§ 7.3 COVARIANT DERIVATIVE

- if time permits this semester, I'd like to explain how this connects to the problem of parallel transport on a Riemannian manifold of dimension n.

- COVARIANT DERIVATIVES ARE PART OF THE INTRINSIC CALCULUS OF A GEOMETRIC SURFACE... WE NEED THEM TO STUDY CURVES ETC...

Given geometric surface M and $V, W \in \mathfrak{X}(M)$ we define $\nabla_V W \in \mathfrak{X}(M)$ and naturally ∇ must satisfy linear, Leibniz rules as before however $\bullet$ replaced by $\langle , \rangle$ for Leibniz on $\langle V, W \rangle$ as opposed to $V \bullet W$ recall the identity with $X[V \bullet W]$ see pg. 83. etc...

oh, given frame field E_1, E_2 the connection form ω_{12} measures rate at which E_1 turns towards E_2 hence we require, for $(M, \langle , \rangle)$ geom. surface.

$$\boxed{\omega_{12}(V) = \langle \nabla_V E_1, E_2 \rangle}$$

Lemma (3.1) (p. 338)

If ∇ satisfies the usual linear, Leibniz (p. 82 Th$^{\underline{m}}$ 5.3) and $\omega_{12}(V) = \langle \nabla_V E_1, E_2 \rangle$ for $E_1, E_2 = V$ then ∇ obeys the <u>connection</u> <u>equations</u>

$$\nabla_V E_1 = \omega_{12}(V) E_2$$

$$\nabla_V E_2 = \omega_{21}(V) E_1$$

Moreover, if $W = f_1 E_1 + f_2 E_2$ then

the covariant derivative formula.

$$\boxed{\nabla_V W = (V[f_1] + f_2 \omega_{21}(V)) E_1 + (V(f_2) + f_1 \omega_{12}(V)) E_2}$$

Proof: pg. 338 - 339. //

Proof: (of Lemma 3.1)

$$\nabla_V E_1 = \underbrace{\overset{0}{\cancel{\langle \nabla_V E_1, E_1 \rangle}}}_{\langle E_1, E_1 \rangle = 1} E_1 + \underbrace{\langle \nabla_V E_1, E_2 \rangle}_{W_{12}(V)} E_2$$

$$V[\langle E_1, E_1 \rangle] = \langle \nabla_V E_1, E_1 \rangle + \langle E_1, \nabla_V E_1 \rangle$$

$$\hookrightarrow \underset{0}{\cancel{V[1]}} = 2\langle \nabla_V E_1, E_1 \rangle \qquad \therefore \langle \nabla_V E_1, E_1 \rangle = 0.$$

Notice that, we find:

$$\underline{\nabla_V E_1 = W_{12}(V) E_2}.$$

And by almost the same argument,

$$\nabla_V E_2 = W_{21}(V) E_1$$

Next, we derive the covariant derivative formula, let $W = f_1 E_1 + f_2 E_2$ and calculate, using Leibniz props,

$$\begin{aligned}
\nabla_V W &= \nabla_V (f_1 E_1) + \nabla_V (f_2 E_2) \\
&= V[f_1] E_1 + f_1 \nabla_V E_1 + V[f_2] E_2 + f_2 \nabla_V E_2 \\
&= V[f_1] E_1 + f_1 W_{12}(V) E_2 + V[f_2] E_2 + f_2 W_{21}(V) E_1 \\
&= \underline{(V[f_1] + f_2 W_{21}(V)) E_1 + (V[f_2] + f_1 W_{12}(V)) E_2}.
\end{aligned}$$ //

To be explicit (Thm 5.3 p. 82 modified for $\langle , \rangle$

(1.) $\nabla_{av+bw} Y = a \nabla_v Y + b \nabla_w Y$

(2.) $\nabla_v (aY + bZ) = a \nabla_v Y + b \nabla_w Y$

(3.) $\nabla_v (fY) = v[f] Y + f \nabla_v Y$

(4.) $v[\langle Y, Z \rangle] = \langle \nabla_v Y, Z \rangle + \langle Y, \nabla_v Z \rangle$

These are the linear / Leibniz rules for ∇.

