§7.4 GEODESICS

LECTURE 26

Geodesics for geometric surface M are like lines for Euclidean spaces. In particular we generalize the notion they have zero acceleration, as seen through perspective of COVARIANT DIFFERENTIATION (this is not usually a plain 2^{nd} derivative, this notation hides $\alpha'' = (\underbrace{\alpha'}_{\frac{d\alpha}{dt}})'$

$\frac{d\alpha}{dt}$ is vector field along α

$(\alpha')'$ is covariant derivative of vector field $\alpha' \in \mathfrak{X}(\alpha)$.

Def^n/ A curve γ in a geometric surface is a geodesic provided $\gamma'' = 0$.

Lemma I.: geodesics have constant speed.

Proof: $\|\gamma'\|^2 = \langle \gamma', \gamma' \rangle$

However $\frac{d}{dt}\langle \gamma', \gamma' \rangle = 2\langle \gamma', \gamma'' \rangle = 2\langle \gamma', 0 \rangle = 0$

hence $\|\gamma'\|^2 = \text{constant} \Rightarrow \|\gamma'(t)\| = c > 0 \quad \forall t \in I$

where $I = \text{dom}\, \gamma$. //

Lemma II: If $F : M \longrightarrow N$ is an isometry then γ a geodesic of $M \Rightarrow F \circ \gamma$ is geodesic of N

Proof: your hwk? (see #7 on pg. 346)

How to calculate geodesics?

- Well, perhaps § 7.5 is where the nicest formulas are found, but this § is a brute force start.

Consider a curve α on a geometric surface M. We seek conditions necessary for α to be a geodesic. Let us use the notations: for frame $E_1, E_2 \in \underbrace{\cancel{\mathcal{F}(M)}}_{\text{frames on } M}$

$$\alpha' = V_1 E_1 + V_2 E_2$$

$$\alpha'' = A_1 E_1 + A_2 E_2$$

Clearly, by definition, α geodesic iff

$$\boxed{A_1 = A_2 = 0}$$

Observe, we have, by $Eq^{\underline{n}}$ (del=) at top of pg. 341 $Y' = (f_1' + f_2 w_{21}(\alpha'))E_1 + (f_2' + f_1 w_{12}(\alpha'))E_2$ applied to our context $Y = \alpha'$,

$$\boxed{\begin{aligned} A_1 &= V_1' + V_2 w_{21}(\alpha') \\ A_2 &= V_2' + V_1 w_{12}(\alpha') \end{aligned}}$$

In terms of $X(u,v)$ with $E = \langle X_u, X_u \rangle = 1 = \langle X_v, X_v \rangle = G$, $F = 0$,

Th$^{\underline{m}}$ (4.2) Geodesic $Eq^{\underline{s}}$ in the orthogonal patch case in terms of E, F, G notation ($F = 0$ by assumption)

$$A_1 = a_1'' + \frac{1}{2E}\left(E_u (a_1')^2 + 2E_v a_1' a_2' - G_u (a_2')^2\right) = 0$$

$$A_2 = a_2'' + \frac{1}{2G}\left(-E_v (a_1')^2 + 2G_u a_1' a_2' + G_v (a_2')^2\right) = 0.$$

<u>Proof</u>: calculation! See pg. 348. //

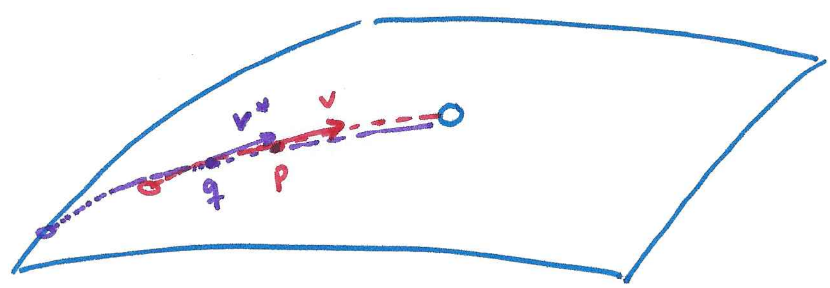
$Th^{\underline{m}}$ (4.3) Given a tangent vector v to M at p $\exists!$ geodesic γ defined on an interval I around 0 s.t.

