Def^n/ Let $\alpha : [a,b] \longrightarrow M$ be a regular curve segment in an oriented geometric surface M. The <u>total geodesic curvature</u> of α is:

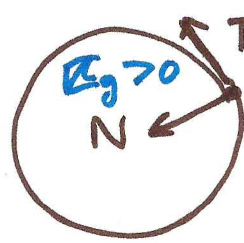
$$\int_\alpha \kappa_g \, ds = \int_{s(a)}^{s(b)} \kappa_g(s(t)) \frac{ds}{dt} \, dt$$

<u>Example</u>: $\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^2$, $\alpha(t) = (r\cos t, r\sin t)$

$$\bar{\alpha}(s) = \left(r\cos\left(\tfrac{s}{r}\right), r\sin\left(\tfrac{s}{r}\right)\right)$$

$$\bar{\alpha}'(s) = \left(-\sin(s/r), \cos(s/r)\right) \Rightarrow \|\bar{\alpha}'(s)\| = 1.$$

$$T(s) = \left(-\sin(s/r), \cos(s/r)\right)$$

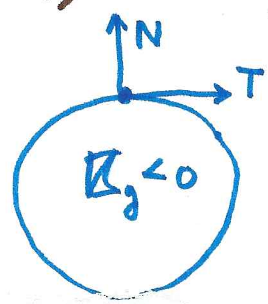


$$N = J(T) = \left(-\cos(s/r), -\sin(s/r)\right)$$

$$\kappa_g = \langle T', N \rangle = \left\langle \left(-\tfrac{1}{r}\cos(s/r), -\tfrac{1}{r}\sin(s/r)\right), \left(-\cos(s/r), -\sin(s/r)\right) \right\rangle$$

$$= \frac{1}{r}\left(\cos^2(s/r) + \sin^2(s/r)\right)$$

$$= \frac{1}{r} \Rightarrow \kappa_g(s(t)) = \frac{1}{r}$$



$$\therefore \int_\alpha \kappa_g \, ds = \int_0^{2\pi r} \frac{ds}{r}$$

$$= \frac{2\pi r}{r}$$

$$= \boxed{2\pi.}$$

CCW-oriented circle in Euclidean plane $\mathbb{R}^2$.

Remark: once CW $\Rightarrow \int_\alpha \kappa_g \, ds = -2\pi$
generally, if wind n-CCW times then $\int \kappa_g \, ds = 2\pi n$.

Recall Cor. 4.6: $\kappa_g = \frac{d\varphi}{ds} + \omega_{12}(\beta')$

we integrate to discover the following:

Lemma 6.2: Let $\alpha : [a,b] \longrightarrow M$ be a regular curve segment in a region of M oriented by a frame field E_1, E_2. Then

$$\int_\alpha \kappa_g \, ds = \varphi(b) - \varphi(a) + \int_\alpha \omega_{12}$$

where φ is the angle function from E_1 to α' along α, and ω_{12} is the connection form of E_1, E_2

Proof:
$$\int_\alpha \kappa_g \, ds = \int_{s(a)}^{s(b)} \kappa_g(s(t)) \frac{ds}{dt} \, dt$$

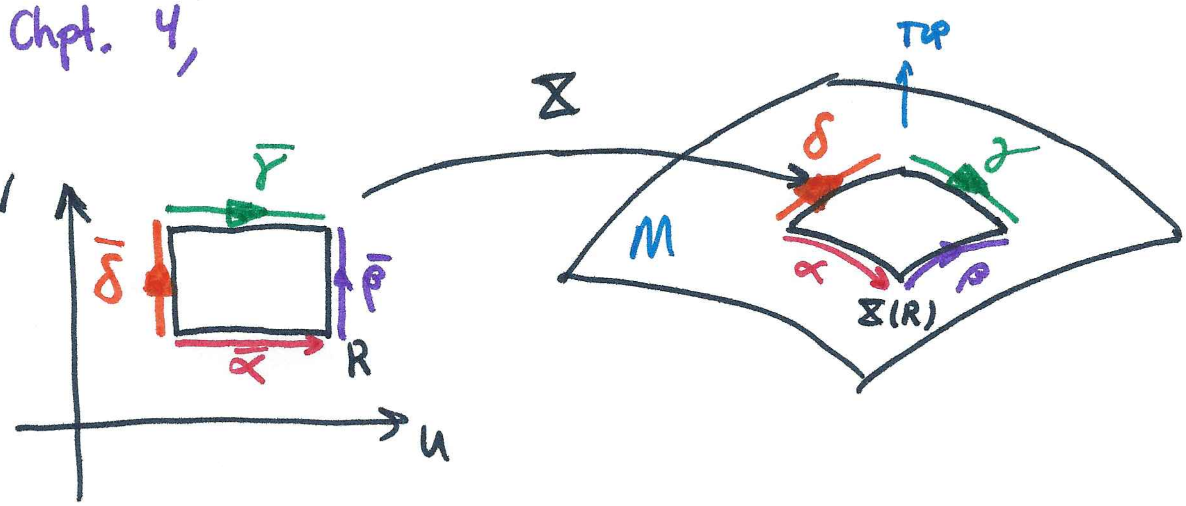
$$= \int_{s_a}^{s_b} \kappa_g(s) \, ds$$

