§7.2 GAUSSIAN CURVATURE

LECTURE 24

In the abstract case we have no shape operator so $\det(S) = K$ is not going to work. However, the 2nd structural eq^{n} generalizes nicely,

$\boxed{\text{Th}^{m}}$ (2.1) Given $(M, \langle , \rangle)$ there is a unique real-valued function K s.t. for every frame field on M the 2nd structural eq^{n}

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

holds. We call K the GAUSSIAN CURVATURE OF M

Proof: For a given frame field, two-dim'l calculus reveals $\exists K$ s.t. $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$ (ω_{12} is 1-form so this is inevitable as $\theta_1 \wedge \theta_2$ is the only two-form.)

Likewise, for $\bar{E}_1, \bar{E}_2$ frame, $d\bar{\omega}_{12} = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2$. We need to show $K = \bar{K}$ if the frames domains overlap.

Recall from last section $\bar{\omega}_{12} = \pm(\omega_{12} + d\varphi)$

and $\bar{\theta}_1 \wedge \bar{\theta}_2 = \pm(\theta_1 \wedge \theta_2)$ + (oriented same way) − (reverse oriented)

Consider then,

$$\begin{aligned} d\bar{\omega}_{12} &= d[\pm(\omega_{12} + d\varphi)] \\ &= \pm\, d\omega_{12} + d(d\varphi) \\ &= \mp K\,\theta_1 \wedge \theta_2 \\ &= \mp K(\pm\,\bar{\theta}_1 \wedge \bar{\theta}_2) \\ &= -K\,\bar{\theta}_1 \wedge \bar{\theta}_2 = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2 \Rightarrow \underline{K = \bar{K}} \; // \end{aligned}$$

Remark: K is determined w/o regard to the orientation of the frame. We can calculate K for non-orientable surfaces.

(8.5)

$$\langle a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v \rangle$$

$$= ac\,\underbrace{\cancel{\langle \mathbf{x}_u, \mathbf{x}_u \rangle}}_{1} + (ad+bc)\,\underbrace{\cancel{\langle \mathbf{x}_u, \mathbf{x}_v \rangle}}_{0} + bd\,\underbrace{\cancel{\langle \mathbf{x}_v, \mathbf{x}_v \rangle}}_{1}$$

$$= \underline{ac + bd}.$$

$$\underset{T_0}{\begin{pmatrix} E = 1 \\ F = 0 \\ G = 1 \end{pmatrix}} \longleftrightarrow \underset{\mathbb{R}^2}{\begin{pmatrix} E = 1 \\ F = 0 \\ G = 1 \end{pmatrix}}$$

Frame Geometry Eq's

$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$
$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

TRIVIAL EXAMPLE:

$M = \mathbb{R}^2$ with U_1, U_2 frame has coframe dx, dy
$\theta_1 = dx$ and $\theta_2 = dy$. Clearly $d\theta_1 = 0$, $d\theta_2 = 0$
thus $\omega_{12} = 0$ and $d\omega_{12} = 0 \Rightarrow \underline{K = 0}$.

EXAMPLE (2.2) A FLAT TORUS $\left(X(u,v) = \left((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u \right) \right)$

Let T be torus of revolution and define

$$\langle X_u, X_u \rangle = 1$$
$$\langle X_u, X_v \rangle = 0$$
$$\langle X_v, X_v \rangle = 1$$

(Fix $u = u_0$ get circle in $z = r\sin u_0$ plane of radius $R + r\cos u_0$.)

these define $(T, \langle , \rangle)$ or simply T_0. Observe $X_*(U_1) = X_u$ and $X_*(U_2) = X_v$ hence $K(T_0) = K(\mathbb{R}^2) = 0$, it's isometric to plane, I should clarify the Euclidean plane.

Remark: $T_0 \subset \mathbb{R}^3$ however, $\langle , \rangle$ is not induced from Euclidean metric on $\mathbb{R}^3$.

Remark: T_0 is compact subset of $\mathbb{R}^3$ as constructed thus (if given geometry induced from its shape in $\mathbb{R}^3$ meaning the metric of T_0 were induced from $\mathbb{R}^3$) cannot be everywhere flat ($K = 0$ on T_0) as Th^{m} (3.5) of Chpt. 6 $\Rightarrow$ $K(P) > 0$ at at least one point of T_0 $\therefore$ the metric we gave to T_0 cannot be realized as a restriction of the Euclidean metric on $\mathbb{R}^3$.

