_3)$.

Recall the SHAPE OPERATOR,

$$-S(v) = \nabla_v U = \sum v[g_i] U_i$$

Thus, by Prop. 7.5 on pg. $F_*(v) = (v[f_1], \dots, v[f_n])_{F(P)}$.

$$G_*(v) = \sum_{i=1}^{3} v[g_i] U_i(G(P)) \quad (\star)$$

Thus $-S(v) \parallel G_*(v)$. We seek to show $K\,dM = G^*(d\Sigma)$

$$\begin{aligned}(K\,dM)(v,w) &= K(P)\,dM(v,w)\\ &= K(P)\,U(P)\cdot(v\times w)\\ &= U(P)\cdot S(v)\times S(w)\end{aligned}$$

Lemma 3.4 of Chpt. 5

Likewise,

$$\begin{aligned}G^*(d\Sigma)(v,w) &= d\Sigma(G_* v, G_* w)\\ &= \bar{U}(G(P))\cdot G_*(v)\times G_*(w)\\ &= U(P)\cdot S(v)\times S(w) \quad \text{by } \star. \;/\!/\end{aligned}$$

Th^m (8.4) (p. 308) THE GAUSSIAN CURVATURE K of an oriented surface $M \subset \mathbb{R}^3$ is the Jacobian of its Gauss map

Proof: given on previous page, a more or less routine calculation.//

Cor. (8.5) The total Gaussian curvature of an oriented surface $M \subset \mathbb{R}^3$ equals the algebraic area of the image of its Gauss map $G : M \longrightarrow \Sigma$

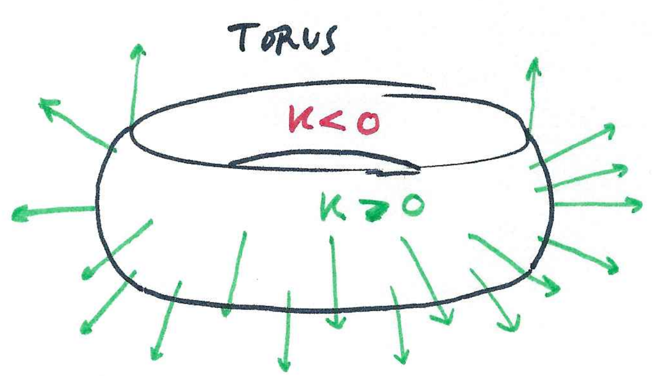
Proof: $\displaystyle\iint_M K\,dM = \iint_M G^*(d\Sigma) = \iint_{G(M)} d\Sigma =$ AREA OF $G(M)$.

Cor. (8.6) Let $\mathcal{R} \subset M \subset \mathbb{R}^3$ where $\mathcal{R}$ is oriented and

(1.) the Gauss map is 1-1 ($U(p) \neq U(q)$ for $p \neq q$.)

(2.) either $K \geq 0$ or $K \leq 0$ (no sign change)

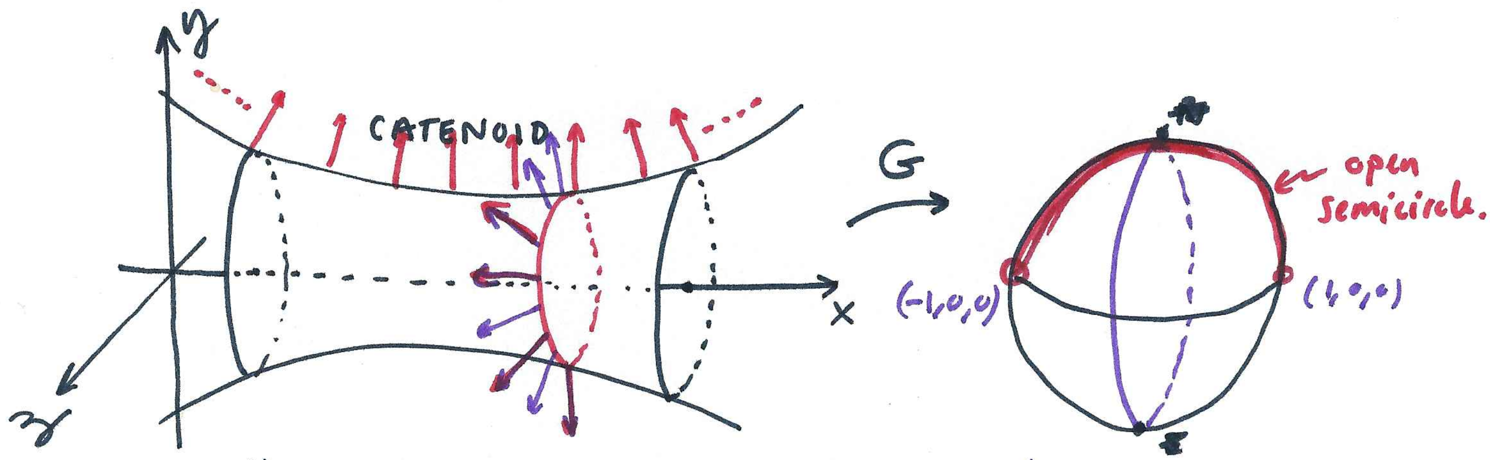
Then the total curvature of $\mathcal{R}$ is $\pm$ area of $G(\mathcal{R})$ where the sign matches K