$$= \int_{s(a)}^{s(b)} \left(\frac{d\varphi}{ds} + \omega_{12}(\alpha') \right) ds$$

$$= \int_{\varphi(a)}^{\varphi(b)} d\varphi + \int_{s(a)}^{s(b)} \omega_{12}(\alpha') \, ds$$

$$= \varphi(b) - \varphi(a) + \int_\alpha \omega_{12}$$

Suppose $\Phi : R \to M$ is one-one and regular on R. This means $\exists\, \tilde{R}$ containing R which is open and Φ is regular on the extension to $\tilde{R}$. We return to the discussion of 2-segments in Chpt. 4,



$$\partial \Phi = \alpha + \beta - \gamma - \delta$$

The natural induced orientation of $\partial\Phi$ makes the edge on your left if you walk on top of M (as defined by $\underline{\underline{dM}}$)

Defn/ Let $\Phi : R \to M$ be a 2-segment in M over the closed rectangle $R = [a,b] \times [c,d]$ So $a \le u \le b$, $c \le v \le d$ and the <u>edge curves</u> of Φ are precisely:

$$\alpha(u) = \Phi(u,c)$$
$$\beta(v) = \Phi(b,v)$$
$$\gamma(u) = \Phi(u,d)$$
$$\delta(v) = \Phi(a,v)$$

(from pg. 177 in Chpt. 4)

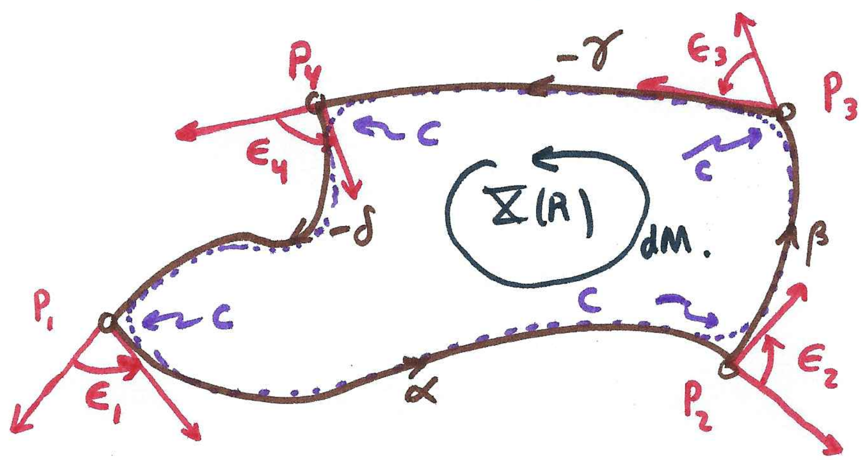
Notice $\partial \Phi = \alpha + \beta - \gamma - \delta$ is a simple-closed-curve on boundary of $\Phi(R) \subset M$. not quite, we have vertices.

What is the total geodesic curvature of $\partial \Phi$?