Th$^{\underline{m}}$/(3.2) ON EACH GEOMETRIC SURFACE M $\exists!$ ∇ with linear/Leibniz (1) - (4) of Cor. 5.4 of Chpt. 2 and satisfying $\omega_{12}(E_i) = \langle \nabla_{E_i} E_1, E_2 \rangle$ for <u>every</u> frame field E_i

<u>Proof</u>: uniqueness is already shown in Lemma 3.1. (p. 338) we now show $\exists$ at least one connection. Following oneil, the proof splits into two parts:

<u>Local Def$^{\underline{n}}$</u>: for frame E_1, E_2 on open $\mathcal{U} \subset M$ we <u>define</u> $\nabla_V W$ by the covariant derivative formula. That is for $V, W \in \mathfrak{X}(M)$ on $\mathcal{U}$,

$$\left.\begin{aligned} \nabla_V W = & \left(V[f_1] + f_2\, \omega_{21}(V) \right) E_1 + \\ & + \left(V[f_2] + f_1\, \omega_{12}(V) \right) E_2 \end{aligned}\right] (\star)$$

I invite the reader to check ∇ so defined is linear Leibniz. (details given on 339). There essentially the same as the calculus on ⑲ of this set of notes.

<u>Consistency</u>: what if $\bar{E}_1, \bar{E}_2$ is another frame on $\bar{\mathcal{U}}$ and $\mathcal{U} \cap \bar{\mathcal{U}} \neq \emptyset$. These define $\bar{\nabla}$ analogus to $(\star)$. We must show $\nabla_V W = \bar{\nabla}_V W$. It suffices to show this for $W = \bar{E}_1, \bar{E}_2$ by linearity already argued in green.

⤵

Continuing from ⑳,

$$\bar{E}_1 = \underbrace{\cos\varphi}_{f_1} E_1 + \underbrace{\sin\varphi}_{f_2} E_2 \qquad \text{"} W = f_1 E_1 + f_2 E_2 \text{"}$$

Hence,

$$\nabla_V \bar{E}_1 = V[\cos\varphi]E_1 + \cos\varphi \nabla_V E_1 + V[\sin\varphi]E_2 + \sin\varphi \nabla_V E_2$$

$$= \underline{-\sin\varphi V[\varphi]E_1} + \underline{\cos\varphi\, \omega_{12}(V) E_2} + \underline{\cos\varphi V[\varphi]E_2} + \underline{\sin\varphi\, \omega_{21}(V) E_1}$$

$$= \sin\varphi\big(\underset{*}{\underline{-V(\varphi) + \omega_{21}(V)}}\big)E_1 + \cos\varphi\big(\underset{**}{\underline{V[\varphi] + \omega_{12}(V)}}\big)E_2$$

Recall, Lemma 1.4 we showed $\bar{\omega}_{12} = \omega_{12} + d\varphi$ and as $d\varphi[V] = V[\varphi]$ we find

$$V[\varphi] = d\varphi(V) = (\bar{\omega}_{12} - \omega_{12})(V)$$

$$V[\varphi] = \bar{\omega}_{12}(V) - \omega_{12}(V)$$

Thus,

$$* \quad -V[\varphi] + \omega_{21}(V) = -\bar{\omega}_{12}(V) + \cancel{\omega_{12}(V)} + \cancel{\omega_{21}(V)}$$

$$** \quad V[\varphi] + \omega_{12}(V) = \bar{\omega}_{12}(V) - \cancel{\omega_{12}(V)} + \cancel{\omega_{12}(V)}$$

Therefore,

$$\nabla_V E_1 = (\underline{-\overset{*}{\sin\varphi} E_1 + \overset{**}{\cos\varphi} E_2})\bar{\omega}_{12}(V)$$

$$= \bar{\omega}_{12}(V)\, \underline{\bar{\bar{E}}_2}$$

$$= \bar{\nabla}_V \bar{E}_1$$

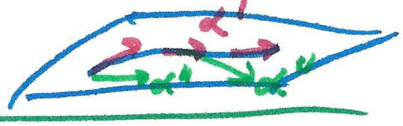
Likewise, $\nabla_V E_2 = \bar{\nabla}_V \bar{E}_2$ hence the cov. der. formula def$^{\underline{n}}$ is independent of the choice of frame. //

Example: Covariant der. of $\mathbb{R}^2$

the natural frame field U_1, U_2 has $\omega_{12} = 0$

thus $\nabla_V W = V[f_1]U_1 + V[f_2]U_2$

Notice how this $\parallel$s— Lemma 5.2 of Chpt. 2 (pg. 82) with $\mathbb{R}^2$ instead of $\mathbb{R}^3$.