BIG PICTURE:

- BEFORE CHAPTER 7: STUDIED SURFACES IN $\mathbb{R}^3$ WHOSE METRIC WAS INDUCED FROM METRIC ON $\mathbb{R}^3$
- CHAPTER 7 and Beyond: METRIC GIVEN TO ABSTRACT SURFACE NEED NOT BE INDUCED FROM AMBIENT CONTEXT. However, it can be as all surfaces studied in previous Chapters are also GEOMETRIC SURFACES

Remark: Given a geometric surface M when does $\exists n \in \mathbb{N}$ and $\overline{M} \subset \mathbb{R}^n$ for which $F : M \to \overline{M} \subseteq \mathbb{R}^n$ is an isometry?

Just because $n = 3$ fails for torus ~~embedding~~ doesn't mean it cannot fit inside larger $\mathbb{R}^n$... see Whitney embedding Th^{m}; and I think for our context, NASH gave the isometric version (Whitney just focused on abstract surface, no geometry)

Cor (2.3): If $\mathbb{R}^2$ has $\langle v,w\rangle_p = \frac{1}{h^2(p)}(v\cdot w)$ then,

$$K = h(h_{uu}+h_{vv}) - (h_u^2+h_v^2) = h^2\Delta \log h$$

Proof: Note $E = 1/h^2 = G$ and $F=0$. Also we found for orthogonal patches $(F=0)$ that

$$K = \frac{-1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}[\sqrt{G}]\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}[\sqrt{E}]\right]\right)$$

$$= -h^2\left(\frac{\partial}{\partial u}\left[h\left[\frac{-1}{h^2}\frac{\partial h}{\partial u}\right]\right] + \frac{\partial}{\partial v}\left[h\frac{\partial}{\partial v}\left[\frac{1}{h}\right]\right]\right)$$

$$= h^2\left(\underline{\frac{\partial}{\partial u}\left[\frac{1}{h}\frac{\partial h}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{h}\frac{\partial h}{\partial v}\right]}\right) *$$

$$= h^2\left[\frac{-1}{h^2}\left(\frac{\partial h}{\partial u}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial u^2} + \left(\frac{-1}{h^2}\right)\left(\frac{\partial h}{\partial v}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial v^2}\right]$$

$$= -\left(\frac{\partial h}{\partial u}\right)^2 - \left(\frac{\partial h}{\partial v}\right)^2 + h\left(\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2}\right). //$$

The fact this can be expressed as $h^2\Delta\log(h)$ follows from Ex. #2 of §6.6 where we learn $\Delta f = f_{uu} + f_{vv}$ the <u>LAPLACIAN</u>.

$$\Delta \log(h) = \frac{\partial^2}{\partial u^2}[\log(h)] + \frac{\partial^2}{\partial v^2}[\log(h)]$$

$$= \underline{\frac{\partial}{\partial u}\left[\frac{1}{h}\frac{\partial h}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{h}\frac{\partial h}{\partial v}\right]} *$$

Hence $\boxed{K = h^2\Delta\log(h)}$. //

<u>Remark</u>: $\mathbb{R}^2$ with $X(u,v) = (u,v)$ and $\langle\,,\rangle$ above is <u>conformal</u>, I mean the "patch is conformal" not isometric to Euclidean plane. See Remark 1.3 pg. 323.

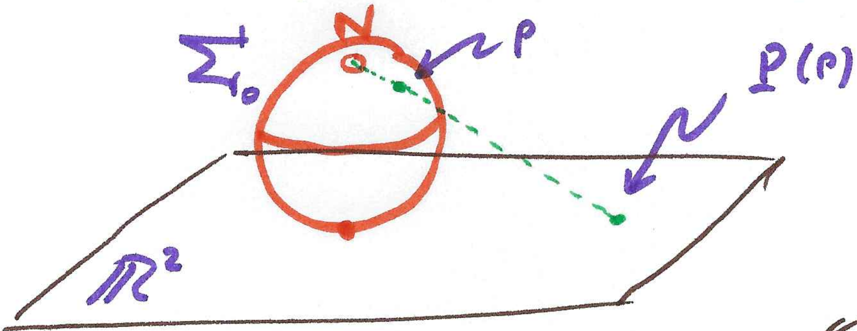
Example 2.4 (pg. 331)

(1.) THE STEREOGRAPHIC SPHERE (see Ex. 5.5 of Chpl. 4)

- Let Σ be unit sphere resting on xy-plane with center at $(0,0,1)$. Let $\underbrace{\Sigma_0 = \Sigma - \{(0,0,\overset{N}{2})\}}_{\text{punctured sphere, is missing North Pole}}$

- imagine light source at top N and for each $p \in \Sigma$ let $P(p)$ be shadow of p in the xy-plane

Here we identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$ for convenience.



$$P(p_1, p_2, p_3) = \left(\frac{2p_1}{2-p_3}, \frac{2p_2}{2-p_3} \right)$$

page 167 has derivation. It's similar Δ and some trig/algebra.