If $\partial \Phi$ was a single curve α we'd be done, but, the corners matter!

$$\int_{\partial \Phi} \kappa_g\, ds = \int_{\alpha} \kappa_g\, ds + \int_{\beta} \kappa_g\, ds + \int_{-\gamma} \kappa_g\, ds + \int_{-\delta} \kappa_g\, ds + \cancel{\text{corner terms}}$$

$$= \int_{\alpha} \kappa_g\, ds + \int_{\beta} \kappa_g\, ds - \int_{\gamma} \kappa_g\, ds - \int_{\delta} \kappa_g\, ds + \cancel{\text{corner terms}}$$



(pg. 366 Oneil)

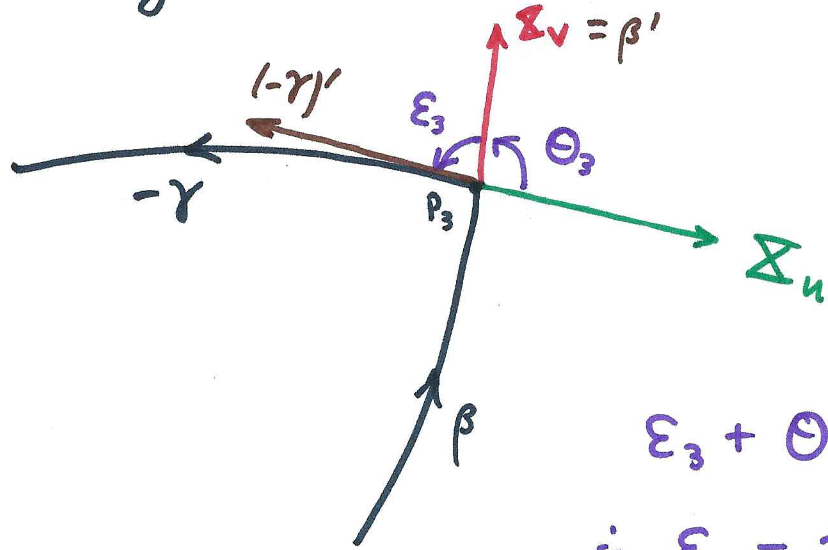
$C = \partial \Phi_{\text{cut corners}} \Rightarrow \int_C \kappa_g\, ds \neq \int_{\partial \Phi} \kappa_g\, ds$ above

the integral over $\partial \Phi$ neglects the turning. The theorem soon to follow adds precision to the $\neq$ above.

Def $\underline{=}$/ Let $\Phi(R)$ be as pictured above. The vertices are P_1, P_2, P_3, P_4. The exterior angle ε_j of Φ at P_j is the turning angle at P_j derived from edge curves $\alpha, \beta, -\gamma, -\delta, \ldots$ in order of occurrence in Φ. The interior angle i_j at P_j is $\pi - \varepsilon_j$.

$\varepsilon_1 = \pi - \theta_1, \quad \varepsilon_2 = \theta_2, \quad \varepsilon_3 = \pi - \theta_3, \quad \varepsilon_4 = \theta_4$

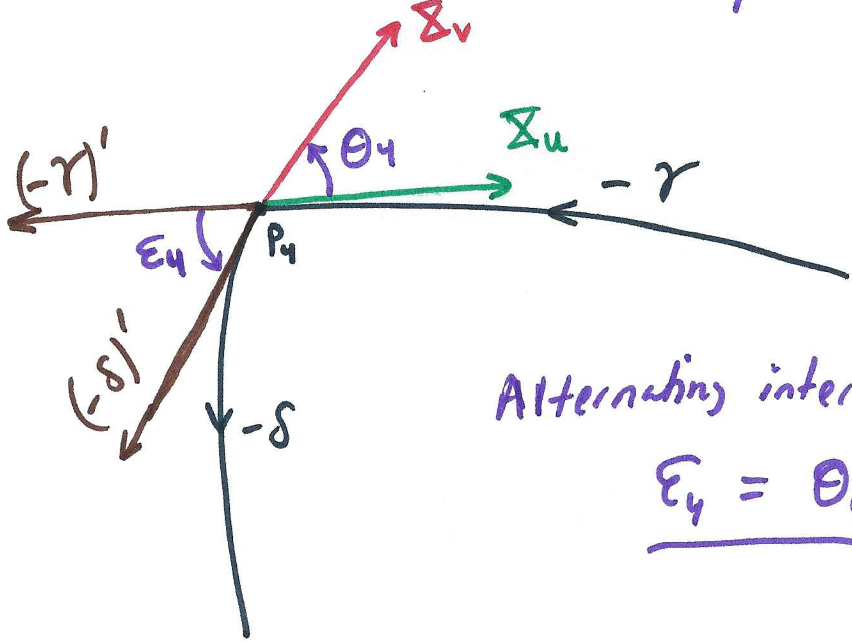
Coordinate angle $0 < \theta < \pi$ from X_u to X_v