Def^n/ If $\alpha : I \longrightarrow M$ is a curve and Y is a vector field along α $\left(t \longmapsto Y_{\alpha(t)} = Y(\alpha(t)) \in T_{\alpha(t)}M \text{ for each } t \in I = \text{dom}\,\alpha \right)$ then the COVARIANT DERIVATIVE OF Y along α is denoted Y' and, is defined by natural adaptation of cov. der. formula,

$$Y' = (f_1' + f_2 \omega_{21}(\alpha'))E_1 + (f_2' + f_1 \omega_{12}(\alpha'))E_2$$

where $Y = f_1E_1 + f_2E_2$ and $f_1' = \frac{df_1}{dt}$, $f_2' = \frac{df_2}{dt}$

Remark: on a geometric surface, it's usually not the case that a vector field $Y \in \mathfrak{X}(M)$ restricted to $\alpha : I \to M$ has $Y' = \frac{dY}{dt}$. Notice One'll uses Y' to implicit a covariant differentiation. Generally, $\frac{dY}{dt} \notin \mathfrak{X}(M)\big|_\alpha = \mathfrak{X}(\alpha)$ however, by construction $Y' \in \mathfrak{X}(\alpha)$. The connection form term removes the normal piece....

Example / Application: Given $\alpha : I \longrightarrow M$ the velocity vector field α' ~~$: I \longrightarrow T_{\alpha(t)}M$~~ is on M so we may calculate α''. This is called the acceleration of α

warning hybrid notation one prime $\frac{d}{dt}$ the other is ∇.

Def^n/ The acceleration of α is the covariant derivative of the ~~covariant deriva~~ velocity field α' on M.

Lemma (3.4) Let E_1, $E_2 = J(E_1)$ be a positively oriented frame field on M where J is the rotation operator as defined in Ex. #3 of §7.1*. Let Y be a vector field of constant length $c > 0$ along a curve α in M. If φ is the angle function from E_1 to Y then

$$Y' = (\varphi' + W_{12}(\alpha'))\,J(Y)$$

Intuitively, $\|Y\| = c$ along $\alpha \Rightarrow Y$ can only change direction, but $J(Y)$ is the only other direction in a two-dim'l surface. Moreover the change comes from angle fnct of Y & twisting of the frame field as quantified by w_{12}.

* [Note, J is defined by insisting $\|J(v)\| = v$, $\langle J(v), v\rangle = 0$ and $dM(v, J(v)) > 0$ for $v \neq 0$. Such a J is derivable for any geometric surface oriented by area form dM.

Proof: Note ~~Let~~ $Y/c = \cos\varphi\, E_1 + \sin\varphi\, E_2$ as $\|Y/c\| = \frac{1}{c}\|Y\| = \frac{c}{c} = 1$.

Apply cov. der. f-la,

$$Y'/c = \text{~~$+\sin\varphi\,\varphi' E_1 + \cos\varphi\,\omega$~~}$$

$$= (-\sin\varphi\,\varphi' E_1 + \sin\varphi\, W_{21}(\alpha')E_1) + (\cos\varphi\,\varphi' E_2 + \cos\varphi\, W_{12}(\alpha')E_2)$$

$$= \sin\varphi\,(-\varphi' + W_{21}(\alpha'))E_1 + \cos\varphi\,(\varphi' + W_{12}(\alpha'))E_2$$

$$= (\varphi' + w_{12}(\alpha'))(-\sin\varphi\, E_1 + \cos\varphi\, E_2)$$

$$= (\varphi' + w_{12}(\alpha'))(J(Y/c))$$

Thus, $Y' = (\varphi' + w_{12}(\alpha'))J(Y)$ .//

Distant parallelism along a geometric surface is not usually possible, however, along a curve $\alpha : I \to M$ we may always define:

DEFN/(3.5)(p. 341) A VECTOR FIELD V ALONG A CURVE α IN A GEOMETRIC SURFACE IS PARALLEL PROVIDED $V' = 0$ ONCE MORE, V' DENOTES THE COVARIANT DER. OF V ON M ALONG α.