G(inside torus) $= \Sigma$

G(outside torus) $= \Sigma$

$\therefore K_{inside} + K_{outside} = 0$. (by symmetry)



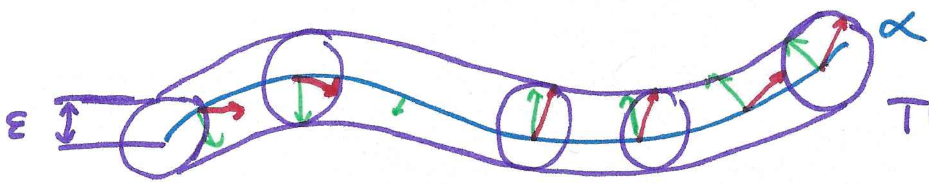
Oneil says G is 1-1, onto Σ modulo ~~$\pm(0,0,1) = N/S$~~ $\pm(1,0,0)$.

$\boxed{\text{Th}^{\underline{m}} \text{ (Fenchel's Th}^{\underline{m}}\text{)(Ex. \#18, §2.4)} \\ \text{A simple closed curve } \alpha \text{ in } \mathbb{R}^3 \text{ has total curvature } \int_\alpha \kappa\, ds \geq 2\pi}$

Proof: Let M be a tube around α with parametrization as described in #17 of §5.4. For $0 < \kappa \leq b$ define:

$$\mathbf{X}(u,v) = \alpha(u) + \varepsilon\left(\cos(v) N(u) + \sin(v) B(u)\right)$$

$B = T \times N$

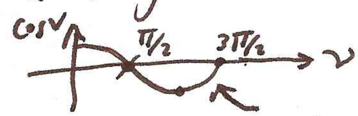


TUBE RADIUS ε.

It is shown $K = \dfrac{-\kappa(u)\cos v}{\varepsilon(1-\kappa(u)\varepsilon\cos(v))}$ in Ex. #17 on p.234 (it is left to you to show it)

note, $K \geq 0$ on $\mathbf{X}(D)$ where

$D: \quad 0 \leq u \leq L, \ \underbrace{\pi/2 \leq v \leq 3\pi/2}_{\cos v < 0 \text{ here}}$ and L is length of α.



$W = \sqrt{EG - F^2}$ (pg. 228, I did not recall this otherwise)

$$\iint_{\substack{B \\ \shortparallel \\ \mathbf{X}(D)}} K\, dM = \int_0^L \int_{\pi/2}^{3\pi/2} \frac{-\kappa(u)\cos v}{\varepsilon(1-\kappa(u)\varepsilon\cos v)} \left\| -\varepsilon(1-\kappa\varepsilon\cos v)\binom{\text{unit}}{\text{vector}}\right\| du\,dv$$

$$= -\int_0^L \int_{\pi/2}^{3\pi/2} \kappa(u)\cos(v)\, du\,dv$$

$$= -\int_0^L \kappa\, du \int_{\pi/2}^{3\pi/2} \cos(v)\,dv$$

$$= 2\int_0^L \kappa\, ds \quad (*)$$

Oneil then argues $K(p) \geq 0 \ \forall p \in B$ and $G: B \to \Sigma$ is onto hence $\underline{\iint_B K\, dM \geq 4\pi}\ (*)$ (I'm not sure I "see" it yet, but I include this for fun.)

Thus $2\int_0^L \kappa\, ds \geq 4\pi \ \therefore \ \int_0^L \kappa\, ds \geq 2\pi \ \therefore \ \int_\alpha \kappa\, ds \geq 2\pi$. //

Angle Measure along surface

Let M be surface with unit-normal $\mathcal{U}$.
If V is tangent to M then

$$J(V) = \mathcal{U} \times V$$

is <u>orthogonal</u> to V and $\|J(V)\| = \|V\|$.
Thus $J^T J = I$ and it can be shown the angle J rotates each vector in T_pM by $90°$

I will prove this by deep magic linear. Oh, maybe not too deep. But,

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \underbrace{\text{trace}(R(\theta)) = 2\cos\theta + 1}_{\text{invariant, if we change coordinates this is still true}}$$

Let $\{E_1, E_2, \mathcal{U}\}$ be frame on M,

Wait, this is two-dim'l, $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
trace $(R(\theta)) = 2\cos\theta$. Set $E_2 = \mathcal{U} \times E_1 \Rightarrow E_1 = E_2 \times \mathcal{U} = -\mathcal{U} \times E_2$

$$J(E_1) = \mathcal{U} \times E_1 = E_2$$
$$J(E_2) = \mathcal{U} \times E_2 = -E_1$$

$$[J]_{\beta,\beta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \underline{\theta = \pi/2}.$$ oh, no need for trace technology.