$$\varepsilon_3 + \theta_3 = \pi$$

$$\therefore \underline{\varepsilon_3 = \pi - \theta_3}.$$

Similar pictures prove the remaining assertions. For example,



Alternating interior angles

$$\underline{\varepsilon_4 = \theta_4}.$$

Th$^{\underline{m}}$/ Let $\mathbf{X}: R \to M$ be a 1-1, regular, 2-segment in a geometric surface M. If dM is the area form determined by $\mathbf{X}$, then

$$\boxed{\iint_{\mathbf{X}} K\, dM + \int_{\partial \mathbf{X}} \kappa_g\, ds + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 2\pi} \quad (*)\ \text{GBe}\angle$$

where ε_j is THE EXTERIOR ANGLE AT VERTEX P_j of $\mathbf{X}$, $j=1,2,3,4$.

$(*)$: Gauss Bonnet formula with exterior angles. In terms of $i_j = \pi - \varepsilon_j$ or $\varepsilon_j = \pi - i_j$ we have the Gauss Bonnet formula with interior angles.

$$\boxed{\iint_{\mathbf{X}} K\, dM + \int_{\partial \mathbf{X}} \kappa_g\, ds = i_1 + i_2 + i_3 + i_4 - 2\pi} \quad \text{GBi}\angle$$

<u>Proof</u>: Based on frame field associated to $\mathbf{X}$. Namely:

$E_1 = \frac{1}{\sqrt{E}}\mathbf{X}_u$ and $E_2 = J(E_1) \Rightarrow dM(E_1, E_2) = 1$.

Recall, the 2$^{\underline{nd}}$ structural eq$^{\underline{n}}$,

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2 = -K\, dM$$

Note, $d\omega_{12} + K\, dM = 0$ thus,

$$\iint_{\mathbf{X}} d\omega_{12} + \iint_{\mathbf{X}} K\, dm = 0$$

By Stoke's Th$^{\underline{m}}$,

$$\Rightarrow \iint_{\mathbf{X}} K\, dM + \int_{\partial \mathbf{X}} \omega_{12} = 0$$

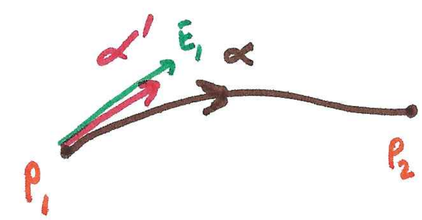
By Lemma to follow on next page,

$$\Rightarrow \iint_{\mathbf{X}} K\, dm + \boxed{\int_{\partial \mathbf{X}} \kappa_g\, ds + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 2\pi} = 0. //$$

Lemma: $\int_{\partial \mathbb{X}} \omega_{12} = \int_{\partial \mathbb{X}} \kappa_g \, ds + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 2\pi$

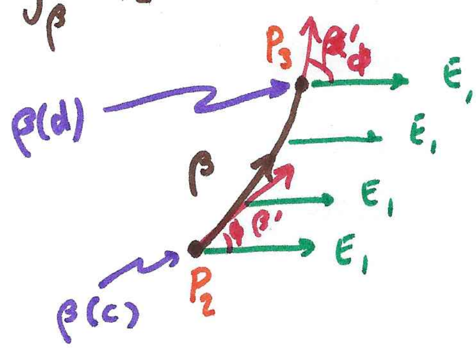
Proof: $$\int_{\partial \mathbb{X}} \omega_{12} = \underbrace{\int_\alpha \omega_{12}}_{\text{I.}} + \underbrace{\int_\beta \omega_{12}}_{\text{II.}} - \underbrace{\int_\gamma \omega_{12}}_{\text{III.}} - \underbrace{\int_\delta \omega_{12}}_{\text{IV.}}$$

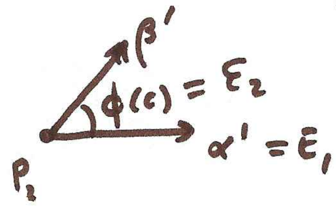
I. $$\int_\alpha \omega_{12} = -(\varphi(b) - \varphi(a)) + \int_\alpha \kappa_g \, ds = \int_\alpha \kappa_g \, ds = \int_\alpha \omega_{12}$$ (I.)