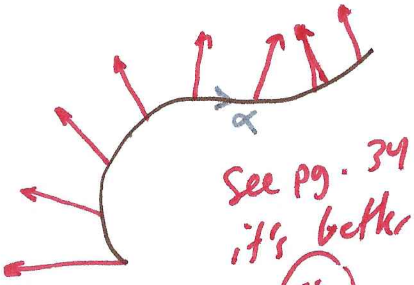
Consider, if $V = f_1 E_1 + f_2 E_2$ along α is $\parallel$ then we have

$\|V\|^2 = \langle V, V \rangle \Rightarrow$ (this is generally true)

And $V' = (f_1' + f_2\, w_{21}(\alpha'))E_1 + (f_2' + f_1 w_{12}(\alpha'))E_2 = 0$

$\Rightarrow f_1' + f_2 w_{21}(\alpha') = 0$ AND $f_2' + f_1 w_{12}(\alpha') = 0$ by LI for E_1, E_2 frame.

$$\Rightarrow f_1' = -f_2 w_{21}(\alpha') = f_2 w_{12}(\alpha')$$
$$f_2' = -f_1 w_{12}(\alpha')$$



See pg. 342 it's better ☺

Notice $\|V\|^2 = f_1^2 + f_2^2$ and

$$\frac{d\|V\|^2}{dt} = 2f_1 \frac{df_1}{dt} + 2f_2 \frac{df_2}{dt}$$
$$= 2f_1(+f_2 w_{12}(\alpha')) + 2f_2(-f_1 w_{12}(\alpha'))$$
$$= 0 \quad \therefore \quad \underline{\|V\|^2 = \text{constant}}.$$

Oneil's Proof: (he's using the prop. alluded to at top of 341, but are never explicitly given... BUT...

$$\frac{d}{dt}\langle V, V \rangle = \langle V', V \rangle + \langle V, V' \rangle$$
$$= 2\langle V', V \rangle$$
$$= 0.$$

THIS MAKES SENSE AS $V' = \frac{d}{dt}$ is v-field along α.

Lemma (3.6) LET α BE A CURVE IN geometric surface M and $v \in T_pM$ such that $p = \alpha(t_o)$ then $\exists!$ parallel vector field V such that $V(t_o) = v$.

Proof: we seek to solve $V'(t) = 0$ $\forall t$ and $V(t_o) = v$. this is immediate from the fund. thm for existence and uniqueness in ODES... but following Oneil lets calculate, [MOTIVATES HOLONOMY !] $\alpha \in \mathcal{N}$ where $E_1, E_2 = J(E_1)$ is pos. oriented frame on $\mathcal{N}$. Since V has constant length $c = \|v\|$ we write $V = c\cos\varphi E_1 + c\sin\varphi E_2$ where φ is T.B.D. By Lemma 3.5 we see

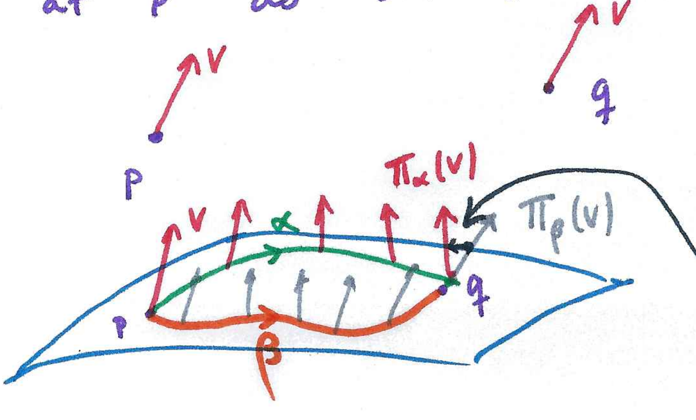
$$V' = 0 \implies \varphi' + \omega_{12}(\alpha') = 0$$

Also, $V(t_o) = v$ iff $\varphi(t_o)$ is angle from $E_1(p)$ to v. The only function to do the above is just

$$\varphi(t) = \varphi(t_o) - \int_{t_o}^{t} \omega_{12}(\alpha')du$$

Defn/ $V(t)$ generated by α & $v \in T_{\alpha(t_o)}M$ is called the parallel translate of v along α. LET'S say $\Pi_\alpha(v)$