Def$^{\underline{n}}$/ Let v, w be unit tangent vectors in T_pM where M is <u>oriented</u>. A number φ is the <u>oriented</u> <u>angle</u> from V to W provided

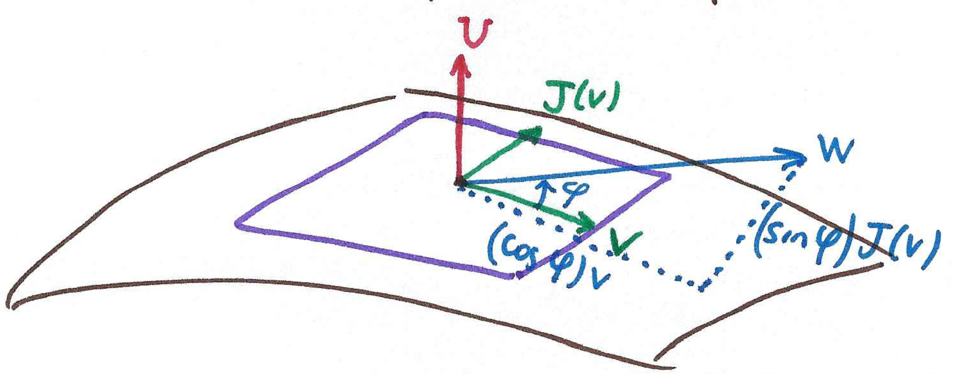
$$W = (\cos\varphi)V + (\sin\varphi)J(V)$$

Def^n/ Let v and w be unit-vectors on an oriented surface M. A number φ is an oriented angle from v to w provided

$$w = (\cos\varphi)v + (\sin\varphi)J(v)$$

uniquely defines φ for $0 \le \varphi \le 2\pi$

No picture in Oneil, I'll attempt one.



not to scale, note $\|v\| = \|w\| = 1$.

$$\|\cos\varphi\, v + (\sin\varphi)J(v)\| = \sqrt{\cos^2\varphi\, v\cdot v + 2\cos\varphi\sin\varphi\, \cancel{v\cdot J(v)}^{\,0} + \sin^2\varphi\, J(v)\cdot J(v)}$$

$$= \sqrt{(\cos^2\varphi + \sin^2\varphi)\, v\cdot v}$$

$$= 1 \cdot \cancel{\|v\|} \to 1$$

It was assumed $\|v\| = \|w\| = 1$.

Lemma (8.8) (p. 311) Let $\alpha : I \to M$ be a curve in an oriented surface M. If V, W are non vanishing tangent vector fields on α then there is a diff. fnct. φ on I s.t. for each $t \in I$, $\varphi(t)$ is an oriented angle from $V(t)$ to $W(t)$

Proof: since V, W are non vanishing wlog $\|V\| = \|W\| = 1$

Note, $\{V, J(V)\}$ is frame field on α. Hence,

$$W = \underbrace{(W\cdot V)}_{f}V + \underbrace{(W\cdot J(V))}_{g}J(V)$$

Continuing proof of Lemma 8.8

$1 = W \cdot W = f^2 + g^2$ maybe I did this one.

now apply Exercise 12 of §2.1 to obtain diff. fnct φ on I s.t. $f = \cos\varphi$, $g = \sin\varphi$. It follows $W(t) = \cos\varphi(t)\, V(t) + \sin\varphi(t)\, J(V(t))$.

That is $\varphi(t)$ is oriented $\angle$ from $V(t)$ to $W(t)$ along α //

Remark $\varphi(t) = \angle(V(t), W(t))$

or just $\varphi = \angle(V, W)$ convenient notation.

NATURAL CHOICES

If M is oriented, use positively oriented patches $dM(\mathbf{x}_u, \mathbf{x}_v) > 0$ and positively oriented frame fields $dM(E_1, E_2) = 1$. In this case, $dM = \theta_1 \wedge \theta_2$. Moreover, if $\hat{u}$ is tangent vector then $\hat{u}, J(\hat{u})$ is a positively oriented frame. Moreover, for any non vanishing vector field on M we define a positively oriented frame field

$$\underbrace{E_1 = \frac{V}{\|V\|} \quad \text{and} \quad E_2 = J(E_1) = \frac{J(V)}{\|V\|}}_{\text{frame associated to } V.}$$