Clearly $P : \Sigma_0 \to \mathbb{R}^2$ is smooth as $p_3 \neq 2$ for $p \in \Sigma_0$. In fact $P^{-1} : \mathbb{R}^2 \to \Sigma_0$ is likewise smooth,

$$P^{-1}(x,y) = (p_1, p_2, p_3)$$

Just solve $x = \dfrac{2p_1}{2-p_3}$ and $y = \dfrac{2p_2}{2-p_3}$

for p_1, p_2, p_3 given $p_3 \geq 0$, $p_1^2 + p_2^2 + (p_3 - 1)^2 = 1$.

$$x(2-p_3) = 2p_1 \longrightarrow 2p_1 + xp_3 = 2x$$
$$y(2-p_3) = 2p_2 \longrightarrow 2p_2 + yp_3 = 2y$$

etc... it's just algebra ☺.

Remark: remind me to give handout from pages 120-122 and Chpt. 15... of John McCleary's "GEOMETRY FROM A DIFFERENTIAL VIEWPOINT"

(1.) STEREOGRAPHIC SPHERE CONTINUED:

- pull-back the Euclidean metric $\underbrace{ds^2 = dx^2 + dy^2}_{g_{Euclidean}}$ on $\mathbb{R}^2$ under the steographic projection map $P : \Sigma_0 \longrightarrow \mathbb{R}^2$. It follows the abstract surface Σ_0 paired with $P^*(g_{Euclid.})$ is FLAT ! $K(\Sigma_0) = 0$.
- "intrinsically" Σ_0 is as flat as the Euclidean plane.

(2.) The STEREOGRAPHIC PLANE : Now turn the tables and pull-back the usual induced (curved) metric on Σ_0 to the plane $\mathbb{R}^2$ (originally thought of w/o geometry!) to obtain $(\mathbb{R}^2, (P^{-1})^*(g_{sphere}))$ the

STEREOGRAPHIC PLANE.

- intrinsically this non-Euclidean plane is curved just like the sphere Σ_0, it has constant Gaussian curvature $K = 1$.

CALCULATION : (see 332) you can show

$$\langle v, w \rangle = (P^{-1})_*(v) \cdot (P^{-1})_*(w) = \left(1 + \frac{\|q\|^2}{4}\right) v \cdot w$$

Hence the steographic plane is conformally related to the Eucidean plane with

$$h(u,v) = 1 + \frac{u^2 + v^2}{4}$$

Rulers get longer as they move further from origin. Small circle at N on Σ_0 gives stupidly large circle in steographic plane. See nbhds of ∞ in $\mathbb{C}$ - variables...

Example 2.5 (THE HYPERBOLIC PLANE)

Use ruler function $h = 1 - \frac{u^2+v^2}{4}$ as opposed to (+) for stereographic plane...

Since $h > 0$ is needed we consider $u^2+v^2 < 4$ given this non-euclidean geometry: the

(H) Poincaré disk model of hyperbolic plane

$$\langle v, w \rangle = \frac{v \cdot w}{h^2} \quad \text{for } v, w \in \{(u,v) \mid u^2+v^2<4\}$$

(oneil does not give us the mapping from $\mathbb{R}^2 \longrightarrow (H, \langle , \rangle)$ which is an isometry) *

However, we showed $K = h^2 \Delta \log h$ hence

$$K = \cancel{h^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\left(1 - \frac{u^2}{4} - \frac{v^2}{4}\right)}$$