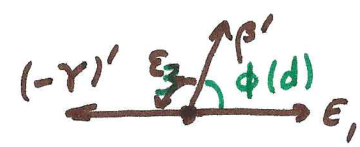


φ is angle fnct. from E_1 to α'.
this is zero along α.
$\therefore \varphi(b) = \varphi(a) = 0.$

II. $$\int_\beta \omega_{12} = \overbrace{\varphi(c) - \varphi(d)}^{\varphi(c)-\varphi(d)} + \int_\beta \kappa_g ds = \varepsilon_2 + (\pi - \varepsilon_3) = \varepsilon_2 + \varepsilon_3 - \pi.$$





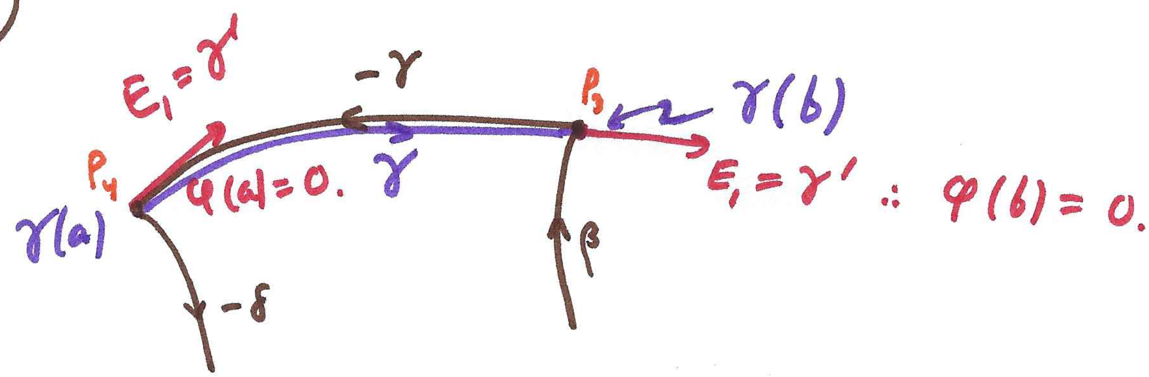


$\phi(d) = \pi - \varepsilon_3$

$$\therefore \int_\beta \omega_{12} = \int_\beta \kappa_g \, ds + \varepsilon_2 + \varepsilon_3 - \pi$$ (II.)

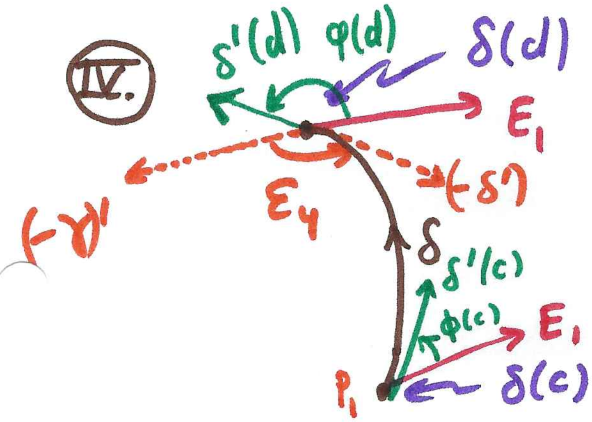
Proof continued

III.



$$\int_\gamma \omega_{12} = \varphi(a) - \varphi(b) + \int_\gamma \kappa_g \, ds = \int_\gamma \kappa_g \, ds = \int_\gamma \omega_{12} \quad \text{III.}$$

IV.



$$\phi(c) + \varepsilon_1 = \pi$$

$$\phi(d) = \varepsilon_4$$

$$\int_\delta \omega_{12} = \varphi(c) - \varphi(d) + \int_\delta \kappa_g \, ds$$

$$= \pi - \varepsilon_1 - \varepsilon_4 + \int_\delta \kappa_g \, ds \quad \text{IV.}$$

