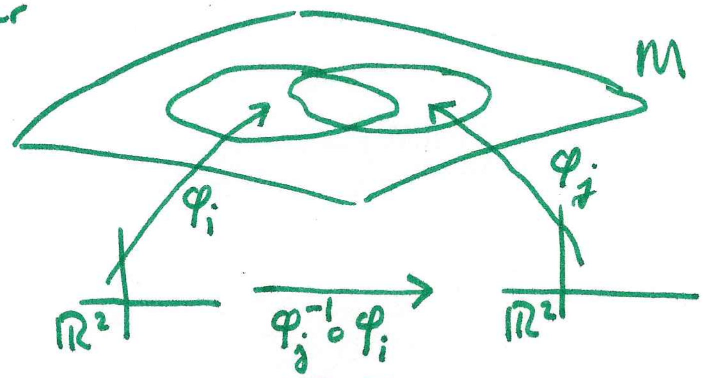
LECTURE 23:

CHAPTER 7: RIEMANNIAN GEOMETRY

Def^n/ M is an abstract surface if $\exists$ patches $\varphi_i : D_i \longrightarrow U_i \subset M$ s.t. $\bigcup_i U_i = M$ and $\varphi_i, \varphi_j, i \neq j$ are smoothly related (compatible) whenever $U_i \cap U_j \neq \phi$; in particular



$\underbrace{\varphi_j^{-1} \circ \varphi_i}$ must be smooth mapping on $\mathbb{R}^2$.

We say M is a geometric surface if M is paired with a metric $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ at each $p \in M$ and $p \longmapsto g_p$ is smooth. The metric $g_p(v, w) = \langle v, w \rangle$ (a usual notation) must be an inner product on T_pM,

1.) $\langle v, w \rangle = \langle w, v \rangle$

2.) $\langle cv_1 + v_2, w \rangle = c\langle v_1, w \rangle + \langle v_2, w \rangle$
$\quad\ \langle w, cv_1 + v_2 \rangle = c\langle w, v_1 \rangle + \langle w, v_2 \rangle$

3.) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

These requirements give us Riemannian geometry. If 1.) or 3.) is modified other types of geometry (symplectic, or Lorentzian etc...) may be obtained.

Remark: replace $\mathbb{R}^2$ with $\mathbb{R}^n$ you get an n-dim'l Riemannian manifold modulo some topological comments...

Given a metric $\langle , \rangle$ we may define lengths and angles ~~of~~ as follows:

$\boxed{\text{Def}^{\underline{n}}/ \quad \|V\| = \sqrt{\langle V, V\rangle} \quad \& \quad \underbrace{\cos\theta = \frac{\langle V, W\rangle}{\|V\|\|W\|}}_{\text{for } V, W \neq 0.}}$

Methods to Construct Metrics

(1.) Distort old metric to form new. On $\mathbb{R}^2$,

$$\langle V, W\rangle = \frac{V \cdot W}{h^2(p)}$$

For $h > 0$ defines geometric surface M conformal with ruler function h

Claim: "locally" every geometric surface is so expressed.

(2.) Pullback: If M is an abstract surface and N is a geometric surface with metric g then we define, via $F: M \to N$ a regular map

$$\langle V, W\rangle_M = \langle F_*(V), F_*(W)\rangle_N$$
$$= g(dF(V), dF(W)).$$

Bilinearity of $\langle , \rangle_M$ is clear from linearity of dF and bilinearity of g. Symmetry & positive definite props also transfer directly from g to $\langle , \rangle_M$.

Notation: $\langle , \rangle_M = F^*(g)$.

NOTE: $\langle V, W\rangle_M = \langle F_*(V), F_*(W)\rangle$ hence $(M, F^*(g))$ is isometric to (N, g). We should refer to a geometric surface as a pair since we'll soon see the same point set supports multiple geometries! The metric determines the geometry.

(3.) Given functions E, F, G on abstract surface M with suitable properties ($E, G, EG > F^2$) we can construct $\langle , \rangle$ by imposing

$$\langle X_u, X_u \rangle = E$$
$$\langle X_u, X_v \rangle = F$$
$$\langle X_v, X_v \rangle = G$$

See Exercise 4. For some indication on <u>how</u> to do this. (I'm a bit stuck on those details at moment.)

We now work on describing how our intrinsic calculus on $M \subset \mathbb{R}^3$ is naturally carried to defining geometric objects (frames, connection form, curvature...) on <u>M</u> abstract.