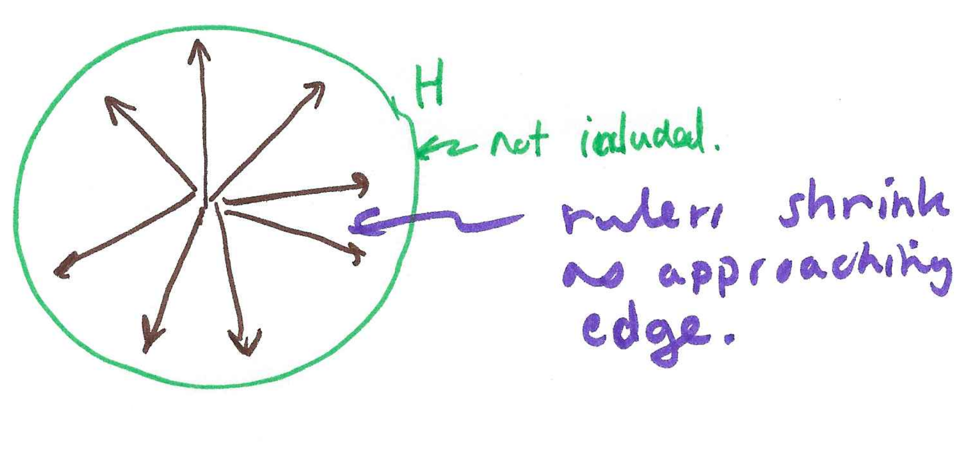
why? instead

$$= h(h_{uu} + h_{vv}) - h_u^2 + h_v^2$$

$$= \left(1 - \frac{1}{4}(u^2+v^2)\right)\left(-\frac{1}{2} - \frac{1}{2}\right) + \frac{u^2}{4} + \frac{v^2}{4}$$

$$= \underline{-1}.$$

Despite * we've said something interesting

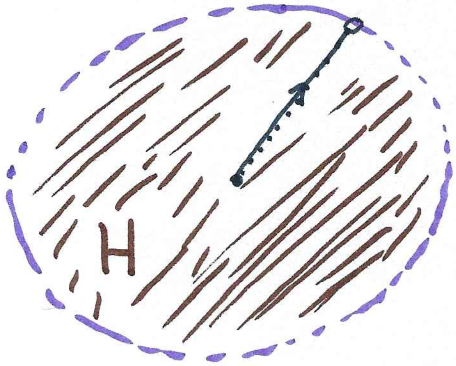


Ex 2.5 : HYPERBOLIC PLANE, CONTINUED

$(\mathbb{R}^2, \langle , \rangle)$ paired here $\langle v, w\rangle = \dfrac{v \cdot w}{h^2}$

$$h(u,v) = 1 - \frac{u^2+v^2}{4}$$

Note, $h > 0 \Rightarrow u^2+v^2 < 4$ we define $H = \{(u,v) \mid u^2+v^2 < 4\}$ paired with $\langle , \rangle$ above as the **HYPERBOLIC PLANE**



$\gamma(t) = (t\cos\theta, t\sin\theta)$

for $0 \le t < 2$

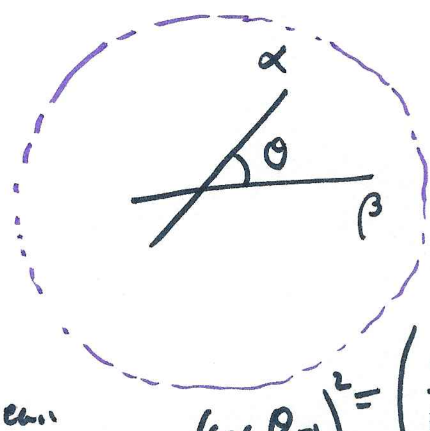
Find length of this arc.

(in hyperbolic sense)

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{\langle \gamma'(t), \gamma'(t)\rangle} = \frac{1}{h(\gamma(t))} = \frac{1}{1 - t^2/4}$$

$$s(t) = \int_0^t \frac{du}{1-(u/2)^2} = 2\tanh^{-1}\left(\frac{t}{2}\right) = \ln\left(\frac{t+2}{2-t}\right)$$

As $t \to 2^-$ we find $s(t) \to \infty$.



$$\underbrace{\theta_{\text{Euclidean}}}_{} = \theta_{\text{hyperbolic}}$$

$$\frac{\alpha' \cdot \beta'}{\|\alpha'\|_E \|\beta'\|_E} = \cos\theta_E \longleftrightarrow \frac{\langle \alpha', \beta'\rangle}{\|\alpha'\|_H \|\beta'\|_H}$$

angles preserved from Euclidean to hyperbolic.

$$(\cos\theta_H)^2 = \left(\frac{\langle \alpha', \beta'\rangle}{\|\alpha'\|_H \|\beta'\|}\right)^2 = \frac{(\alpha' \cdot \beta')^2}{h^4} \frac{1}{\frac{\alpha'\cdot\alpha'}{h^2}\frac{\beta'\cdot\beta'}{h^2}}$$

$$= \frac{(\alpha' \cdot \beta')^2}{(\alpha'\cdot\alpha')(\beta'\cdot\beta')} = (\cos\theta_{Euc})^2 ..$$

PROP. 2.6 (p. 333)

Let $F: M \longrightarrow N$ be regular mapping of a geometric surface M onto a surface N (without geometry). Suppose that whenever $F(P_1) = F(P_2)$ there is an isometry G_{12} from a neighborhood of P_1 to a nbhd. of P_2 s.t. $F \circ G_{12} = F$, $G_{12}(P_1) = P_2$. THEN there is a unique metric tensor on N that makes F a local isometry.