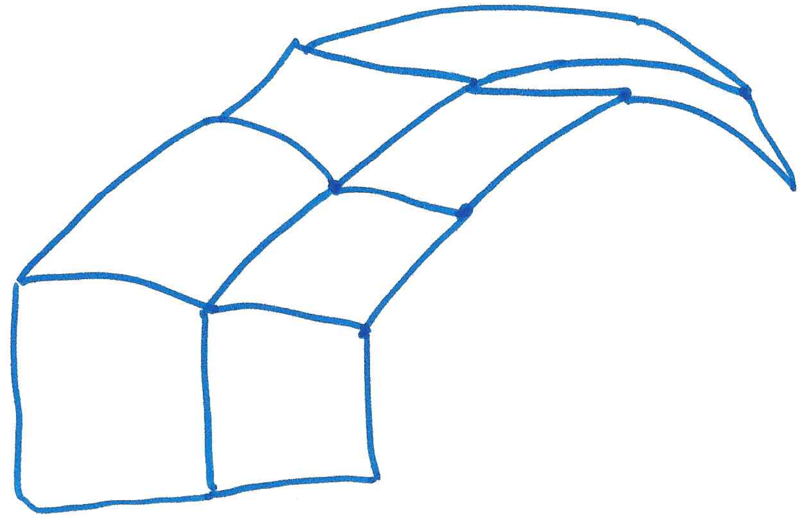
Therefore, to conclude our proof of ~~Lemma~~ begun on 41, we observe: from I, II, III & IV.

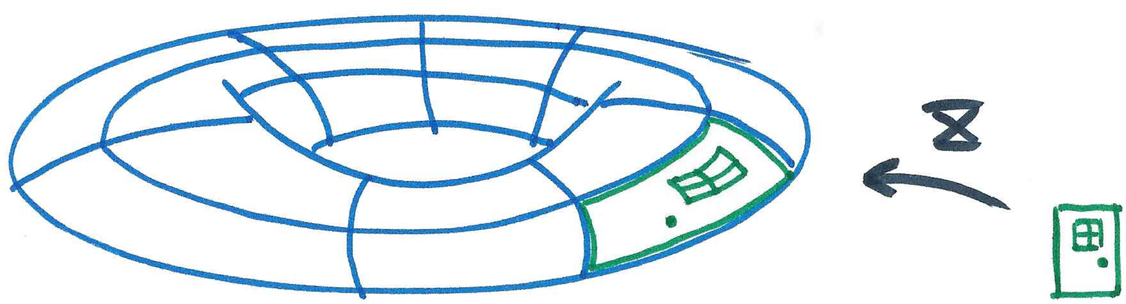
$$\int_{\partial\Sigma} \omega_{12} = \int_\alpha \kappa_g \, ds + \int_\beta \kappa_g \, ds + \varepsilon_2 + \varepsilon_3 - \pi - \int_\gamma \kappa_g \, ds - \left(\pi - \varepsilon_1 - \varepsilon_4 + \int_\delta \kappa_g \, ds\right)$$

$$= \int_{\partial\Sigma} \kappa_g \, ds + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 2\pi. \; //$$

(Noting once again $\partial\Sigma = \alpha + \beta - \gamma - \delta$.)

Rectangularization:





Thm/(6.5) Every Compact Surface has a <u>rectangular</u> decomposition. ← special type of <u>paving</u>

<u>Proof</u>: by scissors (see pg. 369)

<u>Remark</u>: <u>Diffeomorphism</u> $F: M \longrightarrow N$ $\Rightarrow$ <u>F Homeomorphism</u>

smooth with smooth inverse. — continuous with continuous inverse.

Generally, the converse is <u>false</u>: F a homeomorphism $\not\Rightarrow$ F a diffeomorphism. Thus the following is <u>SURPRISING</u>

!! In dimension two, M is homeomorphic to N iff M and N are diffeomorphic !!

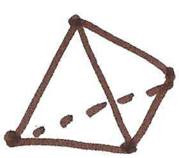
Remark: I'm not sure if the claim M, N homeomorphic iff diffeomorphic for $\dim(M) = \dim(N) = 2$, is ~~correct~~. proved in text... we'll have to think about it...