Defn/ Frame field on $(M, \langle , \rangle)$ consists of two <u>orthonormal</u> vector fields E_1, E_2 defined on some open subset of M. Here we should say $\langle , \rangle$-orthonormal to emphasize the $\langle , \rangle$-dependence:

$$\langle E_1, E_1 \rangle = 1, \quad \langle E_2, E_2 \rangle = 1, \quad \langle E_1, E_2 \rangle = 0.$$

Dual frames also determined same as before:

Defn/ Given frame field E_1, E_2 on $(M, \langle , \rangle)$ we say θ_1, θ_2 is dual frame to E_1, E_2 iff $\theta_i(E_j) = \delta_{ij}$ and $\theta_1^{(p)}, \theta_2^{(p)} : T_pM \to \mathbb{R}$ linear for each $p \in \mathcal{U}$ where $\mathcal{U}$ is the open set on which E_1, E_2 is frame.

Def$^{\underline{n}}$/ Given frame E_1, E_2 with coframe θ_1, θ_2 on $(M, \langle\ ,\ \rangle)$ we define ω_{12} implicitly via

$$d\theta_1 = \omega_{12} \wedge \theta_2 \quad \& \quad d\theta_2 = \omega_{21} \wedge \theta_1$$

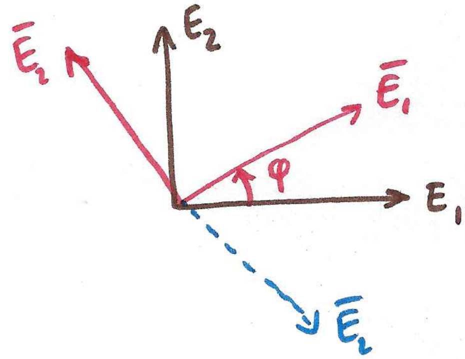
To be explicit, $\omega_{12} = \omega_{12}(E_1)\,\theta_1 + \omega_{12}(E_2)\,\theta_2$

and $\omega_{12} \wedge \theta_1 = \omega_{12}(E_2)\,\theta_2 \wedge \theta_1 = -d\theta_2 \quad \therefore\ d\theta_2 = \omega_{12}(E_2)\theta_1 \wedge \theta_2$

$\omega_{12} \wedge \theta_2 = \omega_{12}(E_1)\,\theta_1 \wedge \theta_2 = d\theta_1$

Hence, $\underline{\omega_{12} = (d\theta_1(E_1, E_2))\theta_1 + (d\theta_2(E_1, E_2))\theta_2}$.

We need to investigate the coordinate dependence of $\theta_1, \theta_2, \omega_{12}$ ... if we had $\bar{E}_1, \bar{E}_2$ another frame then how will $\bar{\theta}_1, \bar{\theta}_2, \bar{\omega}_{12}$ relate to the original coord. system's forms?



$\leftarrow$ both frames orthonormal $\Rightarrow \exists\, \varphi$ s.t.

$$\bar{E}_1 = \cos\varphi\, E_1 + \sin\varphi\, E_2$$

$$\left.\begin{aligned} \bar{E}_2 &= -\sin\varphi\, E_1 + \cos\varphi\, E_2 \\ \bar{E}_2{}' &= \sin\varphi\, E_1 - \cos\varphi\, E_2 \end{aligned}\right\} \text{either choice gives } \langle \bar{E}_i, \bar{E}_j \rangle = \delta_{ij}$$

Note, $\bar{E}_1, \bar{E}_2$ have same orientation as E_1, E_2

However, $\bar{E}_1, \bar{E}_2{}'$ have opposite orientation as E_1, E_2

LEMMA (1.4): Let E_1, E_2 and $\bar{E}_1, \bar{E}_2$ be frame fields on the same region in M. If these frame fields have:

(1.) the same orientation, then

$$\bar{\omega}_{12} = \omega_{12} + d\varphi \quad \text{and} \quad \bar{\theta}_1 \wedge \bar{\theta}_2 = \theta_1 \wedge \theta_2$$

(2.) opposite orientation, then

$$\bar{\omega}_{12} = -(\omega_{12} + d\varphi) \quad \text{and} \quad \bar{\theta}_1 \wedge \bar{\theta}_2 = -\theta_1 \wedge \theta_2$$

Proof: (1.)

$$\bar{E}_1 = \cos\varphi\, E_1 + \sin\varphi\, E_2$$
$$\bar{E}_2 = -\sin\varphi\, E_1 + \cos\varphi\, E_2$$

$$\theta_1 = \theta_1(\bar{E}_1)\bar{\theta}_1 + \theta_1(\bar{E}_2)\bar{\theta}_2 = \cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2$$
$$\theta_2 = \theta_2(\bar{E}_1)\bar{\theta}_1 + \theta_2(\bar{E}_2)\bar{\theta}_2 = \sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2$$