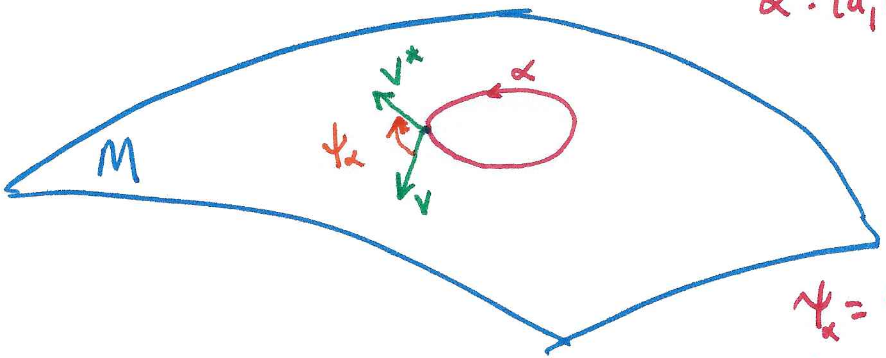
In Euclidean space we maintain parallel vectors by simply freezing the cartesian coordinate of a vector at p as we $\parallel$-translate it to q



we use same v before and after with great success in calculus III & physics!

angle $\neq 0$ as different paths may produce differing parallel translates.

• HOLONOMY CAPTURES NONTRIVIAL PARALLEL TRANSPORT



$\alpha : [a,b] \longrightarrow M$

∥- vector field rotated through angle

$$\psi_\alpha = \varphi(b) - \varphi(a) = -\int_\alpha \omega_{12}$$

↑ holonomy angle of α.

Example (3.7)

Holonomy on a sphere Σ of radius r.

He claims there is no loss of generality in assuming a circle α is a circle of latitude (why?)

$$\alpha(u) = \mathbf{X}(u, v_0) \quad \text{for} \quad 0 \le u \le 2\pi$$

The frame field associated to $\mathbf{X}$ are

$$E_1 = \frac{\alpha'}{\sqrt{E}} \quad \& \quad E_2 = J(E_1)$$

and it is shown on pg. 296, $\theta_1 = r\cos v\,du$, $\theta_2 = r\,dv$ and $\sqrt{E} = r\cos v$ ... $\omega_{12} = \sin(v)\,du$ thus

$$\psi_\alpha = -\int_\alpha \omega_{12} = -\sin(v_0)\int_\alpha du = -2\pi \sin(v_0)$$

This matches geometric intuition, a vector returns to itself under ∥- tranport only when $v_0 = 0$ (equator as Oneil sets-up sphericals in non-standard way, see pg. 86)

Lemma (3.8): If V and W are tangent vector fields on M in $\mathbb{R}^3$, then, (see pg. 344 for picture)

(1.) $\nabla_V W$ is the component of $\tilde{\nabla}_V W$ tangent to M.

(2.) If S is the shape operator of M derived from a unit-normal U then

$$\underbrace{\tilde{\nabla}_V W}_{\substack{\text{covariant}\\ \text{derivative in}\\ \mathbb{R}^3}} = \underbrace{\nabla_V W}_{\substack{\text{covariant}\\ \text{derivative}\\ \text{on } M}} + (S(V)\cdot W)U$$

Proof (2.) Take adopted frame on M, $E_1, E_2, E_3 = U$ and set $E_1 = W$ (If $W = 0$ (2.) is trivially true, so suppose $W \neq 0$ hence wlog we may rescale $\|W\| = 1$.) By Euclidean connection on $\mathbb{R}^3$ for non cartesian frame E_1, E_2, E_3

$$\tilde{\nabla}_V E_1 = \sum_{j=1}^{3} \omega_{1j}(V)E_j$$

$$= \omega_{12}(V)E_2 + \cancel{\omega_{13}(V)E_3}$$

as the frame is adapted we obtain $\omega_{13} = 0$. recall Chapter 6.

duh

Thus,

$$\nabla_V E_1 = \omega_{12}(V)E_2$$

~~$\omega_{13}(V)E_3$~~

But, $\omega_{13}(V)E_3 = ((\tilde{\nabla}_V E_1)\cdot E_3)E_3 = +(S(V)\cdot E_1)U$

$$\Rightarrow \tilde{\nabla}_V E_1 = \nabla_V E_1 + (S(V)\cdot E_1)U$$

similar for E_2 and extends to

$$\underline{\tilde{\nabla}_V W = \nabla_V W + (S(V)\cdot W)U}.$$

(sorry, a bit messy, this is not complicated, note $\tilde{\nabla}$ is $\mathbb{R}^3$ ~~connection~~ cov. der. whereas ∇ is intrinsic M cov. der.

Berry's Phase

$\mathcal{H}$