Th^m (6.6) If $\mathcal{D}$ is a rectangular decomp. of M (a compact surface) let v, e, f be the # of vertices, edges and faces in $\mathcal{D}$. Then the integer $v - e + f$ is the same for all such decompositions. Moreover, this integer is called the Euler Characteristic

$$\chi(M) = v - e + f$$

Proof: given in topology. This equally well applies to polygonial decompositions. See the handout from Lee's Text //

Ex) Sphere Σ has $\chi(\Sigma) = 2$



← triangularization. (blow it up)

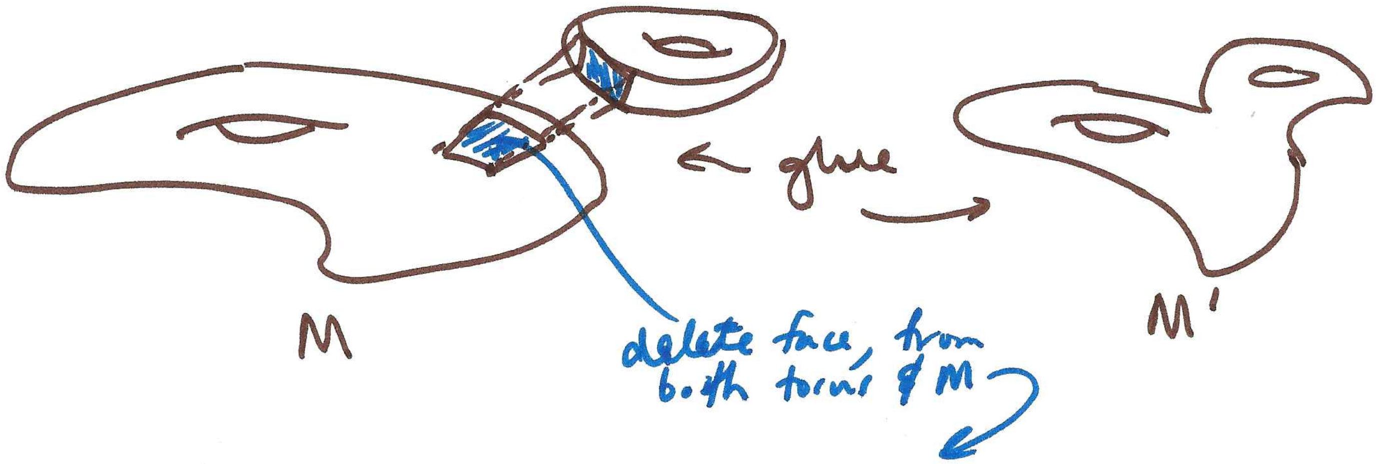
$$\left.\begin{aligned} v &= 4 \\ e &= 6 \\ f &= 4 \end{aligned}\right\} \quad \chi(\Sigma) = 4 - 6 + 4 = \underline{+2}.$$

← rectangularization (blow it up)

$$\begin{aligned} v &= 8 \\ e &= 12 \\ f &= 6 \end{aligned} \quad \chi(\Sigma) = 8 - 12 + 6 = \underline{+2}$$

Ex] TORUS : $\chi(T) = 0$. (I don't see it yet...)

Ex(3) Adding a handle to compact surface reduces its Euler characteristic by 2.



$$\chi(M') = \chi(M) - 2$$

Thm/(6.8) If M is a compact, connected, orientable surface, there is a unique integer $h \geq 0$ such that M is diffeomorphic $\underbrace{\Sigma[h]}_{\text{sphere with } h\text{-handles attached.}}$

$$\chi(M) = 2 - 2h$$

Cor (6.9) Compact orientable surfaces M and N have same Euler characteristic iff they're diffeomorphic.

Proof: If $\chi(M) = \chi(N)$ then M & N have same # of handles $\therefore$ by Thm 6.8 M & N diffeomorphic.

Th$^{\underline{m}}$/(6.10) (GAUSS-BONNET)

The total Gaussian curvature M of a compact orientable geometric surface M is 2π times its Euler characteristic:

$$\iint_M K\, dM = 2\pi\, \chi(M)$$

Proof: orient M by area form dM and let $\mathcal{D}$ be rectangular decomp. of M whose rectangles are consistently oriented with dM. Hence $\mathcal{D}$ is an oriented paving as discussed/defined in §6.7 (p. 302)

By def$^{\underline{n}}$ the total curvature of M is:

$$\iint_M K\, dM = \sum_{i=1}^{f} \iint_{\Phi_i} K\, dM \qquad \left(\begin{array}{l} f = \#\text{ of faces} \\ \quad = \#\text{ of rectangles}\end{array}\right)$$