Observe,

$$\begin{aligned} \theta_1 \wedge \theta_2 &= (\cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2) \wedge (\sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2) \\ &= \cos^2\theta\,\bar{\theta}_1 \wedge \bar{\theta}_2 - \sin^2\varphi\,\bar{\theta}_2 \wedge \bar{\theta}_1 \\ &= \bar{\theta}_1 \wedge \bar{\theta}_2. \end{aligned}$$

Also, ext. diff to obtain,

$$d\theta_1 = -\sin\varphi\, d\varphi \wedge \bar{\theta}_1 - \cos\varphi\, d\varphi \wedge \bar{\theta}_2 + \cos\varphi\, d\bar{\theta}_1 - \sin\varphi\, d\bar{\theta}_2$$
$$d\theta_2 = \cos\varphi\, d\varphi \wedge \bar{\theta}_1 - \sin\varphi\, d\varphi \wedge \bar{\theta}_2 + \sin\varphi\, d\bar{\theta}_1 + \cos\varphi\, d\bar{\theta}_2$$

Oh, now $d\bar{\theta}_1 = \bar{\omega}_{12} \wedge \bar{\theta}_2$ and $d\bar{\theta}_2 = \bar{\omega}_{21} \wedge \bar{\theta}_1$ hence,

$$\begin{aligned} d\theta_1 &= -\sin\varphi\, d\varphi \wedge \bar{\theta}_1 - \cos\varphi\, d\varphi \wedge \bar{\theta}_2 \\ &\quad + \sin\varphi\,\bar{\omega}_{12} \wedge \bar{\theta}_2 + \cos\varphi\,\bar{\omega}_{21} \wedge \bar{\theta}_1 \\ &= (\cos\varphi\,\bar{\omega}_{21} - \sin\varphi\, d\varphi) \wedge \bar{\theta}_1 + (\sin\varphi\,\bar{\omega}_{12} - \cos\varphi\, d\varphi) \wedge \bar{\theta}_2 \\ &= \end{aligned}$$

Proof of $\bar{\omega}_{12} = \omega_{12} + d\varphi$:

Recall $\theta_1 = \cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2$ & $\theta_2 = \sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2$

Thus,

$$\begin{aligned}
d\theta_1 &= (-\sin\varphi\,\bar{\theta}_1 - \cos\varphi\,\bar{\theta}_2)\wedge(-d\varphi) + \cos\varphi\, d\bar{\theta}_1 - \sin\varphi\, d\bar{\theta}_2 \\
&= (\sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2)\wedge d\varphi + \cos\varphi\, d\bar{\theta}_1 - \sin\varphi\, d\bar{\theta}_2 \\
&= \theta_2 \wedge d\varphi + \cos\varphi\,\bar{\omega}_{12}\wedge\bar{\theta}_2 - \sin\varphi\,\bar{\omega}_{21}\wedge\bar{\theta}_1 \\
&= \theta_2\wedge d\varphi - \cos\varphi\,\bar{\theta}_2\wedge\bar{\omega}_{12} - \sin\varphi\,\bar{\theta}_1\wedge\bar{\omega}_{12} \\
&= \theta_2\wedge d\varphi - (\cos\varphi\,\bar{\theta}_2 + \sin\varphi\,\bar{\theta}_1)\wedge\bar{\omega}_{12} \\
&= \theta_2\wedge d\varphi - \theta_2\wedge\bar{\omega}_{12} \\
&= (\bar{\omega}_{12} - d\varphi)\wedge\theta_2 = \omega_{12}\wedge\theta^2
\end{aligned}$$

Likewise,

$$\begin{aligned}
d\theta_2 &= d\varphi\wedge(\cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2) + \sin\varphi\, d\bar{\theta}_1 + \cos\varphi\, d\bar{\theta}_2 \\
&= d\varphi\wedge\theta_1 + \sin\varphi\,\bar{\omega}_{12}\wedge\bar{\theta}_2 + \cos\varphi\,\bar{\omega}_{21}\wedge\bar{\theta}_1 \\
&= d\varphi\wedge\theta_1 + \bar{\omega}_{12}\wedge(\underbrace{\sin\varphi\,\bar{\theta}_2 - \cos\varphi\,\bar{\theta}_1}_{(-)}) \\
&= d\varphi\wedge\theta_1 - \bar{\omega}_{12}\wedge\theta_1 \\
&= (\bar{\omega}_{12} - d\varphi)\wedge\theta_1 = \omega_{21}\wedge\theta^1
\end{aligned}$$

Hence $\omega_{12} = \bar{\omega}_{12} - d\varphi$ // (Recall from Chpt. 6, satisfying ~~the~~ 1st structural eq$^{\underline{n}}$s uniquely determines ω_{12})

Remark: $\omega_{12} = \bar{\omega}_{12} - d\varphi$ shows the choice of frame is tied to choice of angle...