$$\gamma(0) = p \quad , \quad \gamma'(0) = v$$

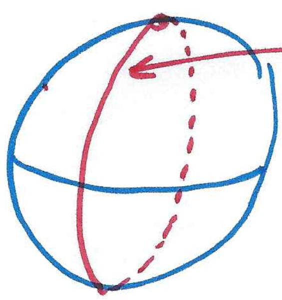
Proof: you write down the ODEs implicit within $Th^{\underline{m}}$ (4.2) then apply the existence & uniqueness $Th^{\underline{m}}$ of ODEs. //

For example: in $\mathbb{R}^n$, just set $\alpha(t) = p + tv$ then in some sense, the same is possible at each $p \in M$ and $v \in T_pM$.

Def$^{\underline{n}}$/ A geometric surface M is complete provided every maximal geodesic is defined on all of $\mathbb{R}$.



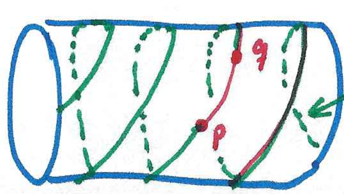
the maximal geodesic is formed by pasting together geodesics which overlap until as large a curve as possible is formed



dom$(\gamma) = \mathbb{R}$ goes on and on, can overlap !

Σ is complete

both cases the geodesic is image of some line in parameter space.



helix winds around cylinder.

In §8.1 we learn $\exp_p(v) \equiv \gamma_v(1)$ and $\exp_p : T_pM \longrightarrow M$ maps lines in T_pM to geodesics in M. $Th^{\underline{m}}$ 4.3 is basis for that def$^{\underline{n}}$...

comment: a complete geometric surface with one point removed will not be complete since the geodesics which run through the deleted point cannot run on forever, they get stuck at the missing point. (idea of my illustration on 30)

Lemma 4.5: Let E_1, E_2 be frame field and α a constant speed curve s.t. α' and E_2 are never orthogonal ($\langle \alpha', E_2 \rangle \neq 0$). If $A_1 = 0$ then $A_2 = 0$ hence α is geodesic.

Proof: $\langle \alpha', \alpha' \rangle = c \Rightarrow \langle \alpha'', \alpha' \rangle = 0$

therefore, using notation A_1, A_2 comp. for α'' & V_1, V_2 for α',

$$0 = \langle A_1 E_1 + A_2 E_2 , \alpha' \rangle$$
$$= A_1 \langle E_1, \alpha' \rangle + A_2 \langle E_2, \alpha' \rangle$$

Suppose $A_1 = 0$ and $\langle E_2, \alpha' \rangle \neq 0 \Rightarrow A_2 = 0$

Hence, $A_1 = A_2 = 0$ $\therefore$ α a geodesic. //

This pursuit begins again in § 7.5 where the Clairaut Parametrization is studied...

Note, Lemma 4.5 paired with Th^{m} 4.2 gives us the rather simple condition

$$E(a_1')^2 + G(a_2')^2 = \text{const} \Rightarrow \alpha \text{ geodesic.}$$

(provided $\langle \alpha', E_2 \rangle \neq 0$)...

FRENET THEORY OF CURVES IN GEOMETRIC SURFACE:

For $\alpha : I \to M$ we cannot (nor would want to) define torsion, however curvature is nicely generalized to geometric surfaces.

If $\beta : I \to M$ is a unit-speed curve in an oriented surface then we define $T = \beta'$ (as usual!) but then $N = J(T)$ where J is the rotation operator induced from dM on M. Moreover, we define $\kappa_g : I \to \mathbb{R}$ to be function s.t.