Proof: If F is a local isometry then we must

require $\underbrace{\langle V_1, W_1 \rangle_M}_{\text{at } P} = \underbrace{\langle F_*(V_1), F_*(W_1) \rangle_N}_{\text{at } q = F(P)}$. Oops he uses V, W for $F_*(V_1), F_*(W_1)$ in both proof. (fixed ☺)

oh, following Oneil, set $V = F_*(V_1)$, $W = F_*(W_1)$ to be vectors in $T_{F(P)}N$ and so define

$$\langle V, W \rangle_N = \langle V_1, W_1 \rangle_M . \ (\star)$$

What if $F(P_2) = F(P) = q$? Let V_2, W_2 be the vectors (unique by regularity which gave 1-1 for F_* at each point) vectors in $T_{P_2}M$ for which $F_*(V_2) = V$ and $F_*(W_2) = W$. Following $(\star)$

$$\langle V, W \rangle_N = \langle V_2, W_2 \rangle_M$$

Obviously, we have a problem if $\langle V_1, W_1 \rangle_M \neq \langle V_2, W_2 \rangle_M$! However, the G_{12} isometry saves the day.

$$(G_{12})_* : T_{P_1}M \longrightarrow T_{P_2}M$$

$$\langle V_1, W_1 \rangle_M = \langle G_{12*}(V_1), G_{12*}(W_1) \rangle_M = \langle V_2, W_2 \rangle_M \ ///$$

Example 2.7 (THE PROJECTIVE PLANE)

projective plane P is abstract surface formed from identifying antipodal points on $\Sigma_1 \subset \mathbb{R}^3$

$$P = \{ \{p, -p\} \mid p \in \Sigma_1 \}$$

See pg. 194 - 195 for a few more comments by Oneil. There he defines $A(p) = -p$ and $F(p) = \{p, -p\}$ and notes $F \circ A = F$. Note, A is an isometry. Thus $F: \Sigma_1 \to P$ together with $A = G_{12}$ (in notation of Prop. 2.6 just proved ↺)

Hence P is given metric by pushing forward the sphere metric of Σ_1, F is not 1-1 but as there is isometry A connecting points in fibers of F the construction of $\langle , \rangle_P$ goes through.

If $dF(v_1) = v$ and $dF(v_2) = w$ then define

$$\langle v, w \rangle_P = \langle v_1, v_2 \rangle_{\Sigma_1}$$

Hence P and Σ_1 are locally isometric and P has $K = 1$ and geodesics are closed (why ?)

Remark: Same construction for Σ_r gives projective plane of radius r with $K = 1/r^2$.

Remark: "P cannot be found in $\mathbb{R}^3$" * It is compact but P is not orientable (all compact surfaces in $\mathbb{R}^3$ are orientable, see pg. 190...)

Question: WHY IS MOBIUS STRIP NOT COMPACT ?

* it seems P being "in" $\mathbb{R}^3$ requires geometry from $\mathbb{R}^3$ being ~~induced on~~ restricted to P...

Example 2.8 (TANGENT SURFACES)

For $n \geq 3$, let β be unit-speed in $\mathbb{R}^n$ with $\kappa = \|T'\| > 0$. Define,

$$\mathbf{X}(u,v) = \beta(u) + vT(u)$$

with $v \neq 0$. Then,

$$\mathbf{X}_u = T + vT'$$
$$\mathbf{X}_v = T$$

Then,

$$E = \mathbf{X}_u \cdot \mathbf{X}_u = 1 + v^2\kappa^2.$$
$$F = \mathbf{X}_u \cdot \mathbf{X}_v = 1.$$
$$G = 1.$$

Observe $\underbrace{EG - F^2}_{\|\mathbf{X}_u \times \mathbf{X}_v\|^2} = v^2\kappa^2 > 0$ (this is why $v \neq 0$ is req^{d})

$$\|\mathbf{X}_u \times \mathbf{X}_v\|^2 > 0 \Rightarrow \mathbf{X}_u \times \mathbf{X}_v \neq 0$$
$$\Rightarrow \mathbf{X} \text{ regular.}$$

Exercise #13 of §5.4 pg. 233 shows the surface so constructed in $\mathbb{R}^3$ is flat. Here, for $\mathbb{R}^n$ we have <u>same</u> E, F, G thus as K is determined by E, G it follows $K = 0$ for this example also.