Apply Gauss Bonnet formula to each face, we have

$$\iint_{\Phi_i} K\, dm = -\int_{\partial \Phi_i} \kappa_g\, ds - 2\pi + i_1 + i_2 + i_3 + i_4$$

~~Every face is adjacent another face hence $\partial\Phi_i$ & $\partial\Phi_j$ for $i \longrightarrow \partial j$~~

~~and $\int_{\alpha_i} \kappa_g\, ds + \int_{\alpha_j} \kappa_g\, ds = 0$ as $\alpha_i = -\alpha_j$.~~

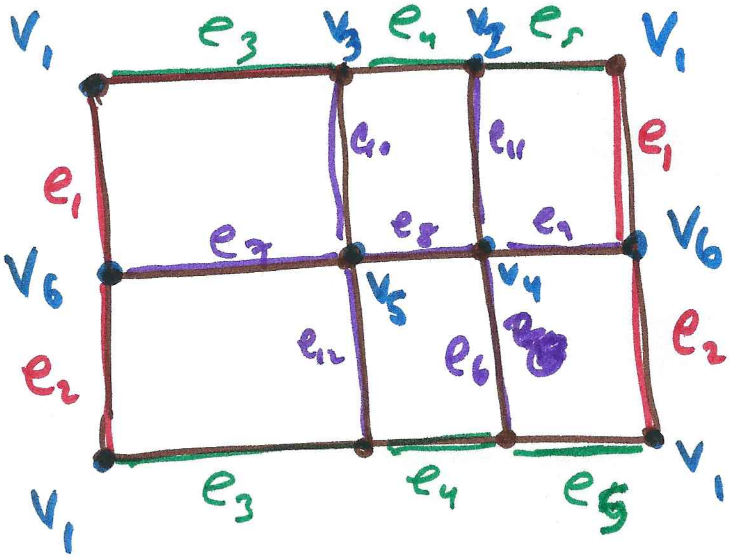
~~So... all the $\int_{\partial\Phi_i} \kappa_g\, ds$ cancel as we Σ over all faces.~~ Moreover $i_1 + i_2 + i_3 + i_4 = 2\pi$ at each of the v-vertices. Thus,

$$\iint_M K\, dm = \sum_{i=1}^{f} \Big(\cancel{-\int_{\partial\Phi_i} \kappa_g\, ds}^{\,\nearrow 0\ (*)} - 2\pi + \underbrace{i_1 + i_2 + i_3 + i_4}_{\text{for } i^{\underline{th}} \text{ face}} \Big)$$

$$= -2\pi f + \sum_{i=1}^{f} \left(\begin{array}{c}\text{interior angles} \\ \text{of } i^{\underline{th}} \text{ face}\end{array}\right)$$

$$= -2\pi f + 2\pi V$$

sum at each vertex 2π



$e = 12$
$f = 6$
$v = 6$

<u>Claim</u>: $4f = 2e$: for my example: $4(6) = 2(12)$

//

<u>Proof continued</u>:

"Each face has 4 edges, but, each edge belongs to <u>two</u> faces. Thus $4f$ counts e twice; that is $4f = 2e$" (p. 374 Oneil)

$-f = f - e$

~~$2f = e$~~

~~$V - f = V - \frac{1}{2}e$~~

Recall $\chi(M) = v - e + f$: by def^{n}

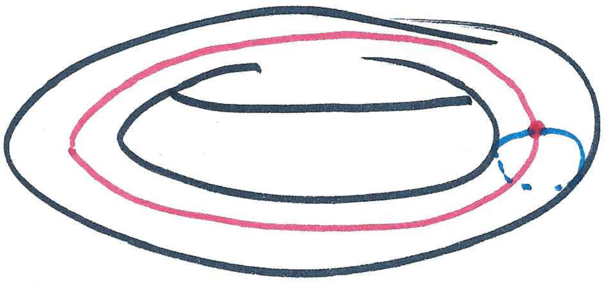
$= v - f$

Therefore, returning to $\iint_M K\,dM = 2\pi(v - f)$

we conclude $\iint_M K\,dM = 2\pi\,\chi(M)$. //

$$\chi(M) = v - e + f$$

Torus $\chi(T) = 1 - 2 + 1 = \underline{0}.$



Σ