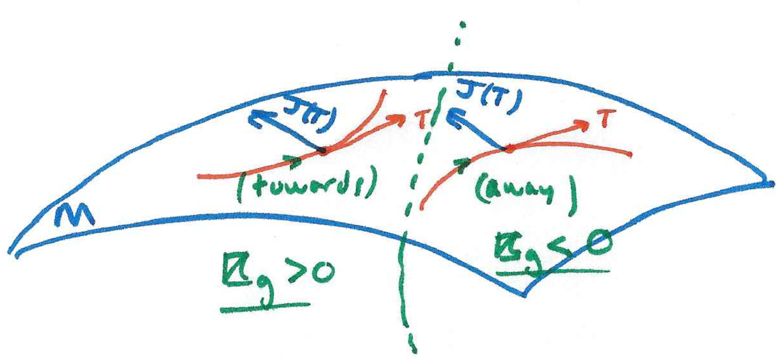
$$T' = \kappa_g N$$

where $\kappa_g = \langle T', N \rangle$ (remember, M paired with $\langle , \rangle$ is geometric surface)

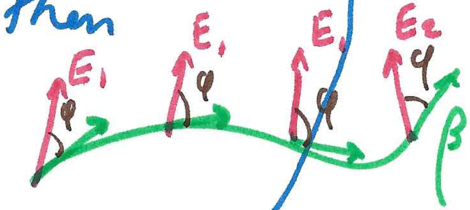
κ_g = geodesic curvature of β

$\kappa_g > 0 \Rightarrow T$ turning towards $J(T)$

$\kappa_g < 0 \Rightarrow T$ turning away from $J(T)$



$\boxed{\text{Cor 4.6: Let } \beta \text{ be unit-speed in oriented region } \mathcal{R} \text{ with frame } E_1, E_2. \text{ If } \varphi \text{ is angle function from } E_1 \text{ to } \beta' \text{ along } \beta \text{ then } \kappa_g = \frac{d\varphi}{ds} + \omega_{12}(\beta')}$



Proof: Set $Y = T$ in Lemma 3.4 where we saw $Y' = (\varphi' + \omega_{12}(\alpha'))J(Y)$ hence, $J(E_1) = E_2$

$$T' = \underline{(\varphi' + \omega_{12}(\beta'))\,J(T)}$$

Hence
$$\begin{aligned}\kappa_g &= \langle T', N \rangle \\ &= \langle \underline{T}', J(T) \rangle \\ &= \varphi' + \omega_{12}(\beta') \\ &= \underline{\frac{d\varphi}{ds} + \omega_{12}(\beta')}.\end{aligned}$$

For $M = \mathbb{R}^2$ with Euclidean metric we have Cartesian frame $E_1 = U_1, E_2 = U_2$ with $\omega_{12} = 0$ thus $\underbrace{\kappa_g = \frac{d\varphi}{ds}}_{\text{Liouville's Formula.}}$.

By same arguments as for space curves in $\mathbb{R}^3$,

$$\boxed{\alpha' = vT, \quad \alpha'' = \frac{dv}{dt}T + \kappa_g v^2 N}.$$

where $v = \|\alpha'\|$ = speed. So... acceleration and both tangent & normal directions $\vec{a} = a_T T + a_N N$.

Lemma (4.7) A regular curve α in M is a geodesic iff if α has constant speed and geodesic curvature $\kappa_g = 0$

Proof: From $\alpha'' = \frac{dv}{dt} T + \kappa_g v^2 N$ note regular $\Rightarrow v > 0$, hence $\alpha'' = \frac{dv}{dt} T + \kappa_g v^2 N = 0$

$$\Rightarrow \underbrace{\frac{dv}{dt} = 0}_{\text{constant speed}} \quad \text{and} \quad \kappa_g v^2 = 0 \Rightarrow \underbrace{\kappa_g = 0}_{\text{geodesic curvature}}.$$ //

Remark: $\kappa_g = 0 \Leftrightarrow \alpha'' = \frac{dv}{dt} T$ so regular curves are geodesic curvature zero iff α'' colinear to $T = \alpha'$.

Def^n/ $\alpha : I \to M$ with $\kappa_g = 0$ is a pregeodesic. (If α is reparametrized to unit-speed then it's an actual geodesic.)

Remark: I'll go to §7.6 at this point. §7.5 is useful, but, technical (and nice) but... Gauss Bonnet calls us... Certainly the point of §7.5 is to make careful the claim of page 360:

HYPERBOLIC PLANE WITH GEODESICS AS AXIOMATIC LINES DOES SATISFY THE PARALLEL POSTULATE.